An-Najah University Journal for Research – A
Natural Sciences

The Approximate Integrals By The Reproducing Kernel Method On The Elliptical Region E_n In The Space $\mathbb{R}^n$

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Accepted Manuscript, In press

Abstract: In this paper, we applied the reproducing kernel method on elliptical region, we establish the formulas of the reproducing kernel from several degrees and generalized those formulas whatever the dimension of space, we obtained all surfaces, points, and their constants for compensation in the cubature formula, tables are included to show our results and examples to show the approximation solution in this method.

Keywords: Reproducing kernel method, Cubature Formula, Orthonormal Polynomials, Algebraic Precision, Integral Points.

1. Introduction

The Reproducing Kernel Method was first discovered in 1968 by the scientist Myscovskikh in order to calculate the double integrals in regions that have a center of symmetry and that do not have a center symmetry [[11], [12]], Bykova used the Reproducing Kernel Method to form the cubature formula as the integral region is symmetric in $\mathbb{R}^2$[2], The important development of the Reproducing Kernel Method was obtained by Möller in 1973, where he was able to prove the theorems and found the cubature formula with $2k+1$ algebraic precision in the case of symmetric regions [7], and $2k$ in the case of asymmetric regions[[8],[9]], In the 1990s, Myscovskikh contributed to the development of the Reproducing Kernel Method and to finding the minimum number of integrative points [[1], [15], [16]], and in 2001, The cubature formula on the square with a weight function $(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}$ was studied [4], in 2003, the formulas of cubature formula was collected to approximate multiple integrals and to study the cubature formula on the unite ball [[3],19]], in 2004, the cubature formula was studied on the simplex area and the surface of the unite ball [18], and in 2006 the cubature formula was studied on the cube [10], in 2008 the cubature formula was studied on the hexagon and the triangle [20], in 2012, conditions were set for the existence of cubature formula [6], in 2014, periodic functions were used to find the cubature formula for the simplex [5], in 2020 the cubature formula was studied using the Gaussian weight function [17].

The current research in this field revolves around the study of properties of the Reproducing Kernel, the application of this method in wider areas of integration , and study this method in Sobolev spaces, and obtaining new generalized cubature formula that can be used to calculate the approximate value of multiple integrals.

2. Material And Methods

Definition 2.1. Cubature Formula: (see [13],[14])The cubature formula is an approximation equality for calculating the approximate value of the multiple integrals by specifying a number of points and finding a number of constants, and it is given as:

$$I(f) := \int_{\Omega} \omega(x)\, f(x)\, dx \cong \sum_{j=1}^{N} C_j f(x^j) \qquad (2.1)$$

Where $x^j = (x_1^j, x_2^j, \dots, x_n^j); j = 1, \dots, N$ are different points two by two and they are called integration points or cubature formula nodes, N the number of integral points, C_j the constants corresponding to those points (in this article $C_j \in \mathbb{R}$), Ω integrative area E_n, $f(x)$ the function to be integrated $dx = dx_1 dx_2 \dots\dots dx_n$ and $\omega(x)$ the weight function even, check:

$$\int_{\Omega} \omega(x) dx > 0,\ \omega(x) \geq 0 \ \&\ x \in \Omega \Longrightarrow -x \in \Omega \ \&\ \omega(x) = \omega(-x)$$

We say (1-1) has an algebraic precision d , if it is transformed into a true equality, when the degree of the integral polynomial $f(x)$ does not exceed d.

Theorem 2.1. Suppose that Ω has internal points, we have two cases:
_if the cubature formula (1-1) has $d = 2k$ algebraic precision, then the number of nodes of cubature formula achieves the inequality:

$$N \geq \chi = M(n,m) = \frac{(n+m)!}{n!\, m!} \quad ; m = \left[\frac{d}{2}\right] \qquad (2.2)$$

_if the cubature formula (1-1) has $d = 2k+1$ algebraic precision, and if θ is not among the nodes of cubature formula, then the number of nodes of cubature formula achieves the inequality:

$$N \geq \begin{cases} 2(\chi - \nu) & ; m: odd\ number \\ 2\nu & ; m: even\ number \end{cases} \qquad (2.3)$$

$m = \left[\frac{d}{2}\right];\ \chi = M(n,m)$

and if θ is among the nodes of cubature formula, then the number of nodes of cubature formula achieves the inequality:

$$N \geq \begin{cases} 2(\chi - \nu) - 1 & ; m: odd\ number \\ 2\nu + 1 & ; m: even\ number \end{cases} \qquad (2.4)$$

Where ν the number of uneven units of term which degree does not exceed k with n variable.

Definition 2.2. Reproducing Kernel: It is a polynomial with $2n$ a variable used in cubature formula with even algebraic precision $d = 2k$ is given by the formula:

$$K_k(u,x) = \sum_{i=1}^{\chi} F_i(u) F_i(x) \qquad (2.5)$$

while the Reproducing Kernel for cubature formula with odd algebraic precision $d = 2k+1$ is given by the formula:

$$\widetilde{K}_k(u,x) = \sum_{i=1}^{\acute{\chi}} F_i(u) F_i(x) \qquad (2.6)$$

Where $u = (u_1, u_2, \dots, u_n), x = (x_1, x_2, \dots, x_n)$, $F_i(x)$Orthonormal polynomials on E_n with n variable, the degree of $F_i(x)$ in (2.6) is $s \leq k$ (if k is an odd number then $s = 1,3,5,..,k$ and if k is an even number then $s = 0,2,4,..,k$), the degree of $F_i(x)$ in (2.5) is $s \leq k (s = 0,1,2,3,..,k)$

To form the cubature formula, we use the following two theorems:

Theorem 2.2. (see [13],[14]) Assuming that $u^1, u^2, \dots, u^n$ points satisfy the condition $K_k(u^i, u^j) = b_i \delta_{ij}$, and $\bigcap_{i=1}^{n} H_i$ consist of points $x^j; j = 1,..,s; s = k^n$ where $H_i := K_k(u^i, u^i) = 0; i = 1,..,n,$ The

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exact cubature formula exists for polynomials whose degrees do not contiguous $2k$:

$$\int_{E_n} \omega(x)f(x)dx \cong \sum_{i=1}^{n} \frac{1}{b_i} f(u^i) + \sum_{j=1}^{s} C_j f(x^j) \qquad (2.7)$$

The number of integral points: is the sum of the number of points x^j plus the number of points u^i

Theorem 2.3. (see [13],[14]) Assuming that $u^1, u^2, \ldots, u^n$ points satisfy the condition $\widetilde{K}_k(u^i, u^j) = b_i \delta_{ij}$, and $\bigcap_{i=1}^{n} H_i$ consists of points $x^j; j = 1, \ldots, s; s = k^n$, then the cubature formula can be formed which algebraic precision $2k + 1$.

$$\int_{E_n} \omega(x)f(x)dx \cong \sum_{i=1}^{n} \frac{1}{2b_i} \left(f(u^i) + f(-u^i) \right) + \sum_{j=1}^{s} C_j f(x^j) \qquad (2.8)$$

The number of integral points: is the sum of the number of points x^j plus the number of points u^i and plus the number of points $-u^i$.

