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Hyers-Ulam-Rassias Stability for Linear and Semi-Linear Systems of Differential Equations

استقرار الأنظمة الخطية وشبه الخطية للمعادلات التفاضلية بمعنى هايرز-أولام-راسياس

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Abstract

This paper considers Hyers-Ulam-Rassias Stability for Linear and Semi-Linear Systems of Differential Equations. We establish sufficient conditions of Hyers-Ulam-Rassias stability and Hyers-Ulam stability for linear and semi-linear systems of differential equations. Illustrative examples will be given.

Keywords: Hyers-Ulam-Rassias Stability, Semi-linear Systems, Differential Equations.

ملخص

هدفت هذه الدراسة إلى استكشاف استقرار أنظمة المعادلات التفاضلية الخطية وشبه الخطية بمفهوم هايرز-أولام-راسياس، فقد تم الحصول على الشروط الكافية لاستقرار أنظمة المعادلة التفاضلية الخطية وشبه الخطية بمفهوم هايرز-أولام وبمفهوم هايرز-أولام-راسياس. وللوصول إلى نتائج الدراسة استخدم الباحث البرهان المباشرمن ثم تقدير القيم العظمى لحلول أنظمة المعادلات التفاضلية الخطية وشبه الخطية ذات المعاملات الثابتة تقريباً، بالإضافة إلى ذلك فقد أثبت الباحث استقرار أنظمة المعادلات التفاضلية الخطية وشبه الخطية بمفهوم هايرز-أولام-راسياس عندما تكون المصفوفة المعاملة متغيرة ومتصلة. ولتوضيح نتائج الدراسة النظرية قدم الباحث بعض الأمثلة.

الكلمات المفتاحية: هايرز-أولام، هايرز-أولام-راسياس، استقرار أنظمة المعادلات التفاضلية، الأنظمة شبه الخطية.

1- Introduction

In 1940, (Ulam, S.M.) posed the stability problem of functional equations. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when (Hyers, D. H.) gave a partial solution to Ulam's problem. During the last two decades very, important contributions to the stability problems of functional equations were given by many mathematicians (Gavruta, P. A; Jun, K. W. Lee, Y. H.; Jung, S. M. (2001); Jung, S. M. (1996); Miura, T. Takahasi S. E & Choda, H.; Park, C. G. (2002).; Park, C. G. (2005); Park, C. G. Cho, Y.-S. & Han M. (2007).; Rassias, T. M.). More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: The differential equation $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0$ has the Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that

$$\left| F(t, y(t), y'(t), ..., y^{(n)}(t)) \right| \leq \varepsilon$$

there exists a solution y_0 of the differential equation such that

$$| y(t) - y_0(t) | \leq K(\varepsilon)$$

and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza (Obloza, M.; Okunuga, S. A. Ehigie, J. O. Sofoluwe, A. B.). Thereafter, Alsina and Ger (Alsina, C. Ger, R.) have studied the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers (Li, Y. & Shen, Y.; Wang, G. Zhou, M. & Sun, L. (2008).) by using the method of integral factors. The results given in (Jung, S. M.; Rus, I. -A,B) have been generalized by Popa and Rus (Popa D. & Rus, I. 2011; 2012). for the linear differential equations of nth order with constant coefficients. (Gordji *et al.* Gordji, M. E. Cho, Y. J. Ghaemi, M. B. & Alizadeh, B.) get sufficient conditions for Hyers-Ulam stability of the first order and the second order nonlinear partial differential equations. Lungu and Craciun (Lungu, N. & Craciun, C.) established results on the Ulam-Hyers stability and the generalized Ulam-Hyers-Rassias stability of nonlinear hyperbolic partial differential equations.

In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order (Gavruta, P. S. Jung, & Li, Y. ; Li, Y.; Miura, T. Miyajima S. & Takahasi. S.-E.; Takahasi, E. Miura T. & Miyajima, S.).

The objective of this article is to investigate the Hyers-Ulam-Rassias stability for the linear systems of differential equations

$$x' = Ax + f(t) \tag{1.1}$$

and

$$x' = (A + B(t))\,x \tag{1.2}$$

Moreover, in this paper we consider the Hyers-Ulam-Rassias Stability for the semi-linear system

$$x' = Ax + g(t, x) \tag{1.3}$$

with the initial condition

$$x(0) = x_0 \tag{1.4}$$

where A is a constant $n \times n$ matrix, $f(t) \in \mathbb{R}^n$ is a continuous vector column, $B(t)$ is a matrix valued on the interval function I, and $g(t, x)$ is continuous in $I \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|g(t, x) - g(t, z_0)| \leq \gamma(t)|x - z_0| \tag{1.5}$$

where $g(t, 0) = 0$ and $I = [0, \infty)$.

