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Solving Higher Order Ordinary Differential Equations in Physics: A New Modified Adomian Decomposition Method Approach to MHD Flows, Elastic Beam, and Sixth-Order Boundary value problem

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Abstract: The study presents a novel approach for solving highe-order ordinary differential equations (ODEs) prevalent in various physics applications, specifically through a modified Adomian Decomposition Method (ADM). This method enhances the traditional ADM by introducing specific modifications that improve its convergence and applicability to complex problems. The research focuses on three primary areas: magnetohydrodynamic (MHD) flows, the dynamics of elastic beams, and sixth-order boundary value problems. The proposed method demonstrates significant effectiveness in deriving analytical solutions that can accurately predict physical behaviors in these domains. By applying the modified ADM, the study not only addresses the challenges associated with higher order ODEs but also offers practical solutions for engineers and physicists working with intricate modeling scenarios. The results indicate that this method provides an efficient and reliable framework for analyzing and solving complex differential equations in the field of physics and engineering.

Keywords: Magnetohydrodynamic (MHD) Flows; Jeffery-Hamel Problem; Elastic Beam Equation; Sixth-Order Boundary Value Problem; Modified Adomian Decomposition Method; Boundary Conditions.

Introduction

Differential equations serve as a fundamental tool across various scientific disciplines, including physics, chemistry, and engineering. They provide a framework for modeling an extensive range of natural phenomena, allowing for a deeper understanding of dynamic systems characterized by either linear or nonlinear behavior. By developing models based on these equations, researchers can derive both analytical and approximate solutions, which have gained significant prominence in recent studies.

Given the inherent challenges in obtaining precise solutions for physical problems represented mathematically by differential equations, the exploration of analytical and numerical techniques becomes imperative. Foundational work in operator calculus, such as that of Mikusiski and RiemannLiouville [1–3], has provided a theoretical basis for manipulating and inverting differential operators, which underpins many decomposition-based methods.

In fluid mechanics, for instance, the Jeffery-Hamel flow, a significant conceptual advancement introduced by Jeffery [4] and Hamel [5] in the early 20th century, has become a pivotal example of how two-dimensional incompressible viscous flows can be examined using Navier-Stokes equations.

The term magnetohydrodynamic (MHD) was first introduced by researcher Parth Bansal, marking a significant development in the field of fluid mechanics [6]. MHD studies the behavior of electrically conducting fluids in the presence of magnetic fields, combining principles from both electromagnetism and fluid dynamics. Bansal's contributions laid the groundwork for exploring various MHD applications, particularly in contexts like astrophysics, engineering, and plasma physics, where magnetic fields interact with fluid flow, influencing behaviors such as turbulence and stability [7].

The diverse physical problems associated with fourth-order differential equations are intrinsically linked to the field of elastic stability theory. These equations play a critical role in modeling the behavior of structural elements under various loading conditions, particularly those that exhibit bending and torsional deformations. When addressing the elastic stability of structures, one encounters phenomena such as buckling, which occurs when a structure is subjected to critical loads that exceed its capacity to maintain its original shape [8]. The analysis of such problems typically involves fourth-order differential equations due to their ability to incorporate the effects of shear deformation and rotational inertia, which are essential for accurately predicting the stability and performance of structural components [9–12].

A comprehensive and uniform framework for sixth-order boundary val-

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ue problems is increasingly recognized as a vital tool for analyzing a diverse array of challenges across fields such as astrophysics, mathematics, and engineering sciences. This framework allows researchers and practitioners to study complex systems and phenomena that exhibit higher-order behavior, which is often crucial for accurately modeling the intricate processes in these domains [13].

In astrophysics, for instance, many phenomena, including the convective stability in stellar structures or the dynamics of astronomical bodies, can be effectively described using sixth-order differential equations. These equations arise from the need to account for higher-order effects that are significant in understanding stability and oscillatory behaviors, allowing for a more nuanced description of astrophysical systems [14]. Given the inherent challenges in obtaining precise solutions for physical problems represented mathematically by differential equations, the exploration of numerical techniques becomes imperative. Among these, the Adomian Decomposition Method (ADM) [15,16] has demonstrated its utility, particularly in addressing ordinary differential equations without the need for discretization or linearization, and its performance has been enhanced by a number of modifications [17–27]. ADM provides an approximate solution of the problem without simplifying it, in contrast to usual methods that include mild nonlinearity and minor perturbations, which alter the physics of the problem through simplification.

The research conducted by various authors illustrates the versatility of the (ADM) and its modified versions in tackling complex differential equations across multiple domains. Odiba's work exemplifies the application of the Modified ADM (MADM) to a third-order ordinary differential equation, showcasing its efficacy in deriving solutions that might be computationally intensive with traditional methods [28]. Meanwhile, I. Hashim's integration of ADM for a fourth-order integro-differential equation highlights the method's adaptability in handling intricate mathematical models, such as those found in engineering and physics [29]. While Ali et al.'s application of ADM to the HIV infection model indicates its relevance in biological systems modeling [30]. Similarly, Coskun et al. demonstrated ADM's utility in studying the behavior of Euler beams under variable stiffness, reinforcing the method's significance in structural analysis [31].

