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# On Pairwise Sublindel öf Spaces
## فضاءات ليندلوف الضعيفة الثنائية

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**Abstract:**

The purpose of the present paper is to introduce some generalizations of pairwise Lindelöf spaces, namely pairwise sublindelöf spaces, and to obtain some related results. We shall introduce two types of pairwise sublindelöf spaces, also a new typs of mappings, $p_1$-paralindelöf and $p_2$-paralindelöf mappings, are introduced.

***Keywords**:* Bitopological spaces, pairwise Lindelöf, $p_1$-paracompact, $p_2$-paracompact, pairwise regular, p-continuous functions.

**ملخص:**

في هذا البحث عرّفنا نوعين من فضاءات ليندلوف الضعيفة الثنائية كتعميم لفضاءات ليندلوف الثنائية ثم درسنا الفضاءات الجديدة من حيث وصفها وامتلاكها لبعض الخواص التبولوجية.

## 1. ***Introduction.***

The concept of bitopological spaces was initiated by Kelly [5]. A set X equipped with two topologies $\tau_1$ and $\tau_2$ is called a bitopological space and is denoted by $(X,\tau_1,\tau_2)$.

In a bitopological space $(X,\tau_1,\tau_2)$, $\tau_1$ is said to be regular with respect to $\tau_2$ [5], if for each point x in X and each $\tau_1$-closed set F with $x \notin F$, there are a $\tau_1$-open set U and a $\tau_2$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$. $(X,\tau_1,\tau_2)$ is pairwise regular [5] if $\tau_1$ is regular with respect to $\tau_2$ and $\tau_2$ is regular with respect to $\tau_1$.

A cover $U'$ of a bitopological space $(X,\tau_1,\tau_2)$ is called $\tau_1\tau_2$-open cover [1] if $U' \subseteq \tau_1 \cup \tau_2$. If, in addition , $U'$ contains at least one non-empty member of $\tau_1$ and at least one non-empty member of $\tau_2$, then $\mathcal{U}$ is called pairwise open, [1].

A bitopological space $(X,\tau_1,\tau_2)$ is called pairwise Lindelöf [1] if each pairwise open cover of X has a countable subcover. This concept also was studed by several authers like Hdeib and Fora, [3].

If $\mathcal{U}$ and $\mathcal{V}$ are $\tau_1\tau_2$-open covers of the bitopological space $(X,\tau_1,\tau_2)$ , then $\mathcal{U}$ is called a refinement of $\mathcal{V}$, [2], if each $U \in \mathcal{U} \cap \tau_i$ is contained in some $V \in \mathcal{V} \cap \tau_i$, i=1,2.

A collection $\mathcal{R}$ of subsets of the bitopological space $(X,\tau_1,\tau_2)$ is called $p_1$-locally finite, [2], if $\mathcal{R} \cap \tau_i$ is locally finite in $(X,\tau_i)$, i=1,2. Also a collection $\mathcal{R}$ of subsets of the bitopological space $(X,\tau_1,\tau_2)$ is called $p_2$-locally finite, [2], if $\mathcal{R} \cap \tau_i$ is locally finite in $(X,\tau_j)$ for each $i\neq j$ ;i,j=1,2.

Hdeib and Fora, [2], initiated a study of pairwise paracompact spaces. A bitopological space $(X,\tau_1,\tau_2)$ is called $p_1$-paracompact ( $p_2$-paracompact respectively ), if each pairwise open cover of X has a $p_1$-locally ( $p_2$-locally respectively ) finite $\tau_1\tau_2$-open refinement.

A function $f: (X,\tau_1,\tau_2) \to (Y,\rho_1,\rho_2)$ is called pairwise continuous, [6], if the functions $f: (X,\tau_1) \to (Y,\rho_1)$ and $f: (X,\tau_2) \to (Y,\rho_2)$ are continuous.

We shall use p- to denote pairwise, e.g. p-Lindelöf stands for pairwise Lindelöf. The $\tau_i$-closure of a set A will be denoted by $cl_iA$. ☐ and ☐ denote the set of real and rational numbers, respectively.