It should be noted that Euclidean n-space $\mathbb{R}^n, n \geq 1$, is equipped with the distance $\left|x - y\right| = \left[\sum_{i=1}^{n} |x_i - y_i|^2\right]^{\frac{1}{2}}$ unless otherwise stated.

2- Preliminaries

We introduce some definitions as follows:

Definition 2.1 (Jung, S. M.**)** Let $\phi(t) = col(\phi_1(t), \phi_2(t), ..., \phi_n(t))$ and $\varphi(t) = |\phi(t)|$ such that $\phi_i : I \to [0,\infty), i = 1,2,...,n$. We say that equation (1.1) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0,\infty)$ if there exists a positive constant $k > 0$ with the following property: For each $x(t) \in C^1(I, \mathbb{R}^n)$, if

$$\left|x' - Ax - f(t)\right| \le \varphi(t), \tag{2.1}$$

then there exists some $z_0(t) \in C^1(I, \mathbb{R}^n)$ of the equation (1.1) such that

$$\|x(t) - z_0(t)\| \le k\varphi(t)\ , t \in [0,\infty) \tag{2.2}$$

Definition 2.2 (Jung, S. M. **)** Let $\phi(t) = col(\phi_1(t), \phi_2(t), ..., \phi_n(t))$ and $\varphi(t) = |\phi(t)|$ such that $\phi_i : I \to [0,\infty), i = 1,2,...,n$. We say that equation (1.2) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi : I \to [0,\infty)$ if there exists a positive constant $k > 0$ with the following property: For each $x(t) \in C^1(I, \mathbb{R}^n)$, if

$$\left|x' - Ax - g(t,x)\right| \le \varphi(t), \tag{2.3}$$

then there exists some $z_0(t) \in C^1(I, \mathbb{R}^n)$ of the equation (1.3) such that

$$\|x(t) - z_0(t)\| \le k\varphi(t)\ \ , t \in I \tag{2.4}$$

Lemma 2.1 (Gronwall's Inequality) Let u(t) and v(t) be nonnegative continuous functions on some interval $0 < t_0 \le t \le t_0 + a$. Also, let the function $h(t)$ be positive, continuous, and monotonically nondecreasing on $[t_{0,} t_0 + a]$ and satisfy the inequality

$$u(t) \leq h(t) + \int_{t_0}^{t} u(s)v(s)ds$$

then, there holds the inequality

$$u(t) \leq h(t)\exp\left(\int_{t_0}^{t} v(s)ds\right), \quad \text{for } t_0 \leq t \leq t_0 + a$$

For the proof of Lemma 2.1, see (Plaat, O.).

Lemma2.2 (Samoilenko, A. M. Krivosheya, S. A. & Perestyuk, N. A.**)** Let A(t) be an $n \times n$ matrix whose elements are functions of parameter t. If

$$A(t).\int_{0}^{t} A(r)dr = \int_{0}^{t} A(r)dr.\,A(t),$$

then the solution of the homogeneous system

$$x' = Ax$$

has the form $x(t) = e^{\int_0^t A(r)dr} x_0$.

3- Main Results on Hyers-Ulam-Rassias Stability

Theorem 3.1 Let A be a constant $n \times n$ matrix, $f(t) \in \mathbb{R}^n$ be a continuous vector column in the interval I. Suppose $x(t)$ satisfies the inequality (1.2) with initial conditions (1.4) on the interval $0 \leq t \leq T \leq \infty$ and $\varphi(t) : [0,\infty) \to (0,\infty)$ is a continuous function such that

$$\int_0^\infty \left\| e^{-A(s-t)} \right\| \varphi(s) ds \leq C\varphi(t), \quad \forall t \geq 0. \tag{3.1}$$

Then the solution of (1.2) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let $x(t)$ be an approximate solution of the initial value problem (1.1), (1.3). We will show that there exists a function $z_0(t)$ satisfying (1.1) and (1.3) such that

$$\|x(t) - z_0(t)\| \leq k\varphi(t)$$

The inequality (2.1) implies that

$$-\phi(t) \leq x' - Ax - f(t) \leq \phi(t). \tag{3.2}$$

Left-multiplying the inequality (3.2) by matrix integrating factor e^{-At}, we get

$$-e^{-At}\phi(t) \leq \frac{d}{dt}\left(e^{-At}x\right) - e^{-At}f(t) \leq e^{-At}\phi(t). \tag{3.3}$$

