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Best Approximation in Cone-Normed Space

التقريب الامثل في فضاءات القياس المخروطي

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Abstract

The question of whether cone metric spaces are real generalizations of metric spaces is proved, in the sense of Best Approximation, not to be affirmative.

Keywords: Cone, Cone metric, Best approximant.

ملخص

نبرهن هنا ان فضاءات القياس المخروطي ليست تعميما لمفهوم فضاء القياس وذلك من وجهة نظر التقريب الأمثل

الكلمات المفتاحية: مخروط، قياس مخروطي، مقرب امثل.

1. Introduction

Cone metric spaces were introduced in (Huang, L.G. & Zhang, X. 2007; 1468 -1476), by means of partially ordering real Banach spaces by specified cones. In (Abdeljawad, T. & *et al.* 2010; 739-753). and (Turkoglo, D. & *et al.* 2012), the notion of cone-normed spaces was introduced. Cone- metric spaces, and hence, cone –normed spaces were shown to be first countable topological spaces. The reader may consult (Turkoglo, D. & Abuloha, M. 2010; 789-796) for this development.

In (Asadi, M. & *et al.* 2011), it was shown that, in a sense, cone-metric spaces are not, really, generalizations of metric spaces .This was the motive for the authors to do further investigations.

As a first result of ours, we point out that:

1. **Fact:** Since, by (Asadi, M. & *et al.* 2011), mutual generations of metrics and cone metrics produce sequentially equivalent topologies, the fact that both topologies are first countable implies that they are the same topology.

Now we put things in place:

2. ***Definition, from*** (Huang, L.G. & Zhang, X. 2007; 1468 -1476)**:** Let (E, $\|.\|$) be a real Banach space and $\wp$ a subset of E. Then $\wp$ is called a cone if:

(a) $\wp$ is closed, convex, nonempty , and $\wp \neq \{0\}$.

(b) $a,b \in R; a,b \geq 0; x,y \in \wp \Rightarrow ax + by \in \wp$

(c) $x \in \wp$ and $-x \in \wp \Rightarrow x = 0$

3. ***Example, from*** (Rezapour, Sh. 2007; 85-88)***:*** Let E = ℓ^1, The absolutely summable real sequences. Then the set $\wp = \{x \in E : x_n \geq 0 \forall n\}$ is a cone in E.

For a cone $\wp \subset E$, we define (on E) the partial order $\leq_\wp$ with respect to $\wp$ as: $x \leq_\wp y \;\; if \;\; y - x \in \wp$. We write x < y to indicate that $x \leq_\wp$ but x $\neq$ y, and $x << y \;\; for \;\; y - x \in \wp^\circ$. (the interior of $\wp$). The cone $\wp$ is called normal if there is a positive number K such that: For x , y$\in E, if \;\; 0 \leq x \leq y$ then $\|x\| \leq K\|y\|$. .The smallest K is called the normal constant of $\wp$. $\wp$ is called strongly minihedral if every subset of E which is bounded above has a supremum. Throughout, we will assume that $\wp$ is a strongly minihedral normal cone with respect to a real Banach space (E, $\|.\|$). It therefore follows that every subset of $\wp$ has an infimum, (Abdeljawad, T. & *et al.* 2010; 739-753).

2. Cone-Normed Spaces

1. *Definition*: Suppose that $\wp$ is a cone in the normed space (E, $\|.\|$), and let X be a nonempty set. The pair (X, $\|.\|_c$) is called a cone- normed space relative to the cone $\wp$ *if* $\|.\|_c : X \to E$ is a function that satisfies:

(a) $o \le \|x\|_c \ \forall x \in X$, and equality holds iff x =0

(b) $\|ax\|_c = |a|\|x\|_c \ \forall a \in R \text{ and } x \in X$

(c) $\|x+y\|_c \le \|x\|_c + \|y\|_c \ \forall x \text{ and } y \in X$

It should be noted that: letting D (x,y) = $\|x-y\|_c$ defines a cone metric on the set X, but not conversely. For a rigorous development of cone metric spaces, we refer the readers to (Huang, L.G. Zhang, X. 2007; 1468 -1476). We construct the following example to show that cone metrics do not necessarily produce cone norms.