3. Results And Discussion

E_n elliptic in the space $\mathbb{R}^n$:

$E_n = \left\{ x \in \mathbb{R}^n;\ \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \cdots + \left(\frac{x_n}{a_n}\right)^2 \le 1 \right\}$, Let's evaluate the integral $P_\alpha = \int x^\alpha\, dx$ on E_n whereas $\alpha = (\alpha_1, \ldots, \alpha_n)$ $x = (x_1, \ldots, x_n)$ if one of the α_i is an odd number, then $P_\alpha = 0$, but if α_i an even numbers, we can write them in the form $\alpha_i = 2q_i$, then we find:

$$P_\alpha = \prod_{i=1}^{n} a_i^{\alpha_i + 1} \frac{\prod_{i=1}^{n} \Gamma\left(\frac{\alpha_i + 1}{2}\right)}{\Gamma\left(\frac{|\alpha| + n}{2} + 1\right)} \quad ; |\alpha| = \alpha_1 + \cdots + \alpha_n \qquad (3.1)$$

By the (3.1) we found the Orthonormal Polynomials (zero, first, second and third degree with n varible) on E_n by Using schmidt's role:

Let it be $\gamma = (n+2)(n+4)(n+6)$, $\mu(E_n) = \frac{a_1 \ldots\ldots a_n \left(\Gamma\left(\frac{1}{2}\right)\right)^n}{\Gamma\left(\frac{n}{2} + 1\right)}$, we find:

$$F_0^0 = \frac{1}{\sqrt{\mu(E_n)}} \qquad (3.2)$$

$$F_i^1 = \sqrt{\frac{n+2}{\mu(E_n)}} \frac{x_i}{a_i} \quad ; i = 1, 2, \ldots, n \qquad (3.3)$$

$$F_{ij}^2 = \begin{cases} \sqrt{\frac{(n+3-i)(n+2)(n+4)}{2a_i^4 (n+2-i)\mu(E_n)}} \left[\frac{x_i^2}{a_i^2} + \frac{1}{n+3-i} \left(\sum_{j=1}^{i-1} \frac{x_j^2}{a_j^2} - 1 \right) \right] & ; i = j \\ \sqrt{\frac{(n+2)(n+4)}{a_i^2 a_j^2 \mu(E_n)}} \ \frac{x_i x_j}{a_i a_j} & ; i = j \end{cases} \qquad (3.4)$$

$$F_{ijk}^3 = \begin{cases} F_{iij}^3 = \sqrt{\frac{(n+5-i)\lambda}{2(n+4-i)a_i^4 a_j^2 \mu(E_n)}} \left[x_i^2 + \frac{a_i^2}{n+5-i} \left(\sum_{s=1}^{i-1} \frac{x_s^2}{a_s^2} - 1 \right) \right] x_j; & i < j \\ F_{iii}^3 = \sqrt{\frac{(n+5-i)\lambda}{6(n+2-i)a_i^2 \mu(E_n)}} \left[x_i^2 + \frac{3a_i^2}{n+5-i} \left(\sum_{s=1}^{i-1} \frac{x_s^2}{a_s^2} - 1 \right) \right] x_i; & i = j, k \\ F_{iij}^3 = \sqrt{\frac{(n+3-i)\lambda}{2(n+2-i)a_i^4 a_j^2 \mu(E_n)}} \left[x_i^2 + \frac{a_i^2}{n+3-i} \left(\sum_{s=1}^{i-1} \frac{x_s^2}{a_s^2} - 1 \right) \right] x_j; & i > j \\ F_{ijk}^3 = \sqrt{\frac{\lambda}{a_i^2 a_j^2 a_k^2 \mu(E_n)}} x_i x_j x_k & ; i \ne j \ne k \end{cases} \qquad (3.5)$$

3.1. Finding The Reproducing Kernel:

3.1.1. Finding The Reproducing Kernel Of The First Degree:

$$\widetilde{K}_1(u, x) = \frac{n+2}{\mu(E_n)} \sum_{j=1}^{n} \frac{u_j x_j}{a_j^2} \qquad (3.6)$$

$$K_1(u, x) = \widetilde{K}_1(u, x) + \widetilde{K}_0(u, x) = \frac{1}{\mu(E_n)} + \frac{n+2}{\mu(E_n)} \sum_{j=1}^{n} \frac{u_j x_j}{a_j^2} \qquad (3.7)$$

3.1.2. Finding The Reproducing Kernel Of The Second Degree:

The Reproducing Kernel is given by the formula:

$$\widetilde{K}_2(u, x) = \frac{1}{\mu(E_n)} + \sum_{j=1}^{n} F_j^2(x) F_j^2(u) + \sum_{i \ne j} F_i^2(x) F_j^2(u)$$

(3.8)

Substitute (3.2) and (3.4) in (3.8), we get:

$$\widetilde{K}_2(u, x) = \frac{1}{\mu(E_n)} + N \sum_{j=1}^{n} \frac{n+3-j}{n+2-j} [U_j][X_j] + \sum_{i \ne j} F_i^2(x) F_j^2(u)$$

(3.9)

Whereas:

$$N = \frac{(n+2)(n+4)}{2\mu(E_n)}, U_j = \frac{u_j^2}{a_j^2} + \frac{1}{n+3-j} \left(\sum_{s=1}^{j-1} \frac{u_s^2}{a_s^2} - 1 \right)$$

(3.10)

$$X_j = \frac{x_j^2}{a_j^2} + \frac{1}{n+3-j} \left(\sum_{s=1}^{j-1} \frac{x_s^2}{a_s^2} - 1 \right)$$

Relationship (3.9) can be written as:

$$\widetilde{K}_2(u, x) = \frac{1}{\mu(E_n)} + N \sum_{j=1}^{n} \frac{n+3-j}{n+2-j} [U_j] \frac{x_j^2}{a_j^2} + N \sum_{j=1}^{n} \frac{[U_j]}{n+2-j} \sum_{s=1}^{j-1} \frac{x_s^2}{a_s^2} - N \sum_{j=1}^{n} \frac{[U_j]}{n+2-j} + \sum_{i \ne j} F_i^2(x) F_j^2(u)$$

$$\widetilde{K}_2(u, x) = \frac{1}{\mu(E_n)} + N(S_1 + S_2 + S_3) + \frac{(n+2)(n+4)}{\mu(E_n)} \sum_{i \ne j} \frac{x_i x_j u_i u_j}{a_i^2 a_j^2}$$