Integrating (3.3) from 0 to t, we have

$$-\int_0^t e^{-As}\phi(s)ds \leq e^{-At}x - x_0 - \int_0^t e^{-As}f(s)ds \leq \int_0^t e^{-As}\phi(s)ds,$$

or, equivalently we get

$$-\int_0^t e^{A(t-s)}\phi(s)ds \leq x - x_0e^{At} - \int_0^t e^{A(t-s)}f(s)ds \leq \int_0^t e^{A(t-s)}\phi(s)ds.$$

It is clear to see that

$$z_0 = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s)ds$$

satisfies Eq. (1.1) with the initial condition (1.3).

Now estimate the difference

$$| x(t) - z_0(t) | = \left| x - e^{At}x_0 - \int_0^t e^{A(t-s)} f(s)ds \right| \le C\varphi(t).$$

Consequently, we have

$$\|x(t) - z_0(t)\| \le C\varphi(t),$$

which completes the proof of Theorem 3.1.

Corollary 3.1 Replacing $\varphi(t)$ by ε in the inequality (3.2) we can get Hyers-Ulam stability for Eq. (1.1) in the interval $0 \le t_0 \le t \le T$, i.e. if $x(t)$ satisfies

$$\left| x' - Ax - f(t) \right| \le \varepsilon,$$

with the initial condition $x(t_0) = x_0$, then there exists some $z_0(t)$ of the equation (1.1) such that

$$\|x(t) - z_0(t)\| \le k\varepsilon .$$

The proof of Corollary 3.1 is quite similar to the proof of Theorem 3.1 and will therefore be omitted.

Corollary 3.2 From Hyers-Ulam-Rassias stability of equation (1.1) (with $f(t) \equiv 0$), one can conclude that the solution $e^{At}x_0$ of (1.1) is bounded in the interval $0 \le t_0 \le t \le T \le \infty$.

Example 3.1 Consider the non-homogeneous system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 4y - 3x \\ \dfrac{dy}{dt} = y - x + t \end{cases} \tag{3.4}$$

with initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

or,

$$\frac{dz}{dt} = Az + f(t)$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}, \; f(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

We find

$$e^{At} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix}$$

with

$$\sup_{t\geq 0}\left\|e^{At}\right\| = \sup_{t\geq 0}\left\|\begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix}\right\| \leq 2e^{-t} + 9te^{-t} \leq 9e^{-(7/9)} = K,$$

where $K = \sup_{t\geq 0}\left\{2e^{-t} + 9te^{-t}\right\} = \max_{t\geq 0}\left\{2e^{-t} + 9te^{-t}\right\} = 4.1348$.

Suppose that

$$\left|\frac{dz}{dt} - Az - f(t)\right| \leq \varphi(t).$$

From which it follows that

$$-\phi(t) \le z' - Az - f(t) \le \phi(t), \tag{3.5}$$

Where

$$\phi(t) = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2t} \\ \frac{1}{\sqrt{2}} e^{2t} \end{pmatrix} \text{and } \varphi(t) = |\phi(t)| = \sqrt{\left(\frac{1}{\sqrt{2}} e^{2t}\right)^2 + \left(\frac{1}{\sqrt{2}} e^{2t}\right)^2} = e^{2t}.$$

Multiplying the inequality (3.5) on the left, by a matrix integrating factor

$$e^{-At} = \begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix},$$

we get

$$-\begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2t} \\ \frac{1}{\sqrt{2}} e^{2t} \end{pmatrix} \le \frac{d}{dt}\left(\begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$-\begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} \le \begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2t} \\ \frac{1}{\sqrt{2}} e^{2t} \end{pmatrix}, \tag{3.6}$$

and integrating (3.6) from 0 to t, we have

$$-\int_0^t \begin{pmatrix} e^s + 2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2s} \\ \frac{1}{\sqrt{2}} e^{2s} \end{pmatrix} ds \le \begin{pmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+\int_0^t \begin{pmatrix} e^s + 2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds \le \int_0^t \begin{pmatrix} e^s + 2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2s} \\ \frac{1}{\sqrt{2}} e^{2s} \end{pmatrix} ds.$$

By multiplying the last inequality on the left by $e^{At} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix}$, we get

$$-\int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds$$

$$\leq \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$- \int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds$$

$$\leq \int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds.$$

It is clear to see that

$$z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+ \int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds$$

$$=\begin{pmatrix} e^{-t}\text{-}2te^{-t} \\ -te^{-t} \end{pmatrix}+\int_0^t \begin{pmatrix} 4s(t\text{-}s)e^{-(t\text{-}s)} \\ se^{-(t\text{-}s)}[1+2(t\text{-}s)] \end{pmatrix} ds=\begin{pmatrix} 4t+9e^{-t}+2te^{-t}-8 \\ 3t+5e^{-t}+te^{-t}\text{-}5 \end{pmatrix},$$

satisfies (3.4) with the initial condition.