However, most existing operator-based extensions of the Adomian Decomposition Method (ADM) focus on polynomial or standard differential operator forms and do not explicitly address exponential-type differential operators. This highlights a gap in the current literature, which our proposed modification seeks to fill by incorporating complex exponential weightings into the operator structure. This study aims to develop and demonstrate a modified Adomian Decomposition Method (ADM) for addressing higher order ordinary differential equations (ODEs) relevant to physics. The focus is on applying this method to specific problems, including magnetohydrodynamic (MHD) flows, the behavior of elastic beams, and sixth-order boundary value problems.

The goal of the research is to provide an effective analytical solution technique that enhances the flexibility and applicability in solving complex ODEs. This approach is intended to bridge theoretical advancements and practical applications in fields such as physics and engineering, ultimately aiming to offer more robust solutions to problems characterized by higher order differential equations in various physical contexts.

The Mathematical Formulation of the Jeffery-Hamel flow and the Beam Equation of Fourth Order

First

As shown in Figure 1, the Figure represent the steady two-dimensional flow of an incompressible, conductive, viscous fluid between two rigid plane walls that meet at an angle of 2α. The velocity of the fluid depends solely on the radial distance (r) and the angle(θ), and it is entirely radial in nature [32]. In polar coordinates, the continuity equation and the Navier-Stokes equations can be expressed as follows:

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru(r,\theta)) = 0, \quad (1)$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\sigma B_0^2}{\rho r^2}u(r,\theta)$$
$$-\mu[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}] = 0, \quad (2)$$

$$\frac{1}{\rho r}\frac{\partial p}{\partial \theta} - \frac{2\mu}{r^2}\frac{\partial u(r,\theta)}{\partial \theta} = 0, \quad (3)$$

note that, ρ the fluid density, μ is the coefficient of kinematic viscosity, σ the conductivity, B_0 the electromagnetic induction and p is the fluid pressure.

Using equation (1)

$$h(\theta) = ru(r,\theta), \quad (4)$$

from dimensionless parameters

$$f(x) = \frac{h(\theta)}{h_{max}}, \quad x = \frac{\theta}{\alpha}, \quad (5)$$

Putting them in (2) and (3), we get nonlinear third order boundary value problem

$$f'''(x) + 2\alpha Ref(x)f'(x) + (4 - Ha)\alpha^2 f'(x) = 0, \quad (6)$$

with boundary conditions

$$f(0) = 1, f'(0) = 0, f(1) = 0. \quad (7)$$

Where Re is Reynolds number

$$Re = \frac{U_{max}}{\mu}\left(\begin{array}{c} divergent: \alpha > 0, U_{max} > 0 \\ convergent: \alpha < 0, , U_{max} < 0 \end{array}\right),$$

and Ha is Hartmann number

$$Ha = \sqrt{\frac{\sigma B_0^2}{\rho\mu}}.$$

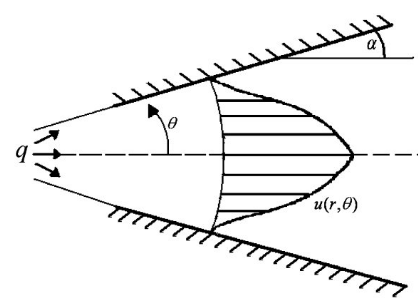


Figure 1: Geometry of the MHD Jeffery-Hamel flow.

Second

When a beam-column that subjected to both the axial load and the a spread load perpendicular to axis as indicated in figure 2 (b) has a cross section distance dx internal forces occur in the element in figure 2 (a). When the equilibrium equation is expressed in the direction of the distributed q, it is given as:

$$q = -\frac{dV}{dx}. \quad (8)$$

The equilibrium equation in the y direction can be written as an ordinary differential equation, as presented in [33–35]

$$-V + qdx + (V + dV) = 0. \quad (9)$$

The sum of the forces acting on each surface of the cross-section must be balanced due to the equilibrium condition. This can be expressed as:

$$M + qdx + \frac{dx}{q} + (V + dV)dx - (M + dM) + p\frac{dy}{dx} = 0. \quad (10)$$

Here M is the bending moment trying to bend the cross-section element, and V is the shear force acting on the surface of the element.
Assuming the rotation is small and neglecting second order terms in dx , equation (10) simplifies to:

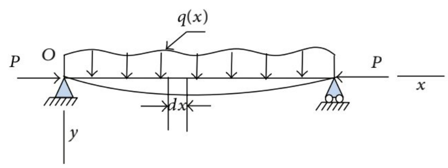
$$V = \frac{dM}{dx} - p\frac{dy}{dx}. \tag{11}$$

Since small rotation is assumed, and considering that $\frac{d^2}{dx^2} = -\frac{M}{EL}$, then (11) written as

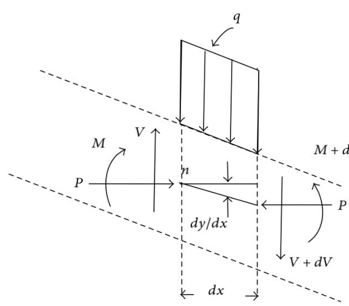
$$-V = EL\frac{d^3y}{dx^3} + p\frac{dy}{dx}. \tag{12}$$

Here, El refers to bending rigidity.Differentiating both sides of equation (12) with respect to x, we obtain a fourth-order differential equation for the elastic curve, given by:

$$EL\frac{d^4y}{dx^4} + p\frac{d^2y}{dx^2} = q(x). \tag{13}$$



(a)



(b)

Figure 2: Cross-section analysis of the column-beam element.