(3.11)

Let's find S_1, S_2, S_3: We take out the indexed sum n in S_1, then we replace each j with a s so we find:

$$S_1 = \frac{3}{2a_n^2} U_n x_n^2 + \sum_{s=1}^{n-1} \frac{n+3-s}{(n+2-s)a_s^2} [U_s] x_s^2$$

(3.12)

We make a substitution in the addition operation in S_2, and we find:

$$S_2 = \sum_{s=1}^{n-1} \frac{x_s^2}{a_s^2} \sum_{j=s+1}^{n} \frac{U_j}{n+2-j} \qquad (3.13)$$

Adding (3.12) and (3.13), we get:

$$S_1 + S_2 = \frac{3}{2a_n^2} U_n x_n^2 + \sum_{s=1}^{n-1} \frac{x_s^2}{a_s^2} S_4 \qquad (3.14)$$

Whereas:

$$S_4 = U_s + \sum_{j=s}^{n} \frac{U_j}{n+2-j} \qquad (3.15)$$

Substituting (3.10) into (3.15), we get:

$$S_4 = \frac{u_s^2}{a_s^2} + \frac{1}{n+3-s} \sum_{t=1}^{s-1} \frac{u_s^2}{a_s^2} - \frac{1}{n+3-s} + \sum_{j=s}^{n} \frac{1}{n+2-j} \frac{u_j^2}{a_j^2} + \sum_{j=s}^{n} \frac{1}{(n+2-j)(n+3-j)} \sum_{t=1}^{j-1} \frac{u_t^2}{a_t^2} - \sum_{j=s}^{n} \frac{1}{(n+2-j)(n+3-j)} \qquad (3.16)$$

We make a substitution in the order of the sum in fifth term, which we will denote by the symbol S_5 in (3.16), so we find:

$$S_5 = \sum_{t=1}^{s-1} \frac{u_t^2}{a_t^2} \sum_{j=s}^{n} \frac{1}{(n+3-j)(n+2-j)} + \sum_{t=s}^{n-1} \frac{u_t^2}{a_t^2} \sum_{j=t+1}^{n} \frac{1}{(n+3-j)(n+2-j)} \qquad (3.17)$$

The sum in (3.17) can be written in the form:

$$\sum_{j=s}^{n} \frac{1}{(n+3-j)(n+2-j)} = \frac{1}{2} - \frac{1}{n+3-s}$$

$$\sum_{j=t+1}^{n} \frac{1}{(n+3-j)(n+2-j)} = \frac{1}{2} - \frac{1}{n+2-t} \qquad (3.18)$$

Substituting (3.18) into (3.17), then substituting the result into (3.16) and then grouping the terms, we find: $S_4 = \frac{u_s^2}{a_s^2} + \frac{1}{2} \sum_{t=s}^{n} \frac{u_t^2}{a_t^2} - \frac{1}{2}$

substituting S_4 into (3.14), we find:

$$S_1 + S_2 = \sum_{s=1}^{n} \frac{x_s^2}{a_s^2} \left(\frac{u_s^2}{a_s^2} + \frac{1}{2} \sum_{s=1}^{n} \frac{u_s^2}{a_s^2} - \frac{1}{2} \right)$$

Let's find S_3:

$$-S_3 = \sum_{j=1}^{n} \frac{[U_j]}{n+2-j} = \sum_{j=1}^{n} \frac{1}{n+2-j} \frac{u_j^2}{a_j^2} + \sum_{j=1}^{n} \frac{1}{(n+2-j)(n+3-j)} \sum_{s=1}^{j-1} \frac{u_s^2}{a_s^2} - \sum_{j=1}^{n} \frac{1}{(n+2-j)(n+3-j)}$$

$$-S_3 = \frac{1}{2} \sum_{s=1}^{n} \frac{u_s^2}{a_s^2} - \frac{n}{2(n+2)}$$

And therefore $s = S_1 + S_2 + S_3$ is:

$$s = \frac{1}{2} \left[2 \sum_{s=1}^{n} \frac{x_s^2 u_s^2}{a_s^4} + \sum_{s=1}^{n} \frac{x_s^2}{a_s^2} \sum_{s=1}^{n} \frac{u_s^2}{a_s^2} - \left(\sum_{s=1}^{n} \frac{u_s^2}{a_s^2} + \sum_{s=1}^{n} \frac{x_s^2}{a_s^2} \right) + \frac{n}{n+2} \right]$$

Substitute in (3.11), we find:

$$\widetilde{K}_2(u, x) = \frac{n+2}{\mu(E_n)} \left\{ \frac{n+2}{4} + (n+4) \sum_{i \ne j} \frac{x_i x_j u_i u_j}{a_i^2 a_j^2} + \frac{n+4}{4} \left[2 \sum_{s=1}^{n} \frac{x_s^2 u_s^2}{a_s^4} + \sum_{s=1}^{n} \frac{x_s^2}{a_s^2} \sum_{s=1}^{n} \frac{u_s^2}{a_s^2} - \left(\sum_{s=1}^{n} \frac{u_s^2}{a_s^2} + \sum_{s=1}^{n} \frac{x_s^2}{a_s^2} \right) \right] \right\}$$

Let's suppose that $(u_1, u_2, \ldots, u_n)$ belong to the elliptic surface, and therefore $\sum_{s=1}^{n} \frac{u_s^2}{a_s^2} = 1$, we substitute in the last relationship, so we find the final formula for $\widetilde{K}_2(u, x)$ is:

$$\widetilde{K}_2(u,x) = \frac{(n+2)(n+4)}{2\mu(E_n)}\left(\sum_{s=1}^{n}\frac{u_s x_s}{a_s^2} + \frac{1}{\sqrt{n+4}}\right)\left(\sum_{s=1}^{n}\frac{u_s x_s}{a_s^2} - \frac{1}{\sqrt{n+4}}\right)$$
(3.19)

In the same way, we find $K_2(u,x)$:

$$\widetilde{K}_2(u,x) = \frac{(n+2)(n+4)}{2\mu(E_n)}\left(\sum_{s=1}^{n}\frac{u_s x_s}{a_s^2} + \frac{1+\sqrt{n+5}}{n+4}\right)\left(\sum_{s=1}^{n}\frac{u_s x_s}{a_s^2} + \frac{1-\sqrt{n+5}}{n+4}\right)$$
(3.20)

3.1.3. The Reproducing Kernel Of the third Order:

Following the same steps as section 3.1.2, we find:

$$\widetilde{K}_3(u,x) = \frac{\gamma}{6\mu(E_n)}\left[(t)^3 - \frac{3}{n+6}t\right] \qquad (3.21)$$

$$\widetilde{K}_3(u,x) = \frac{\gamma}{6\mu(E_n)}\left[(t)^3 + \frac{3}{n+6}(t)^2 - \frac{3}{n+6}\sum_{s=1}^{n} t - \frac{3}{(n+4)(n+6)}\right] \qquad (3.22)$$