## 2. ***Definition and Subspaces of Pairwise Sublindelöf Spaces.***

In the following definition we give two cocepts of pairwise sublindelöf spaces in bitopological spaces where as the concept of sublindelöf spaces was initiated by Hdeib, [4].

***Definition 2.1:*** A bitopological space $(X,\tau_1,\tau_2)$ is called $p_i$-sublindelöf if every $p_i$-locally finite p-open cover of X has a countable subcover for i=1,2.

Obviously, every p-Lindelöf space is $p_1$-sublindelöf and $p_2$-sublindelöf.

The following example shows that a $p_1$-sublindelöf space and a $p_2$-sublindelöf space may fail to be p-Lindelöf also it shows that the $p_1$-sublindelöf property is not a hereditary property.

***Example 2.2:*** Let $X = R \times R$ , $\tau_1 = \tau_s \times \tau_s$ , $\tau_2 = \tau_u \times \tau_u$ where $\tau_s$ is the sorgenfrey line topology and $\tau_u$ is the usual ( standered ) topology.

The space $(X,\tau_1,\tau_2)$ is $p_1$-sublindelöf, indeed, if it is not $p_1$-sublindelöf, then there exists a $p_1$-locally finite p-open cover $\mathcal{U} = \{ U_\alpha : \alpha \in \Lambda \}$ of X such that $\mathcal{U}$ has no countable subcover. Obviously $\mathcal{U}$ must be an uncountable cover of X. Now if $A = \{ (x,y) : x,y \in Q \}$ , then A is dense in $(X,\tau_1)$ and dense in $(X,\tau_2)$. Therefore for each $\alpha \in \Lambda$, $U_\alpha \cap A \neq \phi$. Since A is countable, some elements of A must be contained in uncountably many members of $\mathcal{U}$. So there is an element of A which is contained in uncountably many members of $\mathcal{U} \cap \tau_1$ or uncountably many members of $\mathcal{U} \cap \tau_2$, say an element of A is contained in uncountably many members of $\mathcal{U} \cap \tau_1$ ,so any $\tau_1$-open neighborhood of this element must intersect uncountably many members of $\mathcal{U} \cap \tau_1$ and this contradicts the fact that $\mathcal{U}$ is $p_1$-locally finite.

In a similar way we can show that $(X,\tau_1,\tau_2)$ is $p_2$-sublindelöf.

Now the subspace $L = \{ (x,y) : y= -x \}$ is a $\tau_1$-closed subspace of $(X,\tau_1,\tau_2)$ which is not $p_1$-sublindelöf, because $\{\{x\} : x \in L \} \cup \{ [ (-2, -1) \times (1,2) ] \cap L \}$ is a p-open $p_1$-locally finite cover of L which has no countable subcover. Also, since this subspace is not $p_1$-sublindelöf then it is not p-Lindelöf and so $(X,\tau_1,\tau_2)$ is not p-Lindelöf.

***Definition 2.3:*** A subset F of a bitopological space X is said to be relatively $p_i$-sublindelöf if every $p_i$-locally finite (in X) p-open (in X) cover of F has a countable subcover for i=1,2.

***Definition 2.4:*** A subset F of a bitopological space X is said to be relatively $p_i$-paracompact if every p-open (in X) cover of F has a $p_i$-locally finite (in X) $\tau_1\tau_2$-open (in X) refinement for i=1,2.

It is easy to see that every $\tau_1$-closed and every $\tau_2$-closed subset of a $p_i$-sublindelöf space is relatively $p_i$-sublindelöf for i=1,2.

***Remark 2.5:*** (i) If F is a relatively $p_i$-paracompact, relatively $p_i$-sublindelöf subset of a bitopological space X, then F is a p-Lindelöf subspace of X.

(ii) If F is $p_i$-paracompact, $p_i$-sublindelöf subspace of a bitopological space X, then F is a p-Lindelöf subspace of X.

