Composition operators on de Branges–Rovnyak spaces

مؤثرات التركيب على فضاءات دي برانجز روفنياك

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Abstract

We obtain invariant de Branges–Rovnyak spaces for the composition operator C_φ. We also find the images of the composition operators on the de Branges–Rovnyak spaces, $\mathscr{H}(b)$, for special cases of b.

Keywords:Compsoition operators, Hardy spaces, de Branges–Rovnyak spaces.

ملخص

نعرض في هذه الدراسة بعض فضاءات دي برانجز روفنياك التي لا تتغير تحت تاثير مؤثرات التركيب. كذلك نجد مدى بعض مؤثرات التركيب على حالات خاصة من هذه الفضاءات.

الكلمات المفتاحية : مؤثرات التركيب، فضاءات هاردي، فضاءات دي برانجز روفنياك..

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ان البحث مستل من رسالة الماجستير للطالبة غادة ناصر بعنوان مؤثرات التركيب على فضاء دي برانجز روفنياك والتي تم مناقشتها في جامعة النجاح الوطنية بتاريخ ٢٠١٧-٨-٢٩

1 Introduction

2 Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc $\{z : |z| < 1\}$, and $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$. If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then the composition operator C_φ is the linear operator defined by $C_\varphi f = f \circ \varphi$, f is analytic on $\mathbb{D}$. The composition operator is intensively studied on various function spaces in the past decades; the list of references is too long, for example (Cowen & MacCluer, 1995; Fricain, Karaki, & Mashreghi, 2016; Hammond, 2003; Jafari & Consortium, 1998; Karsisto, 2003; Lefèvre, Li, Queffélec, & Rodríguez-Piazza, 2015; Li, Queffélec, & Rodríguez-Piazza, 2012; Lyubarskii & Malinnikova, 2012; Mashreghi & Shabankhah, 2014, 2013; Sarker & University, 2008; Shapishapirobookro, 1993; Singh & Manhas, 1993). We will focus on the Hilbert-Hardy space H^2 and spaces live inside it, the monographs (Duren, 2000; Koosis, 1998) contain the basic theory of Hardy spaces. We present the basic definitions and properties. The Hardy space H^2 is the space of all analytic functions f in the unit disk $\mathbb{D}$, for which the norm

$$||f||^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \tag{2.1}$$

is finite. The space H^∞ denotes the space of all bounded analytic functions on $\mathbb{D}$ normed by

$$||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

It is well-known that H^2 is a reproducing kernel Hilbert space, that is for f in H^2,

$$f(\lambda) = < f, k_\lambda >,$$

$\lambda \in \mathbb{D}$ with the kernel $k_\lambda(z) = (1 - \bar{\lambda} z)^{-1}$. If f is in H^2 then it can be factorized in a canonical way, specifically, $f(z) = B(z)S(z)O(z)$, where B

is a Blaschke product of the form

$$B(z) = e^{i\gamma} \prod_{j=1}^{\infty} \frac{z_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z},$$

S is a singular inner function and O is an outer function, that is a function of the form

$$O(re^{i\theta}) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} k(e^{it}) dt \right),$$

where k is a real-valued integrable function.

The scene shifts to subspaces in H^2, the so called the de Branges-Rovnyak spaces. Let b be in the closed unit ball of H^∞. Then the de Branges-Rovnyak space $\mathscr{H}(b)$ is the range space of $(I - T_b T_{\bar{b}})^{1/2} H^2$ equipped with the norm which makes $(I - T_b T_{\bar{b}})^{1/2}$ a partial isometry, where T_b is the Toeplitz operator on H^2, $(T_b f = P_+ bf)$. These spaces play an important role in many questions in function theory, operator theory, and in the model theory. For the detailed treatments of $\mathscr{H}(b)$ one can consult (Fricain & Mashreghi, 2016; Sarason, 1994). $\mathscr{H}(b)$ spaces are reproducing kernel Hilbert spaces with reproducing kernel

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)} b(z)}{1 - \bar{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

and $f(\lambda) = \langle f, k_\lambda^b \rangle_b$ for all f in $\mathscr{H}(b)$. If b is an inner function, that is a function in H^∞ of modulus 1 almost everywhere on $\mathbb{T}$, then $\mathscr{H}(b)$ becomes the well-known model space $\mathscr{H}(b) = K_b := H^2 \ominus bH^2$.