$t = \sum_{s=1}^{n}\frac{u_s x_s}{a_s^2}$, $\gamma = (n+2)(n+4)(n+6)$

3.2. The formation of the cubature formula for the Reproducing Kernel $\widetilde{K}_1(u,x), K_1(u,x)$:

- **For $\widetilde{K}_1(u,x)$:**

Let's find the cubature formula in order to algebraic precision $d = 3$, we choose $u^1 = (\alpha_1, 0, \ldots, 0)$, so $-a_1 < \alpha_1 < a_1$, substitute in (3.6), we get:

$$H_1 := \frac{n+2}{\mu(E_n)}\frac{\alpha_1 x_1}{a_1} = 0$$

From H_1: $x_1 = 0$, substitute in $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \le 1$, we get $x_2 = \alpha_2$ $-a_2 < \alpha_2 < a_2$, so $u^2 = (0, \alpha_2, 0, \ldots, 0)$, substitute in (3.6), we get:

$$H_2 := \frac{n+2}{\mu(E_n)}\frac{\alpha_2 x_2}{a_2} = 0$$

Choose u^3 from $E_n \cap (H_1 \cap H_2)$, we get $x_1 = x_2 = 0$ form H_1 and H_2 substitute in $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \le 1$, we find $u^3 = (0, , 0, \alpha_3, 0, \ldots, 0)$, then we find $H_3, H_4, \ldots\ldots, H_n$, $\bigcap_{i=1}^{n} H_i$ is $x^1 = (0,0,\ldots,0)$, and the number of integral points is $2n+1$, and the minimum number of points according to (2.4) is $2n+1$, where $(2b_i)^{-1} = \frac{a_j^2\mu(E_n)}{2(n+2)\alpha_i^2}$.

For $f(x) = 1$ and according theorem (2.8)we find:

$$C = \mu(E_n)\left(\frac{(n+2)\alpha_i^2 - n\,a_j^2}{(n+2)\alpha_i^2}\right)$$

- • **for $K_1(u,x)$:** Let's find the cubature formula in order to algebraic precision $d = 2k$, as in the previous paragraph:

$$u^1 = (a_1, 0, 0, \ldots, 0), u^2 = \left(\frac{-a_1}{n+2}, \frac{a_2}{n+2}\sqrt{(n+1)(n+3)}, 0, \ldots,\right)$$

$$u^3 = \left(\frac{-a_1}{n+2}, \frac{-a_2}{n+2}\sqrt{\frac{n+3}{n+1}}, a_3\sqrt{\frac{n(n+3)}{(n+1)(n+2)}}, 0, \ldots, 0\right)$$

$$u^4 = \left(\frac{-a_1}{n+2}, \frac{-a_2}{n+2}\sqrt{\frac{n+3}{n+1}}, -a_3\sqrt{\frac{(n+3)}{(n+1)(n+2)}}, a_4\sqrt{\frac{(n+3)(n-1)}{n(n+2)}}, 0 \ldots, 0\right)$$

$$u^i = \left(\frac{-a_1}{n+2}, \ldots, -a_{i-1}\sqrt{\frac{(n+3)}{(n+4-(i-1))(n+3-(i-1))(n+2)}}, a_i\sqrt{\frac{(n+3)(n-i+3)}{(n+4-i)(n+2)}}, 0, 0\right)$$

And: $(b_i)^{-1} = [K_1(u^i, u^i)]^{-1} = \frac{\mu(E_n)}{n+3}$

$$x = \left(\frac{-a_1}{n+2}, \frac{-a_2}{n+2}\sqrt{\frac{n+3}{n+1}}, \ldots, -a_{n-1}\sqrt{\frac{n+3}{20(n+2)}}, a_n\sqrt{\frac{3(n+3)}{4(n+2)}}\right)$$

the number of integral points is $n+1$, and the minimum number of is $N \ge M(n,1) = n+1$, For $f(x) = 1$ substitute in (2.7), we find:

$$C = \mu(E_n)\frac{3}{n+3}$$

3.3. The formation of the cubature formula for the Reproducing Kernel $K_2(u,x)$, $\widetilde{K}_2(u,x)$:

- **For $K_2(u,x)$:**

Table (1): points and constants of the Cubature Formula (2.7), for $K_2(u,x), d = 4, n = 2$.

The points	The constant
$u^1 = (a_1, 0), u^2 = \left(\frac{-1+\sqrt{7}}{6}a_1, \frac{\sqrt{28+2\sqrt{7}}}{6}a_2\right)$	$\frac{a_1 a_2}{14}\pi$
$x^1 = \left(\frac{-1+\sqrt{7}}{6}a_1, \frac{-7-2\sqrt{7}}{3\sqrt{28+2\sqrt{7}}}a_2\right)$ $x^3 = \left(\frac{-1-\sqrt{7}}{6}a_1, \frac{-\sqrt{7}}{\sqrt{28+2\sqrt{7}}}a_2\right)$ $x^4 = \left(\frac{-1-\sqrt{7}}{6}a_1, \frac{\sqrt{7}}{\sqrt{28+2\sqrt{7}}}a_2\right)$	$\frac{13-\sqrt{7}}{56}a_1 a_2\pi$
$x^2 = \left(\frac{-1+\sqrt{7}}{6}a_1, \frac{-7+4\sqrt{7}}{3\sqrt{28+2\sqrt{7}}}a_2\right)$	$\frac{9+3\sqrt{7}}{56}a_1 a_2\pi$

The number of integral points is equal to 6, and the minimum number of points according to (2.2) is 6, and we note that all points are located within the elliptic.

Table (2): points and constants of the Cubature Formula (2.7), for $K_2(u,x), d = 4, n = 3$.