New Modified ADM

In this section, we present an innovative modification to the Standard Adomian Decomposition Method (ADM), specifically focusing on alterations made to the differential operator. This enhancement aims to improve the methods applicability and effectiveness in solving nonlinear differential equations.
Consider the following equation:

$$y^{(r+2)} + \nu y^{(r)} = g(x) + N(y), \quad r \in N \cup \{0\}, \tag{14}$$

with the conditions

$$y(0) = a_0, y'(0) = a_1, ..., y^{(r+1)}(0) = a_i, i = 0, 1, ...,$$

in the standard Adomian Decomposition Method (SADM), the linear operator is typically taken as $Ly = y^{(r+2)}$, where $L(y) = y^{(r+2)}$. However, in our proposed modification, we define the linear operator as $Ly = y^{(r+2)} + \nu y^{(r)}$, where $r = n + m$.
The choice $Ly = y^{(r+2)} + \nu y^{(r)}$ improves convergence because it modifies the spectral characteristics of the linear operator, leading to a more balanced and well-conditioned inverse. This structure helps reduce the growth of higher-order terms in the decomposition series, thereby accelerating convergence, especially for stiff or oscillatory problems.
Where $L(y)$ as the following

$$L_1(y) = \frac{d^m}{dx^m} e^{\pm i\sqrt{\nu}x} \frac{d}{dx} e^{\mp 2i\sqrt{\nu}x} \frac{d}{dx} e^{\pm i\sqrt{\nu}x} \frac{d^n}{dx^n}(y), \tag{15}$$

$$L_2(y) = \frac{d^m}{dx^m} e^{\pm i\sqrt{\nu}x} \frac{d}{dx} e^{\mp i\sqrt{\nu}x} \frac{d^n}{dx^n} e^{\mp i\sqrt{\nu}x} \frac{d}{dx} e^{\pm i\sqrt{\nu}x}(y). \tag{16}$$

Where ν is constant and in equation (15) $n, m \in N \cup \{0\}$, and in equation (16) $n \in N$ and $m \in N \cup \{0\}$.
The inverse of eq.(15) and eq.(16) as follows
$L_1^{-1}(y) =$

$$\underbrace{\int_0^x \int_0^x ... \int_0^x}_{n-times} e^{\mp i\sqrt{\nu}x} \int_0^x e^{\pm 2i\sqrt{\nu}x} \int_0^x e^{\mp i\sqrt{\nu}x} \underbrace{\int_0^x \int_0^x ... \int_0^x}_{m-times}(y)$$

$$\underbrace{dxdx...dxdxdxdxdx...dx}_{m+n+2-times}. \tag{17}$$

$L_2^{-1}(y) =$

$$e^{\mp i\sqrt{\nu}x} \int_0^x e^{\pm i\sqrt{\nu}x} \underbrace{\int_0^x \int_0^x ... \int_0^x}_{n-times} e^{\pm i\sqrt{\nu}x} \int_0^x e^{\mp i\sqrt{\nu}x}$$

$$\underbrace{\int_0^x \int_0^x ... \int_0^x}_{m-times}(y) \underbrace{dxdx...dxdxdxdxdx...dx}_{m+n+2-times}. \tag{18}$$

So equation (14) takes the form

$$Ly + Ny = g(x),$$

or equivalently:

$$Ly = g(x) - Ny, \tag{19}$$

to get y, take L^{-1} (L_1^{-1} or L_2^{-1}) to both sides of equations (19). Then

$$y = \kappa(x) + L^{-1}g(x) - L^{-1}N(y), \tag{20}$$

where $\kappa(x)$ arising from auxiliary conditions.
The solution in ADM decomposition to [15]

$$y_0 = \kappa(x) + L^{-1}g(x),$$

and

$$y_{n+1} = -L^{-1}N(y),$$

where

$$N(y) = \sum_{n=0}^{\infty} A_n,$$

A_n called Adomian polynomial and formed by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[N(\sum_{i=0}^{\infty}\lambda^i y_i)\Big], \quad where \quad \lambda = 0, n = 0, 1, 2, 3, ... \quad . \tag{21}$$

That is

$$y_{n+1} = -L^{-1}A_n,$$

then

$$y_1 = -L^{-1}A_0,$$
$$y_2 = -L^{-1}A_1,$$
$$y_3 = -L^{-1}A_2,$$

and so on.
Based on the above, the solution via ADM is given as a series:

$$y(x) = y_0 + y_1 + y_2 + y_3 + ... = \sum_{n=0}^{\infty} y_n. \tag{22}$$

To better illustrate how the proposed NMADM differs from other modified ADM approaches, Table 1 provides a comparative summary of the key features, convergence strategies, and implementation considerations.

Justification for Using Complex Exponential Weightings

The composition of differential operators with complex exponential weightings in equations (15) and (16) is motivated by their ability to reshape the operator's structure and improve its spectral properties. Exponential terms like $e^{\mp 2i\sqrt{\nu}\,x}$ act as integrating factors or transformation kernels, which can reduce oscillatory behavior and enhance the stability of the inverse operator. This strategy is inspired by techniques in spectral methods and perturbation theory, where such weightings are commonly used to simplify the inversion of differential operators and accelerate convergence. As a result, the modified operator becomes better suited for handling nonlinearities and leads to more efficient implementation of the decomposition process.