Now consider the difference

$$| z(t)-z_1(t) |=\left| z-e^{At}z_0-\int_0^t e^{A(t-s)}f(s)ds \right|$$

$$=\left| z(t)-\begin{pmatrix} e^{-t}\text{-}2te^{-t} \\ -te^{-t} \end{pmatrix}-\int_0^t \begin{pmatrix} 4s(t\text{-}s)e^{-(t\text{-}s)} \\ se^{-(t\text{-}s)}[1+2(t\text{-}s)] \end{pmatrix} ds \right|$$

$$\leq \int_0^t \begin{pmatrix} e^{-(t\text{-}s)}\text{-}2(t\text{-}s)e^{-(t\text{-}s)} & 4(t\text{-}s)e^{-(t\text{-}s)} \\ (t\text{-}s)e^{-(t\text{-}s)} & e^{-(t\text{-}s)}(1+2(t\text{-}s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds$$

$$\leq \sup_{t\geq 0}\int_0^t \left\| \begin{matrix} e^{-(t\text{-}s)}\text{-}2(t\text{-}s)e^{-(t\text{-}s)} & 4(t\text{-}s)e^{-(t\text{-}s)} \\ (t\text{-}s)e^{-(t\text{-}s)} & e^{-(t\text{-}s)}(1+2(t\text{-}s)) \end{matrix} \right\| \left\| \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} \right\| ds \leq \frac{K}{2}e^{2t}, \qquad (3.7)$$

where

$$\sup_{t\geq 0}\left\| e^{A(t-s)} \right\| \leq \sup_{t\geq 0}\left\| e^{At} \right\| \leq \mathrm{K}. \qquad (3.8)$$

Consequently, we have

$$\| z(t)-z_1(t) \| \leq \frac{K}{2}e^{2t} \equiv k\varphi(t)$$

Therefore, the non-homogeneous system (3.4) is HUR stable for all $t \geq 0$.

Corollary 3.3 From Hyers-Ulam-Rassias stability of (3.10) (with $f_i(t) \equiv 0, i = 1,2.$), one can conclude that the homogeneous system has with the solution

$$z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-t}-2te^{-t} \\ -te^{-t} \end{pmatrix}$$

is stable the sense of HUR in the interval $0 \le t_0 \le t \le T \le \infty$.

Theorem 3.2 Let A be a constant matrix and let $B(t)$ be a continuous matrix valued function in the interval I. Suppose that $\varphi(t) : I \to (0,\infty)$ is a continuous function such that

$$\int_0^\infty \left\| e^{-A(s-t)} \right\| \varphi(s) ds \le C\varphi(t), \quad \forall t \ge 0 \tag{3.9}$$

If the inequality satisfies

$$\left| x' - Ax - B(t)x \right| \le \varphi(t) \tag{3.10}$$

and the integral

$$\int_0^\infty \| B(t) \| dt < \infty \tag{3.11}$$

converges, then the solution of (1.2) is stable in the sense of Hyers-Ulam-Rassias.

Here the norm $\|B(t)\|$ denotes the sum of the absolute values of elements of the matrix $B(t)$.

Proof. Let $x(t)$ be an approximate solution of the initial value problem (1.1), (1.4). We will show that there exists a function $z_0(t)$ satisfying (1.1) and (1.3) such that

$$\| x(t) - z_0(t) \| \le k\varphi(t)$$

The inequality (3.10) implies that

$$-\phi(t) \le x' - Ax - B(t)x \le \phi(t). \tag{3.12}$$

Multiplying the inequality (3.12) on the left, by a matrix integrating factor e^{-At}, we get

$$-e^{-At}\phi(t) \le \frac{d}{dt}\left(e^{-At}x\right) - e^{-At}B(t)x \le e^{-At}\phi(t). \tag{3.13}$$

Integrating (3.13) from 0 to t, we have

$$-\int_0^t e^{-As}\phi(s)ds \le e^{-At}x - x_0 - \int_0^t e^{-As}B(s)x(s)ds \le \int_0^t e^{-As}\phi(s)ds,$$

or, equivalently we get

$$-\int_0^t e^{A(t-s)}\phi(s)ds \le x - x_0e^{At} - \int_0^t e^{A(t-s)}B(s)x(s)ds \le \int_0^t e^{A(t-s)}\phi(s)ds.$$

It is clear to see that

$$z_0 = e^{At}x_0 + \int_0^t e^{A(t-s)}B(s)z(s)ds$$

satisfies the system (1.2) with the initial condition (1.4).