The points	constant
$u^1 = (a_1, 0, 0), u^2 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{\sqrt{40+2\sqrt{8}}}{7}a_2, 0\right)$ $u^3 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{9\sqrt{8}-16}{7\sqrt{40+2\sqrt{8}}}a_2, \sqrt{\frac{104+64\sqrt{8}}{7(40+2\sqrt{8})}}a_3\right)$	$\frac{a_1 a_2 a_3\pi}{15}$
$x^1 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{-8+9\sqrt{2}}{7\vartheta}a_2, \nu a_3\right)$	ι
$x^2 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{-8+9\sqrt{2}}{7\vartheta}a_2, \eta a_3\right)$ $x^3 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{-8-5\sqrt{2}}{7\vartheta}a_2, \varsigma a_3\right)$ $x^5 = \left(\frac{-\sqrt{8}-1}{7}a_1, \frac{\sqrt{2}}{\vartheta}a_2, \varsigma a_3\right)$ $x^8 = \left(\frac{-\sqrt{8}-1}{7}a_1, \frac{-\sqrt{2}}{\vartheta}a_2, -\varsigma a_3\right)$	ω
$x^4 = \left(\frac{\sqrt{8}-1}{7}a_1, \frac{-8-5\sqrt{2}}{7\vartheta}a_2, -\sigma a_3\right)$ $x^6 = \left(\frac{-\sqrt{8}-1}{7}a_1, \frac{\sqrt{2}}{\vartheta}a_2, -\sigma a_3\right)$ $x^7 = \left(\frac{-\sqrt{8}-1}{7}a_1, \frac{-\sqrt{2}}{\vartheta}a_2, \sigma a_3\right)$	ϱ

Where: $\tau = \sqrt{2268 + 2422\sqrt{8}}, \vartheta = \sqrt{10 + \sqrt{2}}, \varsigma = \frac{14(\sqrt{8}-1)}{\tau}$

$\sigma = \frac{22+6\sqrt{8}}{\tau}, \eta = \frac{-58+2\sqrt{8}}{\tau}, \nu = \frac{-50+22\sqrt{8}}{\tau}, \iota = \frac{175\sqrt{2}+231}{240+1200\sqrt{2}}a_1 a_2 a_3\pi$

$\varrho = \frac{287\sqrt{2}-217}{240+1200\sqrt{2}}a_1 a_2 a_3\pi, \omega = \frac{81\sqrt{2}+173}{240+1200\sqrt{2}}a_1 a_2 a_3\pi$

The number of integral points is equal to 11, and the minimum number of points according to (2.2) is 10 and we note that all points are located within the elliptic.

Table (3): points and constants of the Cubature Formula (2.7), for $K_2(u,x), d = 4, n = 4$.

The points	The constant
$u^1 = (a_1, 0, 0, 0), u^2 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{4}a_2, 0, 0\right)$ $u^3 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{3}{\sqrt{10}}a_3, 0\right)$ $u^4 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{1}{2\sqrt{10}}a_3, \frac{\sqrt{35}}{2\sqrt{10}}a_4\right)$	$\frac{1}{54}a_1 a_2 a_3 a_4\pi^2$
$x^1 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{1}{2\sqrt{10}}a_3, \frac{1}{\sqrt{56}}a_4\right)$	$\frac{1}{18}a_1 a_2 a_3 a_4\pi^2$
$x^2 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{1}{2\sqrt{10}}a_3, \frac{-5\sqrt{2}}{4\sqrt{7}}a_4\right)$ $x^3 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{-2}{\sqrt{10}}a_3, \frac{1}{\sqrt{14}}a_4\right)$ $x^4 = \left(\frac{1}{4}a_1, \frac{-3\sqrt{15}}{20}a_2, \frac{1}{\sqrt{10}}a_3, \frac{1}{\sqrt{14}}a_4\right)$ $x^{5,6} = \left(\frac{-1}{2}a_1, \pm\sqrt{\frac{3}{20}}a_2, \pm\frac{1}{\sqrt{10}}a_3, \pm\frac{1}{\sqrt{14}}a_4\right)$	$\frac{1}{27}a_1 a_2 a_3 a_4\pi^2$
$x^7 = \left(\frac{1}{4}a_1, \frac{\sqrt{15}}{20}a_2, \frac{-2}{\sqrt{10}}a_3, -\sqrt{\frac{2}{7}}a_4\right)$ $x^8 = \left(\frac{1}{4}a_1, \frac{-3\sqrt{15}}{20}a_2, \frac{1}{\sqrt{10}}a_3, -\sqrt{\frac{2}{7}}a_4\right)$ $x^{9,10} = \left(\frac{1}{4}a_1, \frac{-3\sqrt{15}}{20}a_2, \frac{-3}{2\sqrt{10}}a_3, \pm\frac{3}{2\sqrt{14}}a_4\right)$ $x^{11,12} = \left(\frac{-1}{2}a_1, \pm\sqrt{\frac{3}{20}}a_2, \pm\frac{1}{\sqrt{10}}a_3, \mp\sqrt{\frac{2}{7}}a_4\right)$ $x^{13,14} = \left(\frac{-1}{2}a_1, \sqrt{\frac{3}{20}}a_2, \frac{-3}{2\sqrt{10}}a_3, \pm\frac{3}{2\sqrt{14}}a_4\right)$ $x^{15,16} =$ $\left(\frac{-1}{2}a_1, -\sqrt{\frac{3}{20}}a_2, \pm\frac{3}{2\sqrt{10}}a_3, \pm\frac{3}{2\sqrt{14}}a_4\right)$	$\frac{1}{54}a_1 a_2 a_3 a_4\pi^2$

The number of integral points is equal to 20, and the minimum number of points according to (2.2) is 15 and we note that all points are located within the elliptic.

- • **For $\widetilde{K}_2(u,x)$:**

Table (4): points and constants of the Cubature Formula (2.8), for $\widetilde{K}_2(u,x), d = 5, n = 2$.

The points	The constants
$u^1 = (a_{1,0}), u^2 = \left(-\frac{a_1}{\sqrt{6}}, \sqrt{\frac{5}{6}}a_2\right)$	$\frac{a_1 a_2\pi}{20}$

$x^1=\left(-\frac{a_1}{\sqrt{6}},-\frac{a_2}{\sqrt{5}}\left(1+\frac{1}{\sqrt{6}}\right)\right),x^4=\left(\frac{a_1}{\sqrt{6}},\frac{a_2}{\sqrt{5}}\left(1+\frac{1}{\sqrt{6}}\right)\right)$	$\frac{8\sqrt{6}-3}{40\sqrt{6}}a_1a_2\pi$
$x^2=\left(-\frac{a_1}{\sqrt{6}},\frac{a_2}{\sqrt{5}}\left(1-\frac{1}{\sqrt{6}}\right)\right),x^3=\left(\frac{a_1}{\sqrt{6}},\frac{a_2}{\sqrt{5}}\left(-1+\frac{1}{\sqrt{6}}\right)\right)$	$\frac{8\sqrt{6}+3}{40\sqrt{6}}a_1a_2\pi$

The number of integral points is equal to 8, and the minimum number of points according to (2.3) is 8, and we note that all points are located within the elliptic.

Table (5): points and constants of the Cubature Formula (2.8), for $\widetilde{K}_2(u,x),d=5,n=3$.