The exact contents of the de Branges-Rovnyak spaces $\mathscr{H}(b)$, for general b, are not clear. Recently authors of (Fricain, Hartmann, & Ross, 2016) characterized de Branges–Rovnyak spaces for b is rational or $b = q^r$, where q is a rational outer function in the unit ball of H^∞ and $r \geq 0$. They precisely determined which functions belong to $\mathscr{H}(b)$ in such cases. For example we have,

$$\mathscr{H}\left(\frac{1}{2}(1+z) \right) = (z-1)H^2 \oplus \mathbb{C},$$

and

$$\mathscr{H}\left(\frac{1}{2}(1-z)(1+z)\right) = (z-i)(z+i)H^2 \oplus \bigvee\{z+i, z-i\}.$$

These are examples from (Fricain, Hartmann, & Ross, 2016).

3 Composition operators on $\mathscr{H}(b)$ into itself

In this section we will study the composition operator C_φ on the space $\mathscr{H}(b) = (z-\zeta)H^2 \oplus \mathbb{C}$ where φ is analytic and maps the unit disc $\mathbb{D}$ into itself, $\zeta \in \mathbb{T}$. We have obtained sufficient and necessary conditions for C_φ to map $\mathscr{H}(b)$ into itself. We have,

Theorem 3.1. *Let $\zeta \in \mathbb{T}$, and $\mathscr{H}(b) = (z-\zeta)H^2 \oplus \mathbb{C}$. If φ is an analytic self–map of $\mathbb{D}$ and $\varphi(\zeta) = \zeta$ then*

$$C_\varphi : \mathscr{H}(b) \longrightarrow \mathscr{H}(b).$$

Proof. Let $f \in \mathscr{H}(b)$ such that $f = (z-\zeta)g + c$, $g \in H^2$ and c is constant, then

$$\begin{aligned} f \circ \varphi &= (\varphi(z) - \zeta)g(\varphi(z)) + c \\ &= (z-\zeta)h(z)g(\varphi(z)) + c \in \mathscr{H}(b), \end{aligned}$$

$g(\varphi(z)) \in H^2$, and $h(z) \in H^\infty$. □

The converse of the previous theorem is still true if we assume that φ is rational.

Theorem 3.2. *Suppose $\mathscr{H}(b) = (z-\zeta)H^2 \oplus \mathbb{C}$, $\zeta \in \mathbb{T}$. Let φ be a rational analytic function, such that φ maps the unit disc $\mathbb{D}$ into itself, then*

$$C_\varphi : \mathscr{H}(b) \longrightarrow \mathscr{H}(b)$$

if and only if $\varphi(\zeta) = \zeta$.

Proof. Theorem 3.1 proves the sufficiency. For the converse, suppose $C_\varphi : \mathscr{H}(b) \longrightarrow \mathscr{H}(b)$. Let

$$\begin{aligned} f = k_0^b &= 1 - \overline{b(0)}b(z) \\ &= 1 - \frac{1}{c^2}(1+\gamma z) \\ &= 1 - \frac{1}{c^2} - \frac{1}{c^2}\gamma z \\ &= 1 - \frac{1}{c^2} - \frac{\gamma}{c^2}(z - \zeta + \zeta) \\ &= \frac{-\gamma}{c^2}(z-\zeta) + 1 - \frac{\zeta\gamma + 1}{c^2} \in \mathscr{H}(b), \end{aligned}$$

then

$$C_\varphi f = \frac{-\gamma}{c^2}(\varphi(z) - \zeta) + 1 - \frac{\zeta\gamma+1}{c^2} \in \mathscr{H}(b),$$

therefore, we can write $(\varphi(z) - \zeta)$ as $(z-\zeta)h$, where $h \in H^2$, thus $\varphi(\zeta) = \zeta$. □

4 Composition operator on $\mathscr{H}(b)$

In this section we will give several examples of composition operators C_φ that map de Branges-Rovnyak spaces to different de Branges-Rovnyak spaces.

Theorem 4.1. *If* $B(z) = \left(\frac{a-z}{1-\bar{a}z}\right)^2$ *then*

$$C_B : \mathscr{H}\left(\frac{1}{2}(1+z)\right) \longrightarrow \mathscr{H}\left(\frac{1}{2}(z-i)(z+i)\right)$$

where $a \in (-1,1)$.