Now consider the difference

$$\begin{aligned} | x(t) - z_0(t) | &\le \left| x - e^{At}x_0 - \int_0^t e^{A(t-s)}B(s)x(s)ds \right| \\ &+ \int_0^t \left\| e^{A(t-s)} \right\| \| B(s) \| |x(s) - z(s)| ds. \end{aligned} \tag{3.14}$$

From Corollary 3.2, we can find a constant $K > 0$ so that $\sup_{t \ge 0} \left\| e^{At} \right\| \le K.$

Then applying Gronwall's inequality for (3.14) we get

$$\|x(t)-z_0(t)\| \le C\varphi(t)\exp\left(K\int_0^\infty \|B(t)\|dt\right) \quad \equiv k\varphi(t)$$

Therefore, the problem (1.2), (1.4) is HUR stable for all $t \ge 0$, which completes the proof of Theorem 3.2.

Example 3.2 Consider the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 4y - 3x \\ \dfrac{dy}{dt} = y - x + \dfrac{1}{1+t^2}\,x, \end{cases} \tag{3.15}$$

with initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

or

$$,\frac{dz}{dt} = Az + B(t)z$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix},\ A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix},\ B(t) = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{1+t^2} & 0 \end{pmatrix}.$$

We find

$$e^{At} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix},$$

with

$$\sup_{t\geq 0}\left\|e^{At}\right\| = \sup_{t\geq 0}\left\|\begin{pmatrix} \mathrm{e}^{-t}\text{-}2t\mathrm{e}^{-t} & 4t\mathrm{e}^{-t} \\ \text{-}t\mathrm{e}^{-t} & \mathrm{e}^{-t}+2t\mathrm{e}^{-t} \end{pmatrix}\right\| \leq 2\mathrm{e}^{-t} + 9t\mathrm{e}^{-t} \leq 9\mathrm{e}^{-(7/9)} = K,$$

where $K = \sup_{t\geq 0}\left\{2\mathrm{e}^{-t} + 9t\mathrm{e}^{-t}\right\} = \max_{t\geq 0}\left\{2\mathrm{e}^{-t} + 9t\mathrm{e}^{-t}\right\} = 4.1348$.

Suppose that

$$\left|\frac{dz}{dt} - Az - B(t)z\right| \leq \varphi(t).$$

From which it follows that

$$-\phi(t) \leq z' - Az - B(t)z \leq \phi(t), \tag{3.16}$$

Where

$$\phi(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2t} \\ \frac{1}{\sqrt{2}}e^{2t} \end{pmatrix} \text{and } \varphi(t) = |\phi(t)| = \sqrt{\left(\frac{1}{\sqrt{2}}e^{2t}\right)^2 + \left(\frac{1}{\sqrt{2}}e^{2t}\right)^2} = e^{2t}.$$

Multiplying the inequality (3.16) on the left, by a matrix integrating factor

$$e^{-At} = \begin{pmatrix} \mathrm{e}^{t}+2t\mathrm{e}^{t} & -4t\mathrm{e}^{t} \\ t\mathrm{e}^{t} & \mathrm{e}^{t}(1\text{-}2t) \end{pmatrix},$$

we get

$$-\begin{pmatrix} e^t+2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{2t} \\ \frac{1}{\sqrt{2}}e^{2t} \end{pmatrix} \le \frac{d}{dt}\left(\begin{pmatrix} e^t+2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$-\begin{pmatrix} e^t+2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{1+t^2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\le \begin{pmatrix} e^t+2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{2t} \\ \frac{1}{\sqrt{2}}e^{2t} \end{pmatrix}. \tag{3.17}$$

Integrating (3.17) from 0 to t, we have

$$-\int_0^t \begin{pmatrix} e^s+2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds \le \begin{pmatrix} e^t+2te^t & -4te^t \\ te^t & e^t(1\text{-}2t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-\int_0^t \begin{pmatrix} e^s+2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} ds$$

$$\le \int_0^t \begin{pmatrix} e^s+2se^s & -4se^s \\ se^s & e^s(1\text{-}2s) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds.$$