2. *Example*: Let X = ℓ^1, $\wp = [0,\infty)$, and let E =R

Let, for x, y $\in$ X, d (x,y) = $\sum_{n=1}^{\infty} |x_n - y_n|$, then let D (x,y) = $\frac{d(x,y)}{1+d(x,y)}$.

It is easy to see that D is a cone metric relative to the cone $\wp$ which is not compatible with any cone norm.

3. Best Approximation in Normed and Cone-Normed Spaces

The problem of best approximation is the problem of finding, in a given subset G of a normed space (X, $\|.\|$), or a cone- normed space (X, $\|.\|$), for a given point x $\in$X, an element which is the closest to x among all elements of G. This problem began, for normed spaces, in 1853 by

P.L. Chebychev who considered, for X , the spaces of all continuous real – valued functions on a closed interval [a,b]. For a best development of this topic, readers always consider (Singer, I. 1970).

In the year 2000, the first author supervised together with W. Deeb, a master's thesis at An-Najah University written by the second author (Al – Dwaik, S. 2000), where some major problems in best approximation thery in normed spaces were solved. The authors here will pick few results and in a senese, do the same in cone-normed spaces. Having done so, we enforce the feeling that cone –normed spaces are not really generalizations of normed spaces, and hence, cone – normed spaces are not really generalizations of normed spaces, and in turn, cone- metric spaces are not really generalizations of metric spaces.

1. ***Definition*:** Let (X, $\|.\|$ be a cone – normed space, G a nonempty set in X, and $x \in X$. We say that $g_\circ \in G$ is a cone-best approximant of x if $\|x-g_\circ\|_c \leq \|x-g_\circ\|_c \; \forall g \in G.$ We denote the set of best approximants of x in G by P_c(x,G). If $\forall x \in X, P_c(x,G) \neq \phi$, then G is called c-proximal in X and if $\forall x \in X$, P_c (x, G) is a singleton, G is called c-Chebychev.

In this manuscript, G is assumed to be a subspace of the cone-normed space (X,$\|.\|_C$).

2. ***Definition***: from (Huang, L.G. & Zhang, X. 2007; 1468 -1476): For x $\in$X, we define the cone distance $d_c(x,G) = \inf$

$$\{\|x-g\|_c : g \in G\}.$$

The definition makes sense because every subset of $\wp$ has an infimum as pointed out earlier. We now give our main results.

3. ***Theorem***: Let (X, $\|.\|$)$_c$ be a cone- normed space and G a subspace in X ,then:

(a) d_c(x+g,G) = d_c (x,G) ($\forall x \in X, g \in G$)

(b) $d_c(x+y,G) \le d_c(x,G)+d_c(y,G)$ ($\forall x, y \in X$)

(c) $d_c(ax,G) = |a| d_c(x,G) (\forall a \in R, x \in X)$

(d) $\|d_c(x,G) - d_c(y,G)\|_C \le \|x-y\|_C$ $(\forall x, y \in X)$.

Proof

For (a); Let $x \in X, g \in G,$ *and* $a >> 0$. Then by the definition of the infimum, $\exists g_\circ \in G$ such that $\|x - g_\circ\|_c \le d_c(x,G) + a$. So we have:

$$d_c(x+g,G) \le \|x+g-(g+g_\circ\|_c = \|x-g_\circ\|_c \le d_c(x,G)+a$$

But x,g and a were arbitrary, so by the minihedrality of the cone $\wp$, we conclude that :

$$d_c(x+g,G) \le d_c(x,G) (\forall x \in X, g \in G)......................(1)$$

Now, applying this relation to x+g in place of x , and –g in place of g, we get that:

$$d_c(x,G) \le d_c(x+g,G) (\forall x \in X, g \in G)......................(2)$$

Combining (1) and (2) we get the equality.

For (b); Let x,y$\in X$, and a>>0 , so $\frac{a}{2} = \frac{1}{2}$ a>>0 also .

$\exists g1, g2 \in G$ such that $\|x-g_1\|_c < d_c(x,G) + \frac{a}{2}, and \|y-g_2\|_c < d_c(y,G) + \frac{a}{2}$.