Proof. Let $f \in \mathscr{H}(\frac{1}{2}(1+z))$ such that $f = (z-1)g + c$ where $g \in H^2$,

$c \in \mathbb{C}$ then

$$
\begin{aligned}
C_B f = f(B(z)) &= (B(z)-1)g + c \\
&= \left(\left(\frac{a-z}{1-\bar{a}z} \right)^2 - 1 \right) g + c \\
&= \left(\frac{(a-z)^2 - (1-\bar{a}z)^2}{(1-\bar{a}z)^2} \right) g + c \\
&= \left(\frac{z^2(1-\bar{a}^2) + z(2\bar{a}-2a) + a^2 - 1}{(1-\bar{a}z)^2} \right) g + c \\
&= \left(\frac{z^2(1-a^2) - (1-a^2)}{(1-az)^2} \right) g + c \ \ (since\, a\, is\, real) \\
&= (z^2-1)\frac{(1-a^2)g}{(1-az)^2} + c \\
&= (z-1)(z+1)\frac{(1-a^2)g}{(1-az)^2} + c \\
&\in \mathscr{H}(\frac{1}{2}(z-i)(z+i)) \\
&= (z-1)(z+1)H^2 \oplus \bigvee\{1+z, 1-z\}.
\end{aligned}
$$

□

Using the same technique one can prove each of the followings,

Theorem 4.2. *If* $b = \frac{1}{2}(z+1)$ *and* $B(z) = \frac{z-a_1}{1-\overline{a_1}z} \cdot \frac{z-a_2}{1-\overline{a_2}z}$ *then*

$$C_B : \mathscr{H}(\frac{1}{2}(z+1)) \longrightarrow \mathscr{H}(\frac{1}{2}(z-i)(z+i))$$

where a_1, a_2 *are real and* $|a_1| \le 1$, $|a_2| \le 1$.

Theorem 4.3. *If* $b = \frac{1}{2}(1+z)$ *and* $B(z) = \left(\frac{a-z}{1-\bar{a}z}\right)$ *then*

$$C_B : \mathscr{H}(\frac{1}{2}(1+z)) \longrightarrow \mathscr{H}(\frac{1}{2}(1-z))$$

where a *is real and* $|a| \le 1$.

Theorem 4.4. *If* $b = \frac{1}{2}(1-z)$ *and* $B(z) = z\left(\frac{a-z}{1-\bar{a}z}\right)$ *then*

$$C_B : \mathscr{H}(\frac{1}{2}(1-z)) \longrightarrow \mathscr{H}(\frac{1}{2}(z-i)(z+i))$$

where a *is real and* $|a| \leq 1$.

5 Composition operator on $\mathscr{H}(b)$, polynomials

Theorem 5.1. *(Fricain, Hartmann, & Ross, 2016, Corollary 5.10) Suppose* q *is a polynomial outer function of degree* s *and let* a *be the Pythagorean mate for* q. *Let* N *be the number of zeros of* a *on* $\mathbb{T}$ *counted with multiplicities. Then the following are equivalent:*

1. $\mathscr{H}(q) = \mathscr{M}(a) \oplus \mathscr{P}_{N-1}$
2. $N = s$.

Theorem 5.2. *Suppose we have* q *that satisfies Theorem 5.1. If* φ *is an analytic self–map of* $\mathbb{D}$ *and of the form*

$$\varphi(z) = a(z)h(z) + z,$$

then

$$C_\varphi : \mathscr{H}(q) \to \mathscr{H}(q)$$

Proof. Let q be a polynomial of degree N, Suppose that its Pythagorean is of the form

$$a(z) = \prod_{k=1}^{n}(z-\zeta_k)^{m_k}h(z),$$

and $\sum_{n=1}^{n} m_j = N$.

Take $f \in \mathscr{H}(q)$. Say

$$f = \underbrace{\prod_{k=1}^{n}(z-\zeta_k)^{m_k}h(z)}_{f_1} + \underbrace{c_0 + c_1 z + \ldots + c_{N-1}z^{N-1}}_{f_2}.$$

Then, the composition with f_1 is:

$$\begin{aligned}(C_\varphi f_1)(z) &= \prod_{k=1}^{n}(a(z)h(z)+z-\zeta_k)^{m_k}h(z)\\ &= \prod_{k=1}^{n}(z-\zeta_k)^{m_k}\left(\frac{a(z)h(z)}{z-\zeta_k}+1\right)^{m_k}h(z)\\ &= \prod_{k=1}^{n}(z-\zeta_k)^{m_k}g(z)\in\mathscr{M}(a).\end{aligned}$$

And the composition with f_2 is

$$(C_\varphi f_2)(z) = c_0+c_1\Big(a(z)h(z)+z\Big)+\ldots+c_{N-1}\Big(a(z)h(z)+z\Big)^{N-1}.$$

The Binomial theorem easily implies that the last equality takes the form,

$$(C_\varphi f_2)(z) = a(z)g(z)+\alpha_0+\alpha_1 z+\ldots+\alpha_{N-1}z^{N-1}\in\mathscr{H}(q).$$

So, $C_\varphi f = C_\varphi f_1 + C_\varphi f_2 \in \mathscr{H}(q)$.

$\square$

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