Left-multiplying the last inequality by $e^{At} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix}$,

we get

$$-\int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds \le \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-\int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ds$$

$$\le \int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{2s} \\ \frac{1}{\sqrt{2}}e^{2s} \end{pmatrix} ds.$$

It is clear to see that

$$z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{-t}-2te^{-t} & 4te^{-t} \\ -te^{-t} & e^{-t}+2te^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+\int_0^t \begin{pmatrix} e^{-(t-s)}-2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}-2te^{-t} \\ -te^{-t} \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{4(t-s)e^{-(t-s)}}{1+s^2} & 0 \\ \frac{e^{-(t-s)}(1+2(t-s))}{1+s^2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} ds$$

satisfies (3.15) with the initial condition.

Now consider the difference

$$| z(t) - z_1(t) |$$

$$\leq \left| z - e^{At} z_0 - \int_0^t e^{A(t-s)} B(s) z_1(s) ds \right| + \int_0^t \left\| e^{A(t-s)} \right\| \| B(s) \| | z(s) - z_1(s) | ds$$

$$\leq \int_0^t \begin{pmatrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2s} \\ \frac{1}{\sqrt{2}} e^{2s} \end{pmatrix} ds$$

$$+ \int_0^t \left\| \begin{matrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{matrix} \right\| | z(s) - z_1(s) | ds$$

$$\leq \int_0^t \begin{pmatrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2s} \\ \frac{1}{\sqrt{2}} e^{2s} \end{pmatrix} ds$$

$$+ \int_0^t \left\| \begin{matrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{-(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{matrix} \right\| | z(s) - z_1(s) | ds$$

$$\leq \sup_{t \geq 0} \int_0^t \left\| \begin{pmatrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \right\| \left\| \begin{pmatrix} \frac{1}{\sqrt{2}} e^{2s} \\ \frac{1}{\sqrt{2}} e^{2s} \end{pmatrix} \right\| ds$$

$$+ \int_0^t \left\| \begin{pmatrix} e^{-(t-s)} - 2(t-s)e^{-(t-s)} & 4(t-s)e^{-(t-s)} \\ (t-s)e^{(t-s)} & e^{-(t-s)}(1+2(t-s)) \end{pmatrix} \right\| \left\| \begin{pmatrix} 0 & 0 \\ \frac{1}{1+s^2} & 0 \end{pmatrix} \right\| | z(s) - z_1(s) | ds$$

$$\leq \frac{K}{2} e^{2t} + K \int_0^t \frac{1}{1+s^2} | z(s) - z_1(s) | ds, \qquad (3.18)$$

where

$$\sup_{t \geq 0} \left\| e^{A(t-s)} \right\| \leq \sup_{t \geq 0} \left\| e^{At} \right\| \leq \mathrm{K}.$$

(3.19)

Then applying Gronwall's inequality for (3.18) we get

$$\|z(t)-z_1(t)\| \leq \frac{K}{2}e^{2t}\exp\left(K\int_0^{\infty}\|B(t)\|dt\right)$$

$$= 2.\ 0674\, e^{2t}\exp\left(4.1348\int_0^{\infty}\frac{1}{1+t^2}dt\right) = C\varphi(t)\exp(2.\ 0674\pi) \equiv k\varphi(t)$$

Therefore, the system of equations (3.15) is HUR stable for all $t \geq 0$.

Corollary 3.3 Replacing $\varphi(t)$ by ε in the inequality (3.2) we can get Hyers-Ulam stability for Eq. (1.2) in the interval $0 \leq t_0 \leq t \leq T < \infty$, i.e. if $x(t)$ satisfies

$$|x' - Ax - f(t)| \leq \varepsilon,$$

with the initial condition $x(t_0) = x_0$, , then there exists some $z_0(t)$ of the equation (1.2) such that

$$\|x(t) - z_0(t)\| \leq k\varepsilon\,.$$

The proof of Corollary 3.3 is quite similar to the proof of Theorem 3.2 and will therefore be omitted.

In the following theorem we establish the Hyers-Ulam-Rassias stability for (1.3) in the interval $0 < t_0 \leq t \leq T \leq \infty$.