The points	The constants
$u^1=(a_1,0,0),u^2=\left(\frac{a_1}{\sqrt{7}},\sqrt{\frac{6}{7}}a_2,0\right)$ $u^3=\left(\frac{a_1}{\sqrt{7}},\frac{a_2}{\sqrt{6}}\upsilon,\frac{a_3}{\sqrt{3}}\sqrt{2+\frac{1}{\sqrt{7}}}\right)$	$\frac{2a_1a_2a_3\pi}{45}$
$x^1=\left(\frac{a_1}{\sqrt{7}},\frac{a_2}{\sqrt{6}}\upsilon,\frac{a_3}{\sqrt{3}}\xi\right),x^8=\left(\frac{-a_1}{\sqrt{7}},\frac{-a_2}{\sqrt{6}}\upsilon,\frac{-a_3}{\sqrt{3}}\xi\right)$	$\frac{(3\sqrt{7}+9)a_1a_2a_3\pi}{90}$
$x^2=\left(\frac{a_1}{\sqrt{7}},\frac{a_2}{\sqrt{6}}\upsilon,\frac{-a_3}{\sqrt{3}}\frac{2+\sqrt{7}}{\sqrt{14-\sqrt{7}}}\right),x^3=\left(\frac{a_1}{\sqrt{7}},\frac{-a_2}{\sqrt{6}}\rho,\delta a_3\right)$ $x^4=\left(\frac{a_1}{\sqrt{7}},\frac{-a_2}{\sqrt{6}}\rho,-\delta a_3\right),x^5=\left(\frac{-a_1}{\sqrt{7}},\frac{a_2}{\sqrt{6}}\rho,-\delta a_3\right)$ $x^6=\left(-\frac{a_1}{\sqrt{7}},\frac{a_2}{\sqrt{6}}\rho,\delta a_3\right),x^7=\left(\frac{-a_1}{\sqrt{7}},\frac{-a_2}{\sqrt{6}}\upsilon,\frac{a_3}{\sqrt{3}}\frac{2+\sqrt{7}}{\sqrt{14-\sqrt{7}}}\right)$	$\frac{(-\sqrt{7}+9)a_1a_2a_3\pi}{90}$

where $\upsilon=\left(1-\frac{1}{\sqrt{7}}\right),\rho=\left(1+\frac{1}{\sqrt{7}}\right),\xi=\frac{4-\sqrt{7}}{\sqrt{14-\sqrt{7}}},\epsilon=\sqrt{\frac{3}{14-\sqrt{7}}}$

The number of integral points is equal to 14, and the minimum number of points according to (2.3) is 14, and we note that all points are located within the elliptic.

Table (6): points and constants of the Cubature Formula (2.8), for $\widetilde{K}_2(u,x),d=5,n=4$.

The points	constants
$u^1=(a_1,0,0,0)$ $u^2=\left(-\frac{a_1}{\sqrt{8}},\sqrt{\frac{7}{8}}a_2,0,0\right)$ $u^3=\left(-\frac{a_1}{\sqrt{8}},\frac{-1-\sqrt{8}}{\sqrt{56}}a_2,\sqrt{\frac{40-2\sqrt{8}}{56}}a_3,0,0\right)$ $u^3=\left(-\frac{a_1}{\sqrt{8}},\frac{-1-\sqrt{8}}{\sqrt{56}}a_2,\frac{-16-9\sqrt{8}}{\beta}a_3,\sqrt{\frac{13-9\sqrt{8}}{40-2\sqrt{8}}}a_4\right)$	$\frac{1}{84}a_1a_2a_3a_4\pi^2$
$x^{1,2}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1+\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{16+9\sqrt{8}}{\beta}a_3,\pm\varepsilon a_4\right)$	A
$x^{3,4}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1-\sqrt{8}}{\sqrt{56}}a_2,\mp\frac{\sqrt{7}}{\gamma}a_3,\pm\zeta a_4\right)$ $x^{5,6}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1+\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{16-5\sqrt{8}}{\beta}a_3,\mp\zeta a_4\right)$ $x^{3,4}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1-\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{\sqrt{7}}{\gamma}a_3,\mp\zeta a_4\right)$	B
$x^{9,10}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1-\sqrt{8}}{\sqrt{56}}a_2,\mp\frac{\sqrt{7}}{\gamma}a_3,\mp\delta a_4\right)$ $x^{11,12}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1-\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{\sqrt{7}}{\gamma}a_3,\pm\delta a_4\right)$ $x^{13,14}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1+\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{16-5\sqrt{8}}{\beta}a_3,\pm\delta a_4\right)$ $x^{15,16}=\left(\pm\frac{a_1}{\sqrt{8}},\pm\frac{1+\sqrt{8}}{\sqrt{56}}a_2,\pm\frac{16+9\sqrt{8}}{\beta}a_3,\pm\frac{88+25\sqrt{8}}{\chi}a_4\right)$	C

Where: $\chi=\sqrt{8}\sqrt{40-2\sqrt{8}}\sqrt{13-9\sqrt{8}},\varepsilon=\frac{8+29\sqrt{8}}{\chi},\zeta=\frac{24-11\sqrt{8}}{\chi}$

$\gamma=\sqrt{40-2\sqrt{8}},\delta=\frac{56+7\sqrt{8}}{\chi},\beta=\sqrt{56}\gamma,A=\frac{700+231\sqrt{8}}{336(80+4\sqrt{8})}a_1a_2a_3a_4\pi^2$

$B=\frac{1148-217\sqrt{8}}{336(80+4\sqrt{8})}a_1a_2a_3a_4\pi^2,C=\frac{324+173\sqrt{8}}{336(80+4\sqrt{8})}a_1a_2a_3a_4\pi^2$

The number of integral points is equal to 24, and the minimum number of points according to (2.3) is 22, and we note that all points are located within the elliptic.

3.4. The formation of the cubature formula for the Reproducing Kernel $K_3(u,x),\widetilde{K}_3(u,x)$:

- **For $K_3(u,x)$:**

Table (7): points and constants of the Cubature Formula (2.7), for $K_3(u,x),d=6,n=2$.

The points	The constant
$u^1=(a_1,0),u^2=\left(\frac{-381}{500}a_1,\frac{\sqrt{104839}}{500}a_2\right)$	$\frac{\pi r^2}{30}$
$x^1=\left(\frac{-381}{500}a_1,\frac{-295339}{500\sqrt{104839}}a_2\right)$	$-0.01467068687\,\pi r^2$
$x^2=\left(\frac{-381}{500}a_1,\frac{29661}{500\sqrt{104839}}a_2\right)$	$0.0961346475\,\pi r^2$
$x^3=\left(\frac{-381}{500}a_1,\frac{-143039}{500\sqrt{104839}}a_2\right)$	$0.074657325\,\pi r^2$
$x^4=\left(\frac{269}{500}a_1,\frac{-88011}{500\sqrt{104839}}a_2\right)$	$0.1441217863\,\pi r^2$
$x^5=\left(\frac{269}{500}a_1,\frac{236989}{500\sqrt{104839}}a_2\right)$	$0.00326883097\,\pi r^2$
$x^6=\left(\frac{269}{500}a_1,\frac{64289}{500\sqrt{104839}}a_2\right)$	$0.1852513736\,\pi r^2$
$x^7=\left(\frac{-191}{1250}a_1,\frac{-549021}{1250\sqrt{104839}}a_2\right)$	$0.01281529675\,\pi r^2$
$x^8=\left(\frac{-191}{1250}a_1,\frac{263479}{1250\sqrt{104839}}a_2\right)$	$0.1845697863\,\pi r^2$
$x^9=\left(\frac{-191}{1250}a_1,\frac{-168271}{1250\sqrt{104839}}a_2\right)$	$0.2471840999\,\pi r^2$

we note that the points x^2,x^4,x^6,x^8,x^9 are located inside the elliptic, and the rest of the points are located outside it, The number of integral points is equal to 11, and the minimum number of points according to (2.2) is 10