Existence and Uniqueness of Solution

Let $\aleph : \hbar \rightarrow \hbar$ is an operator, where $\hbar$ is the Banach space $(C(I_0), \|.\|)$, the space of all continuous functions on I_0 with norm $\|y\| = \max_{x\in I_0} |y(x)|$, and N satisfied Lipschitz condition ($|N(y) - N(z)| \leq \ell|y - z|$ where $y, z \in \hbar$).

Theorem 1 *The equation (14) has a unique solution $y \in \hbar$ on $I_0 = [0, T]$ when $0 < \varrho < 1$ where $\varrho = \ell\gamma$, and $\gamma = \sum_{i=0}^{\infty}(-1)^i \nu^i \frac{T^{(r+2)+2i}}{((r+2)+2i)!}$.*

proof *The mapping $\aleph : \hbar \rightarrow \hbar$ is defined as*

$$(\aleph y)(x) = \kappa(x) + L^{-1}g(x) - L^{-1}N(y),$$
$$(\aleph z)(x) = \kappa(x) + L^{-1}g(x) - L^{-1}N(z),$$

where L^{-1} define in equations (17,18).

$$\|(\aleph y)(x) - (\aleph z)(x)\| = \max_{x\in I_0} |L^{-1}N(y) - L^{-1}N(z)|$$
$$= \max_{x\in I_0} |L^{-1}(N(y) - N(z))|$$
$$\leq \ell \max_{x\in I_0} |y - z| L^{-1}(1)$$
$$\leq \ell\gamma \max_{x\in I_0} |y - z|$$
$$\leq \varrho\|y - z\|$$

.
Since $0 < \varrho < 1$, Then the mapping $\aleph$ is contraction. That is, there exists a unique solution $y \in \hbar$ by Banach contraction principle.

Convergent and Error Estimate

In this section, we will study the Convergent and Error Estimate of the solution.

Proof of Convergent

Theorem 2 *The series solution (22) of the equation (14) using NMADM converges if $| y_1 | < \infty$ and $0 < \varrho < 1$, $\varrho = \ell\gamma$.*

Proof *Let $S_n = \sum_{i=0}^{n} y_i$ is the sequence of partial sum. Since,*

$$N(y) = N(\sum_{i=0}^{\infty} y_i) = \sum_{i=0}^{\infty} A_i,$$

so,

$$N(S_n) = \sum_{i=0}^{n} A_i.$$

Now, we prove that S_n is a Cauchy sequence in Banach space $\hbar$. Let S_n,S_m be two arbitrary partial sums such that $n \geq m$, then,

$$\|S_n - S_m\| = \max_{x\in I_0} | S_n - S_m | = \max_{x\in I_0} | \sum_{i=m+1}^{n} y_i(x) |$$
$$= \max_{x\in I_0} | \sum_{i=m+1}^{n} L^{-1}Ny |$$
$$= \max_{x\in I_0} | \sum_{i=m+1}^{n} L^{-1}A_i |$$
$$= \max_{x\in I_0} | \sum_{i=m}^{n-1} L^{-1}A_i |$$
$$= \max_{x\in I_0} | L^{-1}[N(S_n) - N(S_m)] |$$
$$\leq \max_{x\in I_0} L^{-1} | [N(S_n) - N(S_m)] |$$
$$\leq \max_{x\in I_0} L^{-1}\ell | S_n - S_m |$$
$$\leq \gamma\ell\|S_n - S_m\|$$
$$\leq \varrho\|S_n - S_m\|.$$

Let $n = m + 1$ then,

$$\|S_{m+1} - S_m\| \leq \varrho\|S_m - S_{m-1}\|$$
$$\leq \varrho^2\|S_{m-1} - S_{m-2}\| \leq ... \leq \varrho^m\|S_1 - S_0\|.$$

Based on the triangle inequality, we get

$$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| +$$
$$\|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\|$$
$$\leq (\varrho^m + \varrho^{m+1} + ...\varrho^{n-1})\|S_1 - S_0\|$$
$$\leq \varrho^m(1 + \varrho + ...\varrho^{n-m-1})\|S_1 - S_0\|$$
$$\leq \varrho^m(\frac{1 - \varrho^{n-m}}{1 - \varrho})\|y_1\|.$$

Since $0 < \varrho < 1$, $(\varrho^{n-m}) \leq 1$. Then

$$\|S_n - S_m\| \leq \frac{\varrho^m}{1 - \varrho}\|y_1\|$$

$$\leq \frac{\varrho^m}{1 - \varrho} \max_{x\in I_0} | y_1 | .$$

Since $| y_1 | < \infty$, then, $\|S_n - S_m\| \rightarrow 0$ hence, S_n is a Cauchy sequence in Banach space $\hbar$, that is $\sum_{n=0}^{\infty} y_n$ converges and this completes the proof.

Error Estimate

The following theorem allows us to estimate the maximum absolute truncation error of the Adomian's series solution for NMADM.