Theorem 3.3 Suppose that $x(t)$ satisfies the inequality (2.3) with initial conditions (1.4). Let A be a constant matrix and $\varphi(t) : I \to (0,\infty)$ be a continuous function such that

$$\int_0^{\infty}\left\|e^{-A(s-t)}\right\|\varphi(s)ds \leq C\varphi(t), \quad \forall t \geq 0 \tag{3.20}$$

If (1.5) holds and the integral

$$\int_0^{\infty}\gamma(t)ds < \infty, \tag{3.21}$$

then the solution of (1.3) is stable in the sense of Hyers-Ulam-Rassias as $t \to \infty$.

Proof. Let $x(t)$ be an approximate solution of the initial value problem (1.3), (1.4). We wish to show that there exists a function $z_0(t)$ satisfying (1.2) and (1.3) such that

$$\|x(t) - z_0(t)\| \le k\varphi(t).$$

Then it follows from the inequality (2.3) that

$$-\phi(t) \le x' - Ax - g(t,x) \le \phi(t). \tag{3.22}$$

Multiplying (3.22) by matrix integrating factor e^{-At}, gives

$$-e^{-At}\phi(t) \le \frac{d}{dt}\left(e^{-At}x\right) - e^{-At}g(t,x) \le e^{-At}\phi(t).$$

Integrating the later inequality from 0 to t, we have

$$\begin{aligned} -\int_0^t e^{A(t-s)}\phi(s)ds \le x - x_0e^{At} - \int_0^t e^{A(t-s)}g(s,x))ds \\ \le \int_0^t e^{A(t-s)}\phi(s)ds, \end{aligned} \tag{3.23}$$

or, equivalently we get

$$\left| x - e^{At}x_0 - \int_0^t e^{A(t-s)}g(s,x))ds \right| \le \int_0^\infty \left\|e^{A(t-s)}\right\|\varphi(s)ds.$$

One can easily show that

$$z_0(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}g(s,z_0))ds$$

satisfies (1.2) with the initial condition (1.3).

Now, let us estimate the difference

$$
\begin{aligned}
&| x(t) - z_0(t) | \\
&= \left| x - e^{At}x_0 + \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,z)ds \right| \\
&\le \left| x - e^{At}x_0 - \int_0^t e^{A(t-s)}g(s,x)ds \right| + \left| \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,z)ds \right| \\
&\le \int_0^\infty \left\| e^{A(t-s)} \right\| \varphi(s)ds + \left| \int_0^t e^{A(t-s)}(g(s,x) - g(s,z))ds \right| \\
&\le C\varphi(t) + \int_0^t \gamma(t) \left\| e^{A(t-s)} \right\| |x(s) - z_0(s)|\, ds,
\end{aligned}
$$

so, we obtain

$$
\|x(t) - z_0(t)\| \le C\varphi(t) + \int_0^t \gamma(t) \left\| e^{A(t-s)} \right\| |x(s) - z_0(s)|\, ds.
$$

Now, since $\sup_{t \ge 0} \left\| e^{At} \right\| \le K$ then using Gronwall's inequality we get

$$
\|x(t) - z_0(t)\| \le C\varphi(t) \exp\left(K \int_0^\infty \gamma(t) ds \right) \equiv k\varphi(t),
$$

which means that (2.4) holds true for all $t > 0$.

Corollary 3.3 Replacing $\varphi(t)$ by ε in the inequality (3.2) we can get Hyers-Ulam stability for Eq. (1.3) in the interval $0 \le t_0 \le t \le T < \infty$, i.e. if $x(t)$ satisfies

$$
\left| x' - Ax - g(t,x) \right| \le \varepsilon,
$$

with the initial condition $x(t_0) = x_0,$, then there exists some $z_0(t)$ of the equation (1.3) such that

$\|x(t) - z_0(t)\| \le k\varepsilon$.

The proof of Corollary 3.3 is quite similar to the proof of Theorem 3.3 and will therefore be omitted.

Now we will prove HUR stability for the system (1.3) in which $A(t), t \in [0,\infty),$ is an $n \times n$ matrix of real continuous functions on I. In this case one can similarly get HUR stability for the systems (1.2), (1.3).

Theorem 3.4 Suppose that $x(t)$ satisfies the inequality (2.3) with initial condition (1.4), and A(t) is a continuous matrix function commuting with its integral.

Then a sufficient condition for problem (1.3-1.4) to be stable in the sense of Hyers-Ulam-Rassias as $t \to \infty$ is that

$$(i) \qquad \int_0^\infty \left\| e^{A(t-s)} \right\| \varphi(s)ds \le C\varphi(t), \quad \forall t \ge 0 \tag{3.24}$$

$$(ii) \qquad \int_0^\infty \gamma(t)ds < \infty, \tag{3.25}$$

$$(iii) \qquad \sup_{t\ge 0} \left\| e^{A(t)} \right\| \le c_1 \tag{3.26}$$

where c_1 is a constant and $\varphi(t) : I \to (0,\infty)$ is a continuous function for $t \in [0,\infty)$.