- where $n=3$: $u^1=(a_1,0,0),u^2=\left(\frac{-443}{625}a_1,\frac{\sqrt{194376}}{625}a_2,0\right)$
$u^3=\left(\frac{-443}{625}a_1,\frac{-473124}{625\sqrt{194376}}a_2,1.565417474\ i\,a_3\right)$, we note that we obtained complex values, and therefore it is impossible to obtain cubature formula with real points and that is for $d=6,n=3$.

- where $n=4$: : $u^1=(a_1,0,0,0),u^2=\left(\frac{-6656}{10000}a_1,\frac{\sqrt{55657719}}{10000}a_2,0,0\right)$
$u^3=\left(\frac{-6656}{10000}a_1,\frac{-110932281}{10000\sqrt{55657719}}a_2,1.286247125\,i\,a_3,0\right)$
$u^4=\left(\frac{-6656}{10000}a_1,\frac{-110932281}{10000\sqrt{55657719}}a_2,1.545995034\,i\,a_3,2.011102263\,i\,a_4\right)$
it is impossible to obtain cubature formula in the method of the reproducing kernel with real points and that is for $d=6,n=4$, therefore it is impossible to obtain cubature formula with real points and that is for $d=6,n\geq 3$.

- • **For $\widetilde{K}_3(u,x)$:**

Table (7): points and constants of the Cubature Formula (2.8), for $\widetilde{K}_3(u,x),d=7,n=2$.

The points	The constants
$u^1=\left(\frac{a_1}{2},\frac{\sqrt{3}}{2}a_2\right),u^2=\left(\frac{-\sqrt{3}}{2}a_1,\frac{1}{2}a_2\right)$	$\frac{a_1a_2\pi}{40}$
$x^1=(0,0)$	$\frac{7a_1a_2\pi}{54}$
$x^2=\left(\frac{-3}{4\sqrt{2}}a_1,\frac{\sqrt{3}}{4\sqrt{2}}a_2\right),x^3=\left(\frac{3}{4\sqrt{2}}a_1,\frac{-\sqrt{3}}{4\sqrt{2}}a_2\right)$ $x^4=\left(\frac{-3}{4\sqrt{2}}a_1,\frac{-\sqrt{3}}{4\sqrt{2}}a_2\right),x^7=\left(\frac{3}{4\sqrt{2}}a_1,\frac{\sqrt{3}}{4\sqrt{2}}a_2\right)$	$\frac{16a_1a_2\pi}{135}$
$x^5=\left(\frac{-3-\sqrt{3}}{4\sqrt{2}}a_1,\frac{\sqrt{3}-3}{4\sqrt{2}}a_2\right)$ $x^6=\left(\frac{3-\sqrt{3}}{4\sqrt{2}}a_1,\frac{-\sqrt{3}-3}{4\sqrt{2}}a_2\right)$ $x^8=\left(\frac{-3+\sqrt{3}}{4\sqrt{2}}a_1,\frac{\sqrt{3}+3}{4\sqrt{2}}a_2\right)$ $x^9=\left(\frac{3+\sqrt{3}}{4\sqrt{2}}a_1,\frac{-\sqrt{3}+3}{4\sqrt{2}}a_2\right)$	$\frac{2a_1a_2\pi}{27}$

The number of integral points is equal to 13, and the minimum number of points according to (2.4) is 13, and we note that all points are located within the elliptic.

Table (8): points and constants of the Cubature Formula (2.8), for $\widetilde{K}_3(u,x),d=7,n=3$.

The points	constant
$u^1=(a_1,0,0)$ $u^2=(0,a_2,0)$ $u^3=(0,0,a_3)$	$\frac{2a_1a_2a_3\pi}{105}$
$x^1=(0,0,0)$	$\frac{8a_1a_2a_3\pi}{105}$
$x^{2,3}=\left(\pm\frac{1}{\sqrt{3}}a_1,0,0\right),$ $x^{4,5}=\left(0,\pm\frac{1}{\sqrt{3}}a_2,0\right)$	$\frac{2a_1a_2a_3\pi}{35}$

$x^{6,7} = \left(0,0 \pm \frac{1}{\sqrt{3}} a_3\right)$	
$x^{8,9} = \left(\pm \frac{1}{\sqrt{3}} a_1, \pm \frac{1}{\sqrt{3}} a_2, 0\right)$ $x^{10,11} = \left(\pm \frac{1}{\sqrt{3}} a_1, 0, \pm \frac{1}{\sqrt{3}} a_3\right)$ $x^{12,13} = \left(0, \frac{\pm 1}{\sqrt{3}} a_2, \frac{\pm 1}{\sqrt{3}} a_3\right)$ $, x^{14,15} = \left(\pm \frac{1}{\sqrt{3}} a_1, \mp \frac{1}{\sqrt{3}} a_2, 0\right)$ $x^{16,17} = \left(\frac{\pm 1}{\sqrt{3}} a_1, 0, \frac{\mp 1}{\sqrt{3}} a_3\right),$ $x^{18,19} = \left(0, \pm \frac{1}{\sqrt{3}} a_2, \mp \frac{1}{\sqrt{3}} a_3\right)$	$\frac{2a_1a_2a_3\pi}{35}$
$x^{20,21} = \left(\frac{\mp 1}{\sqrt{3}} a_1, \frac{\mp 1}{\sqrt{3}} a_2, \frac{\mp 1}{\sqrt{3}} a_3\right),$ $x^{22,23} = \left(\frac{\pm 1}{\sqrt{3}} a_1, \frac{\mp 1}{\sqrt{3}} a_2, \frac{\mp 1}{\sqrt{3}} a_3\right)$	$\frac{a_1a_2a_3\pi}{70}$

The number of integral points is equal to 33, and the minimum number of points according to (2.4) is 27, and we note that all points are located within the elliptic.