Theorem 3 *The series solution (22) to the problem (14) is estimated to have a maximum absolute truncation error given by*

$$\max_{x\in I_0}|y(x)-\sum_{i=0}^{m}y_i(x)|\leq\frac{\varrho^m}{1-\varrho}\max_{x\in I_0}\mid y_1\mid$$

Proof *Theorem (2) gives us*

$$\|S_n-S_m\|\leq\frac{\varrho^m}{1-\varrho}\max_{x\in I_0}\mid y_1\mid .$$

Since $S_n=\sum_{i=0}^{n}y_i$, as $n\to\infty$, then $S_n\to y(x)$ so,

$$\|y(x)-S_m\|\leq\frac{\varrho^m}{1-\varrho}\max_{x\in I_0}\mid y_1\mid .$$

Thus, in the interval I_0, the maximum absolute truncation error is

$$\max_{x\in I_0}|y(x)-\sum_{i=0}^{m}y_i(x)|\leq\frac{\varrho^m}{1-\varrho}\max_{x\in I_0}\mid y_1\mid .$$

And thus the proof is finished.

Before starting the application, we will give the following remarks.

Remark 1 *Although the current analysis is conducted in the Banach space $C([0,T])$, it is worth noting that the approach may be extended to other functional settings. In particular, weighted spaces and Sobolev spaces $W^{k,p}$ may provide a more natural framework for problems involving boundary layers or beam-type structures. These spaces allow for more accurate modeling of the regularity and boundary behavior of solutions and could enhance both the theoretical and numerical aspects of the method.*

Remark 2 *Equation (15), and equation (16) give many differential operators, for instant, if the differential equation of third order we have six operators by giving the different value of n and m.*

$$L_1(y)=e^{-i\sqrt{\nu}\,x}\frac{d}{dx}e^{2i\sqrt{\nu}\,x}\frac{d}{dx}e^{-i\sqrt{\nu}\,x}\frac{d}{dx}(y),$$
$$L_2(y)=e^{i\sqrt{\nu}\,x}\frac{d}{dx}e^{-2i\sqrt{\nu}\,x}\frac{d}{dx}e^{i\sqrt{\nu}\,x}\frac{d}{dx}(y),$$
$$L_3(y)=\frac{d}{dx}e^{-i\sqrt{\nu}\,x}\frac{d}{dx}e^{2i\sqrt{\nu}\,x}\frac{d}{dx}e^{-i\sqrt{\nu}\,x}(y),$$
$$L_4(y)=\frac{d}{dx}e^{i\sqrt{\nu}\,x}\frac{d}{dx}e^{-2i\sqrt{\nu}\,x}\frac{d}{dx}e^{i\sqrt{\nu}\,x}(y),$$
$$L_5(y)=e^{-i\sqrt{\nu}\,x}\frac{d}{dx}e^{i\sqrt{\nu}\,x}\frac{d}{dx}e^{i\sqrt{\nu}\,x}\frac{d}{dx}e^{-i\sqrt{\nu}\,x}(y),$$
$$L_6(y)=e^{i\sqrt{\nu}\,x}\frac{d}{dx}e^{-i\sqrt{\nu}\,x}\frac{d}{dx}e^{-i\sqrt{\nu}\,x}\frac{d}{dx}e^{i\sqrt{\nu}\,x}(y).$$

All of them give the differential equation $Ly=y'''+\nu y'$.

Remark 3 *All above differential operators give the same result.*

Proof *We aim to show that all the defined differential operators yield the same result.*
Starting with the operator $L_1(y)$, we compute:

$$\begin{aligned}L_1(y)&=e^{-i\sqrt{\nu}x}\frac{d}{dx}\left(e^{2i\sqrt{\nu}x}\frac{d}{dx}\left(e^{-i\sqrt{\nu}x}\frac{d}{dx}(y)\right)\right)\\&=e^{-i\sqrt{\nu}x}\frac{d}{dx}\left(e^{2i\sqrt{\nu}x}\frac{d}{dx}\left(-i\sqrt{\nu}e^{-i\sqrt{\nu}x}y'+e^{-i\sqrt{\nu}x}y''\right)\right)\\&=e^{-i\sqrt{\nu}x}\frac{d}{dx}\left(-i\sqrt{\nu}e^{i\sqrt{\nu}x}y'+e^{i\sqrt{\nu}x}y''\right)\\&=e^{-i\sqrt{\nu}x}\left(\nu e^{i\sqrt{\nu}x}y'-i\sqrt{\nu}e^{i\sqrt{\nu}x}y''+i\sqrt{\nu}e^{i\sqrt{\nu}x}y''\right.\\&\left.+e^{-i\sqrt{\nu}x}y'''\right)\\&=e^{-i\sqrt{\nu}x}\left(\nu e^{i\sqrt{\nu}x}y'+e^{-i\sqrt{\nu}x}y'''\right)=y'''+\nu y'\end{aligned}$$

Similarly, it can be shown that:

$$L_2(y)=L_3(y)=L_4(y)=L_5(y)=L_6(y)=y'''+\nu y'$$

Now consider the inverse operator of L_1:

$$L_1^{-1}(y)=\int_0^x e^{i\sqrt{\nu}x}\left(\int_0^x e^{-2i\sqrt{\nu}x}\left(\int_0^x e^{-i\sqrt{\nu}x}y(x)\,dx\right)dx\right)dx$$