***Definition 2.6 [2] :*** Let $(X, \tau_1, \tau_2)$ be a bitopological space. A subset D of X is called p-dense if $cl_1D = cl_2D = X$ and for every p-open (in X) $p_2$-locally finite (in X) countable cover $\mho$ of D we have

$$cl_1 \cup \{V : V \in \mho \cap \tau_1\} \subseteq cl_2 \cup \{V : V \in \mho \cap \tau_1\} \text{ or}$$
$$cl_2 \cup \{V : V \in \mho \cap \tau_2\} \subseteq cl_1 \cup \{V : V \in \mho \cap \tau_2\}.$$

It is proved in Hdeib and Fora [2] that a p-regular $p_2$-paracompact space X with a p-dense p-Lindelöf subspace A is p-Lindelöf. The following theorem is a generalization of this result.

***Theorem 2.7:*** Let $(X,\tau_1,\tau_2)$ be a p-regular $p_2$-paracompact space having a p-dense relatively $p_2$-sublindelöf subset A. Then X is p-Lindelöf.

***Proof:*** Let $U'$ be any p-open cover of X. Then there exist $U_0 \neq \phi$ and $V_0 \neq \phi$ such that $U_0 \in U' \cap \tau_1$ and $V_0 \in U' \cap \tau_2$. Since $(X,\tau_1,\tau_2)$ is p-regular and $p_2$-paracompact, $U'$ has a $p_2$-locally finite $\tau_1\tau_2$-open refinement $\mho$ such that for each $V \in \mho \cap \tau_1$; $cl_2V$ is contained in some $U \in U' \cap \tau_1$ and for each $V \in \mho \cap \tau_2$; $cl_1V$ is contained in some $U \in U' \cap \tau_2$. Now $\mho$ is a p-open cover of A ( add $U_0$ to $\mho$ if $\mho \cap \tau_1 = \phi$ and add $V_0$ to $\mho$ if $\mho \cap \tau_2 = \phi$ ). Since A is relatively $p_2$-sublindelöf there exists a countable subcover $\mho_0$ of $\mho$ which covers A. We may assume that $\mho_0$ is p-open. But A is p-dense in X, so we have two cases to consider:

Case 1. $cl_1 \cup \{ V : V \in \mho_0 \cap \tau_1 \} \subseteq cl_2 \cup \{ V : V \in \mho_0 \cap \tau_1 \}$: In this case we have

$X = cl_1A$

$\subseteq cl_1\cup \{ V : V\in \mho_0 \}$

$= cl_1\cup \{ V : V\in \mho_0\cap\tau_1 \} \cup cl_1\cup\{ V : V\in \mho_0\cap\tau_2 \}$

$\subseteq [ cl_2\cup\{ V : V\in \mho_0\cap\tau_1 \} ] \cup [ cl_1\cup\{ V : V\in \mho_0\cap\tau_2 \} ]$

$= [ \cup\{ cl_2V : V\in \mho_0\cap\tau_1 \} ] \cup [ \cup\{ cl_1V : V\in \mho_0\cap\tau_2 \} ]$

Case 2. $cl_2\cup \{ V : V\in \mho_0\cap\tau_2 \} \subseteq cl_1\cup \{ V : V\in \mho_0\cap\tau_2 \}$: In this case we have

$X = cl_2A$

$\subseteq cl_2\cup \{ V : V\in \mho_0 \}$

$= cl_2\cup \{ V : V\in \mho_0\cap\tau_1 \} \cup cl_2\cup\{ V : V\in \mho_0\cap\tau_2 \}$

$\subseteq [ cl_2\cup\{ V : V\in \mho_0\cap\tau_1 \} ] \cup [ cl_1\cup\{ V : V\in \mho_0\cap\tau_2 \} ]$

$= [ \cup\{ cl_2V : V\in \mho_0\cap\tau_1 \} ] \cup [ \cup\{ cl_1V : V\in \mho_0\cap\tau_2 \} ]$

Now in each case we get

$X \subseteq [ \cup\{ cl_2V : V\in \mho_0\cap\tau_1 \} ] \cup [ \cup\{ cl_1V : V\in \mho_0\cap\tau_2 \} ]$

For each $V\in \mho_0\cap\tau_i$ choose one element $U_V\in \mathcal{U}$ such that $cl_jV\subseteq U_V$ ( $i\neq j$; $i,j=1,2$ ). Then $\{ U_V : V\in \mho_0 \}$ is a countable subcover of $\mathcal{U}$ for X. Hence the result.