3.5. Generalization of the reproducing kernel formula:

• **For $K_k(u,x)$:**

1- The method (1):

Let's put $E_m(u,x) = K_m(u,x) - K_{m-1}(u,x)$ and suppose that $t = \sum_{j=1}^{n} \frac{u_j x_j}{a_j^2}$ we get:

$$E_1(t) = \frac{n+2}{\mu(E_n)} t$$

$$E_2(t) = \frac{(n+2)(n+4)}{2\mu(E_n)} \left(t^2 - \frac{1}{n+2}\right)$$

$$E_3(t) = \frac{(n+2)(n+4)(n+6)}{6\mu(E_n)} \left(t^3 - \frac{3}{n+4} t\right)$$

and in general: $E_m = \frac{(n+2m)!!}{n!!m!\,\mu(E_n)} P_{m00\ldots0}(t)$

Where $P_{m00\ldots0}(t)$ are basic polynomials on $B_n^{(1)}$ (a ball of radius 1 in $\mathbb{R}^n$) (see p172 in [13]), and we can write it by the next formula:

$$P_{m00\ldots0}(t) = t^m + \sum_{k=1}^{\left[\frac{m}{2}\right]} (-1)^k \frac{m!}{2^k k!\,(m-2k)!} \frac{[n+2(m-k-1)]!!}{[n+2(m-1)]!!} t^{m-2k} \qquad ; m \geq 2$$

then we find:

$$K_m(t) = E_m(t) + K_{m-1}(t)$$

$$K_m(t) = \frac{(n+2m)!!}{n!!\,m!\;\mu(E_n)} P_{m00\ldots0}(t) + E_{m-1}(t) + K_{m-2}(t)$$

$$K_m(t) = \frac{(n+2m)!!}{n!!m!\,\mu(E_n)} P_{m00\ldots0}(t) + \frac{(n+2m-2)!!}{n!!(m-1)!\,\mu(E_n)} P_{m-100\ldots0}(t) + \ldots + K_0(t)$$

$$K_m(t) = \sum_{j=0}^{m} \frac{(n+2j)!!}{n!!\,j!\,\mu(E_n)} P_{m00\ldots0}(t)$$

2- The method (2):

Let's put $n = 2$ in $K_1(u,x), K_2(u,x), K_3(u,x)$, and let's assume that:

$$\vartheta = \left(\frac{u_1 x_1}{a_1^2} + \frac{u_2 x_2}{a_2^2}\right), \lambda = 2$$

we find that the relations of $K_1(u,x), K_2(u,x), K_3(u,x)$ are written in the form:

$$K_1(u,x) = \frac{1}{a_1 a_2 \pi} [2\lambda\vartheta + 1]$$

$$K_2(u,x) = \frac{1}{a_1 a_2 \pi} [2\lambda(1+\lambda)\vartheta^2 + 2\lambda\vartheta - \lambda]$$

$$K_3(u,x) = \frac{1}{a_1 a_2 \pi} \left[\frac{4}{3}\lambda(1+\lambda)(2+\lambda)\vartheta^3 + 2\lambda(1+\lambda)\vartheta^2 - 2\lambda(1+\lambda)\vartheta - \lambda\right]$$

The polynomials in the right side are Gegenbauer polynomials, and in general for $\vartheta = \sum_{i=1}^{n} \frac{u_i x_i}{a_i^2}$ and $n = 2,3,4,\ldots$, we find:

$$K_k(u,x) = \frac{1}{\mu(E_n)} \left[C_k^{\frac{n+2}{2}}(\vartheta) + C_{k-1}^{\frac{n+2}{2}}(\vartheta)\right] \quad ; k = 1,2,\ldots$$

• • **For $\widetilde{K}_k(u,x)$:**

In the same way as before, we find:

$$\widetilde{K}_k(u,x) = \frac{1}{\mu(E_n)} C_k^{\frac{n+2}{2}}(\vartheta) \quad ; k = 1,2,\ldots$$

4. Examples:

Table (9): Examples

Integral	Approximation solution	Exact solution
$I_1 = \iint_{E_2} ln\left(\sqrt{x^2+y^2} + 1\right) dxdy$ $a_1 = a_2 = 1$	$d = 5$ $I_1 = 0.59930582\pi$ $d = 7$ $I_1 = 0.5111610834\pi$	0.5π
$I_2 = \iint_{\Omega} e^{x^2+y^2} dxdy$ is the quarter of the episodeΩ $\Omega = \left\{x, y; \frac{1}{4} \leq \frac{x^2}{4} + \frac{y^2}{4} \leq 1\right\}$	$d = 5$ $I_2 = 14.90205909\pi$ $d = 7$ $I_2 = 13.23574473\pi$	12.96996705π
$I_3 = \iint_{E_2} (x^2+y^2)^{3/2} dxdy$ $a_1 = a_2 = 2$	$d = 5$ $I_3 = 11.08275398\pi$ $d = 7$ $I_3 = 12.84212162\pi$	$12.8\,\pi$

5. Conclusions

The reproducing kernel method is distinguished from other methods in that it can be applied regardless of the dimension of space and whatever the shape of the studied area, and Through the above, we find that it is useful to increase the degree of polynomials to obtain a cubature formula with higher algebraic precision, and the formula of reproducing kernel can be generalized for any degree of the polynomial to obtain it without conclusion, we have been able through what we previously concluded, that cubature formulas (2.7) and (2.8) can be written in the form:

$$\int_{\Omega} \omega(x) f(x) dx \cong \sum_{i=1}^{n} \frac{1}{b_i} f(u^i) + \sum_{j=1}^{s} C_j f(x^j)$$

$$\int_{\Omega} \omega(x) f(x) dx \cong \sum_{i=1}^{n} \frac{\mu(\Omega)}{C} f(u^i) + \sum_{j=1}^{s} \mu(\Omega)\, h_i f(x^j)$$

$$\int_{\Omega} \omega(x) f(x) dx \cong \mu(\Omega) \left(\sum_{i=1}^{n} \frac{1}{C} f(u^i) + \sum_{j=1}^{s} h_i f(x^j)\right)$$

$$\int_{\Omega} \omega(x) f(x) dx \cong mess(\Omega) \left(\sum_{i=1}^{n} \frac{1}{C} f(u^i) + \sum_{j=1}^{s} h_i f(x^j)\right)$$

$$\frac{1}{mess(\Omega)} \int_{\Omega} \omega(x) f(x) dx \cong \sum_{i=1}^{n} \frac{1}{C} f(u^i) + \sum_{j=1}^{s} h_i f(x^j)$$

And in this way:

$$\frac{1}{mess(\Omega)} \int_{\Omega} \omega(x) f(x) dx \cong \sum_{i=1}^{n} \frac{1}{2C} \left(f(u^i) + f(-u^i)\right) + \sum_{j=1}^{s} h_i f(x^j)$$

$mess(\Omega)$: area size

Author's contribution

All authors reviewed the results and approved the final version of the manuscript.

Funding

nothing

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article

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