Applying this operator to $y'''+\nu y'$, and using `Mathematica`, we obtain:

$$L_1^{-1}(y'''+\nu y')=-f(0)+f(x)-\frac{\sin(x\sqrt{\nu})}{\sqrt{\nu}}f'(0)+\frac{-1+\cos(x\sqrt{\nu})}{\nu}f''(0)$$

Following the same procedure for the inverse operators $L_2^{-1},L_3^{-1},\ldots,L_6^{-1}$, we find that they all yield the same expression:

$$-f(0)+f(x)-\frac{\sin(x\sqrt{\nu})}{\sqrt{\nu}}f'(0)+\frac{-1+\cos(x\sqrt{\nu})}{\nu}f''(0)$$

Thus, all the differential operators $L_1,L_2,\ldots,L_6$ produce the same result.
Hence, the proof is complete.

Application

Example 1 *Equation (6) [38–41] with boundary conditions (7) can be written as*

$$Ly=-2\alpha Ref(x)f'(x), \tag{23}$$

we can get Ly by putting $m=0,n=1$ or $m=1,n=0$ in equation (15), $m=0,n=1$ in equation (16), and $\nu=(4-Ha)\alpha^2$. In this example we used when $m=0,n=1$ in equation (15) that is

$$Ly=e^{\pm i\,\sqrt{(4-Ha)\alpha^2}\,x}\frac{d}{dx}e^{\mp 2i\,\sqrt{(4-Ha)\alpha^2}\,x}\frac{d}{dx}e^{\pm i\,\sqrt{(4-Ha)\alpha^2}\,x}\frac{d}{dx}(y), \tag{24}$$

L^{-1} of equation (24) is

$$L^{-1}(y)=\int_0^x e^{\mp i\,\sqrt{(4-Ha)\alpha^2}\,x}\int_0^x e^{\pm 2i\,\sqrt{(4-Ha)\alpha^2}\,x}\int_0^x e^{\mp i\,\sqrt{(4-Ha)\alpha^2}\,x}(y)dxdxdx, \tag{25}$$

applying inverse equation (25) on equation (23), and using Adomian polynomial in equation (21) of non-linear part $-2\alpha Ref(x)f'(x)$ we get

$$\begin{aligned}f_0(x)&=1+\frac{c}{2}x^2-\frac{c\,\alpha^2}{6}x^4+\frac{cHa\alpha^2}{24}x^4+\frac{c\,\alpha^4}{45}x^6-\frac{cHa\alpha^4}{90}x^6\\&+\frac{cHa^2\alpha^4}{720}x^6-\frac{c\,\alpha^6}{630}x^8+\frac{cHa\alpha^6}{840}x^8-\frac{cHa^2\,\alpha^6}{3360}x^8+\frac{cHa^3\,\alpha^6}{40320}x^8,\end{aligned}$$

$$\begin{aligned}f_1(x)&=\frac{-c\,Re\alpha}{12}x^4-\frac{c^2\,Re\alpha}{120}x^6+\frac{cRe\alpha^3}{45}x^6-\frac{cHa,Re\alpha^3}{180}x^6\\&+\frac{c^2Re\alpha^3}{280}x^8-\frac{c^2\,HaRe\alpha^3}{1120}x^8-\frac{cRe\,\alpha^5}{420}x^8\\&+\frac{cHaRe\alpha^5}{840}x^8-\frac{cHa^2Re\alpha^5}{6720}x^8,\end{aligned}$$

$$\begin{aligned}f_2(x)&=\frac{cRe^2\,\alpha^2}{180}x^6+\frac{c^2\,Re^2\,\alpha^2}{560}x^8-\frac{cRe^2\,\alpha^4}{840}x^8\\&+\frac{cHaRe^2\,\alpha^4}{3360}x^8,\end{aligned}$$

$$\begin{aligned}f_3(x)&=-\frac{cRe^3\alpha^3}{5040}x^8-\frac{c^2\,Re^3\alpha^3}{5600}x^{10}+\frac{cRe^3\alpha^5}{28350}x^{10}\\&-\frac{cHaRe^3\alpha^5}{113400}x^{10},\end{aligned}$$

the series solution is given by

$$f(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) =$$

$$1 + \frac{cx^2}{2} - \frac{1}{12}cRe\alpha x^4 - \frac{1}{120}c^2Re\alpha x^6 - \frac{1}{6}c\alpha^2x^4 + \frac{1}{24}cHa\alpha^2x^4 + \frac{1}{180}cRe^2\alpha^2x^6 + \frac{1}{560}c^2Re^2\alpha^2x^8 + \frac{1}{45}cRe\alpha^3x^6 - \frac{1}{180}cHaRe\alpha^3x^6 + \frac{1}{280}c^2Re\alpha^3x^8 - \frac{c^2HaRe\alpha^3}{1120}x^8 - \frac{cRe^3\alpha^3}{5040}x^8 - \frac{c^2Re^3\alpha^3}{5600}x^{10} + \frac{1}{45}c\alpha^4x^6 - \frac{1}{90}cHa\alpha^4x^6 + \frac{1}{720}cHa^2\alpha^4x^6 - \frac{1}{840}cRe^2\alpha^4x^8 + \frac{cHaRe^2\alpha^4}{3360}x^8 - \frac{1}{420}cRe\alpha^5x^8 + \frac{1}{840}cHaRe\alpha^5x^8 - \frac{cHa^2Re\alpha^5}{6720}x^8 + \frac{cRe^3\alpha^5}{28350}x^{10} - \frac{cHaRe^3\alpha^5}{113400}x^{10} - \frac{1}{630}c\alpha^6x^8 + \frac{1}{840}cHa\alpha^6x^8 - \frac{cHa^2\alpha^6}{3360}x^8 + \frac{cHa^3\alpha^6}{40320}x^8.$$