Proof. Let $x(t)$ be an approximate solution of the initial value problem (1.3), (1.4). We wish to show that there exists a function $z_0(t)$ satisfying (1.3) and (1.4) such that

$\|x(t) - z_0(t)\| \le k\varphi(t)$

Then it follows from the inequality (2.3) that

$$-\phi(t) \le x' - A(t)x - g(t,x) \le \phi(t), \tag{3.27}$$

Multiplying (3.27) by matrix integrating factor $e^{-A(t)t}$, gives

$$-e^{-A(t)t}\phi(t) \le e^{-A(t)t}x' - e^{-A(t)t}A(t)x - e^{-A(t)t}g(t,x) \le e^{-A(t)t}\phi(t). \quad (3.28)$$

Since A(t) is a commuting with its integral, then by Lemma 2.2 we can write (3.28) in the following

$$-e^{-At}\phi(t) \le \frac{d}{dt}\left(e^{-At}x\right) - e^{-At}g(t,x) \le e^{-At}\phi(t).$$

Integrating the later inequality from 0 to t, we have

$$-\int_0^t e^{A(t-s)}\phi(s)ds \le x - e^{At}x_0 - \int_0^t e^{A(t-s)}g(s,x))ds$$
$$\le \int_0^t e^{A(t-s)}\phi(s)ds,$$

or, equivalently we get

$$\left| x - e^{At}x_0 - \int_0^t e^{A(t-s)}g(s,x))ds \quad \right| \le \int_0^\infty e^{A(t-s)}\phi(s)ds.$$

One can easily show that

$$z_0(t) = e^{\cdot A(t)}x_0 + \int_0^t e^{A(t-s)}g(s,z_0))ds$$

satisfies Eq. (1.3) with the initial condition (1.4).

Now, let us estimate the difference

$$
\begin{aligned}
|x(t)-z_0(t)| &= \left| x - e^{At}x_0 + \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,z)ds \right| \\
&\le \left| x - e^{At}x_0 - \int_0^t e^{A(t-s)}g(s,x)ds \right| + \left| \int_0^t e^{A(t-s)}g(s,x)ds - \int_0^t e^{A(t-s)}g(s,z)ds \right| \\
&\le \int_0^\infty \left\| e^{A(t-s)} \right\| \varphi(s)ds + \int_0^t \left\| e^{A(t-s)} \right\| |g(s,x)-g(s,z)| ds \\
&\le C\varphi(t) + \int_0^t \gamma(t) \left\| e^{A(t-s)} \right\| |x(s)-z_0(s)| ds.
\end{aligned}
$$

So, we obtain

$$
\|x(t)-z_0(t)\| \le C\varphi(t) + \int_0^t \gamma(t) \left\| e^{A(t-s)} \right\| |x(s)-z_0(s)| \, ds
$$

Using Gronwall's inequality we obtain

$$
\|x(t)-z_0(t)\| \le C\varphi(t) \exp\left(K \int_0^\infty \gamma(t)ds \right) \quad \equiv k\varphi(t)
$$

which means that (2.4) holds true for all $t > 0$.

Corollary 3.4 Replacing $\varphi(t)$ by ε in the inequality (3.2) we can get Hyers-Ulam stability for Eq. (1.3) in the interval $0 \le t_0 \le t \le T < \infty$, i.e. if $x(t)$ satisfies

$$
\left| x' - A(t)x - g(t,x) \right| \le \varepsilon,
$$

with the initial condition $x(t_0) = x_0,$, then there exists some $z_0(t)$ of the system (1.3) such that

$$
\|x(t)-z_0(t)\| \le k\varepsilon .
$$

The proof of Corollary 3.4 is quite similar to the proof of Theorem 3.4 and will therefore be omitted.

Conclusion

This paper considers the problem of Hyers-Ulam-Rassias stability for linear and semi-linear systems of differential equations. Here we use the direct method to obtain some integral sufficient conditions of Hyers-Ulam-Rassias stability for linear systems with almost constant matrix, linear and semi-linear systems of differential equations with constant matrix. Moreover, in this paper we consider the Hyers-Ulam-Rassias stability for the semi-linear system with continuous matrix function commuting with its integral. To illustrate the results, we provided some examples satisfying the assumptions of the proved theorems.

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