So , $d_c(x+y,G) \le \|(x+y-(g_1+g_2)\|_c$

$$\le \|x-g_1\|_c + \|y-g_2)\|_c$$

$$\le d_c(x,G) + \frac{a}{2} + d_c(y,G) + \frac{a}{2}$$

$$= d_c(x,G) + d_c(y,G) + a$$

Since a was arbitrary, we get that: $d_c(x+y,G) \le d_c(x,G) + d_c(y,G)$.

***For* (c);** Let $x \in X$ and $\alpha \neq 0$ be a scalar, and let a >> 0.

Pick $g_\circ \in G$ *for which* $\|x - g_\circ\|_c \le d_c(x,G) + \frac{a}{|\alpha|}$.

So , $d_c(\alpha x, G) \le \|\alpha x - \alpha g_\circ\|_c$

$$= |\alpha| \|x - g_\circ\|_c$$

$$\le |\alpha| d_c(x,G) + a$$

Since a was arbitrary, $d_c(\alpha x, G) \le |\alpha| d_c(x,G)$..................(1)

Now, replacing x by α x and α by $\frac{1}{\alpha}$, we get (by (1)) that:

$d_c(x,G) = d_c(\frac{1}{a}.\alpha x, G) \le \frac{1}{|\alpha|} d_c(\alpha x, G)$, and hence:

$|\alpha| d_c(x,G) \le d_c(\alpha x, G)$......... ....(2)

Combining (1) and (2) gives the required equality.

For (d); Let $x, y \in X$ and let a >> 0

Take $g_\circ \in G$ so that $\|y - g_\circ\|_c \le d_c(y,G) + a$

So, $d_c(x,G) \le \|x - g_\circ\|_c \le \|x - y\|_c + \|y - g_\circ\|_c$

$$\le \|x - y\|_c + d_c(y,G) + a$$

Since a was arbitrary , $d_c(x,G) - d_c(y,G) \le \|x - y\|_c$

Similarly, One gets , $d_c(y,G) - d_c(x,G) \le \|x - y\|_c$. Thus ,

$$\left\| d_c(x,G) - d_c(y,G) \right\|_C \le \left\| x - y \right\|_c$$

4. The Set of Best Approximants

In this section, we introduce some basic properties of the set Pc(x,G) of cone best approximants of x in G. the following example has a dual copy in (Al – Dwaik, S. 2000).

1. ***Example*:** Let E=R, $\wp = [0,\infty)$ and let X=R^2, being equipped with the cone norm $\|x\|_c = |x_1| + |x_2|$.

An easy calculation shows that, for x = (1,-1), and G = $\{(x_1, x_2) : x_1 = x_2\}$, $P_c(x,G) = \{(x_1, x_2) \in G : |x_1| \le 1\}$. Hence, consequently, the set P_c(x,G) is not a subspace of X .

2. ***Theorem*:** Let G be a subspace of a cone-normed space X. Then,

(a) If $x \in G$ then P_c (x,G) = $\{x\}$.

(b) If G is not closed then P_c (x,G) = ϕ

(c) P_c (x,G) is a convex set.

Proof

(a) Let $x \in G$. Then $d_c(x,G) = \inf \{\|y - g\|_c : g \in G\} = 0$

Thus, if $g \in p_c(x,G)$, then d_c(x,g)=0 .

Since X is a cone metric space g = x.

(b) Suppose that G is not closed.

Pick $x \in \bar{G}$/G. Thus, for each a >> 0, $\exists x_a \in G$ such that $\|x - x_a\|_c \le a$

Since $\wp$ is strongly minihedral, $\|x - x_a\|_c = 0$, So x = x_a, which implies that $x \in G$, a contradiction.

(c) Let $\delta = d_c(x, G)$.

The statement holds if P_c (x, G) is empty or a singleton.