And the solution by SADM as follows

$$f_0(x) = 1 + \frac{c}{2}x^2,$$

$$f_1(x) = -\frac{1}{12}cRe\alpha x^4 - \frac{1}{6}c\alpha^2x^4 + \frac{1}{24}cHa\alpha^2x^4 - \frac{1}{120}c^2Re\alpha x^6,$$

$$f_2(x) = \frac{1}{180}cRe^2\alpha^2x^6 + \frac{1}{45}cRe\alpha^3x^6 - \frac{1}{180}cHaRe\alpha^3x^6 + \frac{1}{45}c\alpha^4x^6 - \frac{1}{90}cHa\alpha^4x^6 + \frac{1}{720}cHa^2\alpha^4x^6 + \frac{1}{280}c^2Re^2\alpha^2x^8 + \frac{1}{140}c^2Re\alpha^3x^8 - \frac{1}{560}c^2HaRe\alpha^3x^8 + \frac{13}{75600}c^3Re^2\alpha^2x^{10},$$

$$f_3(x) = \frac{1}{180}cRe^2\alpha^2x^6 + \frac{1}{90}cRe\alpha^3x^6 - \frac{1}{360}cHaRe\alpha^3x^6 + \frac{13}{10080}c^2Re^2\alpha^2x^8 + \frac{1}{504}c^2Re\alpha^3x^8 - \frac{1}{2016}c^2HaRe\alpha^3x^8 - \frac{1}{2520}cRe^2\alpha^4x^8 + \frac{1}{10080}cHaRe^2\alpha^4x^8 - \frac{1}{630}cRe\alpha^5x^8 + \frac{1}{1260}cHaRe\alpha^5x^8 - \frac{1}{10080}cHa^2Re\alpha^5x^8 - \frac{1}{630}c\alpha^6x^8 + \frac{1}{840}cHa\alpha^6x^8 - \frac{1}{3360}cHa^2\alpha^6x^8 + \frac{1}{40320}cHa^3\alpha^6x^8 + \frac{1}{14400}c^3Re^2\alpha^2\eta^{10} - \frac{1}{10800}c^2Re^3\alpha^3x^{10} - \frac{1}{1890}c^2Re^2\alpha^4x^{10} + \frac{1}{7560}c^2HaRe^2\alpha^4x^{10} - \frac{13}{18900}c^2Re\alpha^5x^{10} + \frac{13}{37800}c^2HaRe\alpha^5x^{10} - \frac{13}{302400}c^2Ha^2Re\alpha^5x^{10} - \frac{53}{3326400}c^3Re^3\alpha^3x^{12} - \frac{37}{997920}c^3Re^2\alpha^4x^{12} + \frac{37}{3991680}c^3HaRe^2\alpha^4x^{12} - \frac{89}{165110400}c^4Re^3\alpha^3x^{14},$$

the series solution is given by

$$f(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) =$$

$$1 + \frac{c}{2}x^2 - \frac{1}{12}cRe\alpha x^4 - \frac{1}{6}c\alpha^2x^4 + \frac{1}{24}cHa\alpha^2x^4 - \frac{1}{120}c^2Re\alpha x^6 + \frac{1}{90}cRe^2\alpha^2x^6 + \frac{1}{30}cRe\alpha^3x^6 - \frac{1}{120}cHaRe\alpha^3x^6 + \frac{1}{45}c\alpha^4x^6 - \frac{1}{90}cHa\alpha^4x^6 + \frac{1}{720}cHa^2\alpha^4x^6 + \frac{7}{1440}c^2Re^2\alpha^2x^8 + \frac{23}{2520}c^2Re\alpha^3x^8 - \frac{23}{10080}c^2HaRe\alpha^3x^8 - \frac{1}{2520}cRe^2\alpha^4x^8 + \frac{1}{10080}cHaRe^2\alpha^4x^8$$

$$-\frac{1}{630}cRe\alpha^5x^8 + \frac{1}{1260}cHaRe\alpha^5x^8$$

$$-\frac{1}{10080}cHa^2Re\alpha^5x^8 - \frac{1}{630}c\alpha^6x^8 + \frac{1}{840}cHa\alpha^6x^8 - \frac{1}{3360}cHa^2\alpha^6x^8$$

$$+\frac{1}{40320}cHa^3\alpha^6x^8 + \frac{73}{302400}c^3Re^2\alpha^2x^{10} - \frac{1}{10800}c^2Re^3\alpha^3x^{10}$$

$$-\frac{1}{1890}c^2Re^2\alpha^4x^{10} + \frac{1}{7560}c^2HaRe^2\alpha^4x^{10} -$$

$$\frac{13}{18900}c^2Re\alpha^5x^{10} + \frac{13}{37800}c^2HaRe\alpha^5x^{10} - \frac{13}{302400}c^2Ha^2Re\alpha^5x^{10}$$