***Corollary 2.8:*** Let X be a p-regular $p_2$-paracompact bitopological space having a p-dense $p_2$-sublindelöf subset A. Then X is p-Lindelöf.

***Proof:*** Since every $p_2$-sublindelöf subspace is relatively $p_2$-sublindelöf, the result follows from Theorem 2.7.

Now the result of Hdeib and Fora [2], mentioned before Theorem 2.7, can be obtained directly from Corollary 2.8.

## 3. *Pairwise Sublindelöf Mappings.*

In this section, we define new types of mappings namely $p_1$-paralindelöf and $p_2$-paralindelöf.

***Definition 3.1:*** A p-continuous mapping f from a bitopological space $(X,\tau_1,\tau_2)$ onto a bitopological space $(Y,\rho_1,\rho_2)$ is called $p_i$-paralindelöf if for each $p_i$-locally finite p-open cover $\mathcal{U}$ of X, there is a $p_i$-

locally finite p-open cover Ʊ of Y such that for each V in Ʊ, $f^{-1}(V)$ is contained in the union of countably many members of U for i=1,2.

***Theorem 3.2:*** Let f be a p-continuous mapping from $(X,\tau_1,\tau_2)$ onto a $p_1$-sublindelöf space $(Y,\rho_1,\rho_2)$. Then X is $p_1$-sublindelöf if and only if f is $p_1$-paralindelöf.

***Proof :*** Suppose $(X,\tau_1,\tau_2)$ is $p_1$-sublindelöf. Let U be a $p_1$-locally finite open cover of X. Then U has a countable subcover U′. Now { Y } is a $p_1$-locally finite p-open cover of Y such that $f^{-1}(Y)$ is contained in the union of countably many members of U. Hence f is $p_1$-paralindelöf.

Conversely, suppose that f: X $\rightarrow$ Y is $p_1$-paralindelöf and Y is $p_1$-sublindelöf. Let U be a $p_1$-locally finite p-open cover of X. Then there is a $p_1$-locally finite p-open cover Ʊ of Y such that for each V in Ʊ, $f^{-1}(V)$ is a subset of a union of countably many members of U. Since Y is $p_1$-sublindelöf, Ʊ has acountable subcover Ʊ′. Now X = $\cup$\{ $f^{-1}(V)$ : V$\in$Ʊ′ \} and each $f^{1}(V)$ is covered by a countable subcollection of U. Consequently X is covered by a countable subcollection of U. Then X is $p_1$-sublindelöf .

Similarly we can prove the following theorem:

***Theorem 3.3 :*** Let f be a p-continuous mapping from $(X,\tau_1,\tau_2)$ onto a $p_2$-sublindelöf space $(Y,\rho_1,\rho_2)$. Then X is $p_2$-sublindelöf if and only if f is $p_2$-paralindelöf.

***Theorem 3.4 :*** Let f be a p-continuous mapping from a bitopological space X onto a bitopological space Y. Then Y is $p_1$-sublindelöf if X is so.

***Proof :*** Let X be a $p_1$-sublindelöf space and Ʊ be a $p_1$-locally finite p-open cover of Y. Since f is a p-continuous mapping, $f^{-1}$(Ʊ) is $p_1$-locally finite p-open cover of X. Since X is $p_1$-sublindelöf, $f^{-1}$(Ʊ) has a countable subcover, say $f^{-1}$(Ʊ′). Therefore Ʊ′ is a countable subcover of Ʊ .

Similarly we can prove the followig theorem:

***Theorem 3.5 :*** Let f be a p-continuous mapping from a bitopological space X onto a bitopological space Y. Then Y is $p_2$-sublindelöf if X is so.

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