Suppose that y, z $\in P_c$ (x, G) and y $\neq$ z

For $0 \leq \alpha \leq 1$, let w= α y + (1- α) z. Then:

$$\|x - w\|_c = \|x - (\alpha y + (1-\alpha)z + \alpha x - \alpha x\|_c$$
$$= \|x - \alpha y - (1-\alpha)z + \alpha x - \alpha x\|_c$$
$$= \|\alpha(x - y) + (1-\alpha)(x - z)\|_c$$
$$\leq \alpha\|x - y\|_c + (1-\alpha)\|x - z\|_c$$
$$= \alpha\delta + (1-\alpha)\delta$$
$$= \delta$$

Since G is a subspace, $w \in G$, which implies that $\delta \leq \|x - w\|_c$.

Therefore, $\|x - w\|_c = \delta$ and so P_c (x,G) is convex

3. *Definition*: (This definition was suggested by (Abdeljawad, T. & et al. 2010; 739-753))

Let (X, $\|.\|_c$) be a cone-normed space. Then, a subset A of X is said to be bounded if sup $\{\|x - y\|_c : x, y \in A\}$ exists in E.

4. *Theorem*: Let G be a subspace of a cone-normed space X. Then , for $x \in X$,

(a) P_c (x,G) is a bounded set.

(b) If G is closed then P_c (x,G) is a closed set.

Proof

(a) Let $g_0 \in P_c$ (x ,G) .

$$\|g_\circ\|_c = \|g_\circ - x + x\|_c$$
$$\leq \|g_\circ - x\|_c + \|x\|_c$$
$$\leq \|0 - x\|_c + \|x\|_c \qquad \text{(since } 0 \in G \text{)}$$
$$= 2\|x\|_c \in E$$

So, $P_c(x,G)$ is bounded.

(b) Suppose that $\delta = d_c$ (x, G), and let (g_n) be a sequence in P_c (x,G) which converges in (X, $\|.\|_C$) to g.

Since G is closed, $g \in G$

Now, for each $n \in N$, $\|x - g_n\|_c = \delta$. But, since the function,

$\|.\| : (X, \|.\|_C) \to E$ is continuous (easy to see) , $\|x - g\|_c = \delta$

Thus, P_c (x, G) is closed.

We conclude this work with the following theorem whose analogue can be found in (Singer, I. 1970). For this we need the following:

5. ***Definition*:** Let $(X, \|.\|_C)$ be a cone- normed space, $G \subset X$, and $x \in X$.

We say that x is orthogonal to G $(x \perp G)$ *if* $\|x\|_c \leq \|x + \alpha g\|_c$ for all scalars α and $x \in X$.

6. ***Theorem*:** Let $(X, \|.\|_C)$ be a cone-normed space, $G \subset X$, and $x \in X / \bar{G}$, and $g_\circ \in G$. Then , $g_\circ \in P_c(x,G)$ if and only if $x - g_\circ \perp G$

Proof: $g_\circ \in P_c(x,G)$ if and only if $\|x-g_\circ\| \le \|x-g_\circ+\alpha g\|_c$ for all scalars α and all $g \in G$. The conclusion now follows since G is a subspace of X.

References

- Huang, L.G. & Zhang, X. (2007). *Cone metric spaces and fixed point theorems of contractive mappings*, J.Math. Anal. Appl., 332, 1468 - 1476.
- Rezapour, Sh. (2007). *Best approximation in cone metric spaces*, Mathematica Moraveca, Vol.11, 85-88.
- Singer, I. (1970). *Best approximation in normed linear spaces by elements of linear subapaces*, Springe Verlag, Berlin.
- Abdeljawad, T. Turkoglo, D. & Abuloha, M. (2010). *Some theorems and examples of cone Banach spaces*, J. Computational Anal. and App. Vol.12. No.4. 739-753.
- Turkoglo, D. & Abuloha, M. (2010). *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta Mathematica, English series, 789-796. Vol.26.No,3.
- Asadi, M. Vaezpour, S.M. & Soleimani, H. (2011). *Metrizability of cone metric spaces*, ar Xiv: 1102.2353, v1[math.FA].
- Al –Dwaik, S. (2000). *The S-property and best approximation (Master's thesis*, An-Najah National University, Palestine).
- Turkoglo, D. Abuloha, M. & Abdeljawad, T. (2012). *Some theorems in cone metric spaces*, Journal of concrete and applicable mathematics, Vol.10.