$$-\frac{53}{3326400}c^3Re^3\alpha^3x^{12} -$$

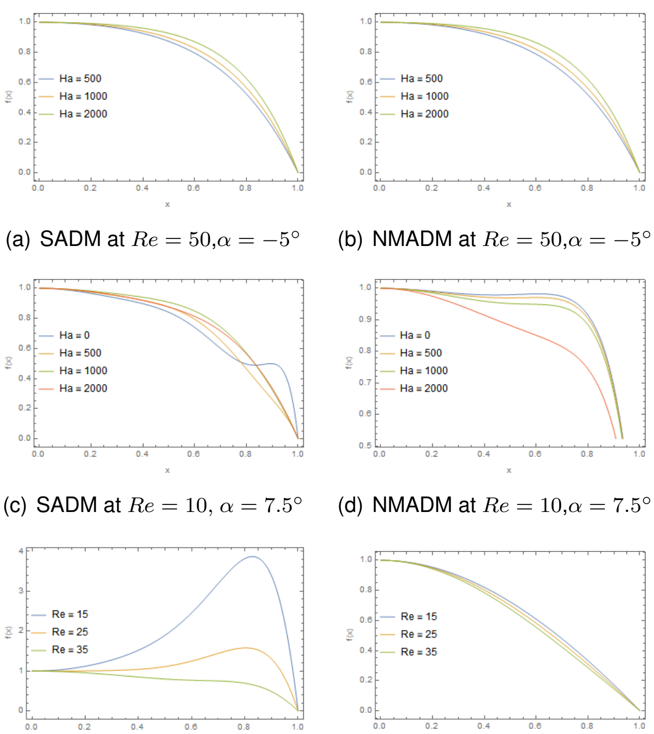
$$\frac{37}{997920}c^3Re^2\alpha^4x^{12} + \frac{37}{3991680}c^3HaRe^2\alpha^4x^{12}$$

$$-\frac{89}{165110400}c^4Re^3\alpha^3x^{14}.$$

Based on the analysis of Figures 3 (a) and (b), which explore the variation of the function (f) for ($\alpha < 0$) under different Hartmann numbers (Ha) with a fixed Reynolds number (Re = 50), the results obtained through NMADM were compared to those derived from the SADM. The comparison indicates a strong agreement between the two methods, with the channel exhibiting a convergent steepness.

In Figures (c) and (d), the focus shifts to the variation of the function (f) for ($\alpha > 0$) with different Hartmann numbers while maintaining a fixed Reynolds number ($Re = 10$). Here, a notable difference between the NMADM and SADM results emerges, highlighting a divergent steepness of the channel.

Additionally, Figures 3 (e) and (f) address the variation of the function (f) for ($\alpha > 0$) across different Reynolds numbers while keeping the Hartmann number fixed at $Ha = 5$. The results indicate a divergence between the NMADM and SADM outcomes, with the steepness of the channel demonstrating divergent behavior in the SADM context, while it remains convergent in the case of NMADM.



(a) SADM at $Re = 50, \alpha = -5°$ (b) NMADM at $Re = 50, \alpha = -5°$

(c) SADM at $Re = 10,\ \alpha = 7.5°$ (d) NMADM at $Re = 10, \alpha = 7.5°$

(e) SADM at $Ha = 50,\ \alpha = 5°$ (f) NMADM at $Ha = 50,\ \alpha = 5°$

Figure 3: NMADM and SADM Solutions for different Hartmann number ($Ha = 0, 50, 500, 1000, 2000$), Reynolds number ($Re = 10, 15, 25, 35, 50$) and($\alpha = -5°, 5°, 7.5°$).

Example 2 *Consider equation (13) [10–12] with $EL = 1, p = -2$, and $q(x) = -8e^x$, that is $\nu = \frac{p}{EL} = -2$ for*

$$y^{(4)} - 2y^{(2)} + y = -8e^x, x \in [0,1], \tag{26}$$

with boundary conditions

$$y(0) = y(1) = 0, y''(0) = 0, y''(1) = -4e. \tag{27}$$

With Exact solution $y(x) = x(1-x)e^x$. Equation (26) in an operator form is

$$Ly = -8e^x - y, \tag{28}$$

where Ly gets when $m = 0, n = 2$ in equation (16)

$$L(y) = e^{\pm i\sqrt{-2}\,x}\frac{d}{dx}e^{\mp i\sqrt{-2}\,x}\frac{d^2}{dx^2}e^{\mp i\sqrt{-2}\,x}\frac{d}{dx}e^{\pm i\sqrt{-2}\,x}(y), \tag{29}$$

L^{-1} of equation (29) is

$$L^{-1}(y) = e^{\mp i\sqrt{-2}\,x}\int_0^x e^{\pm i\sqrt{-2}\,x}\int_0^x\int_0^x e^{\pm i\sqrt{-2}\,x} \int_0^x e^{\mp i\sqrt{p}\,x}(y)dxdxdxdx. \tag{30}$$

Appling equation (30) to (28) equation we have

$$y_0 = a\,x + \frac{b\,x^3}{6} - \frac{x^4}{3} + \left(-\frac{1}{15} + \frac{b}{60}\right)x^5 - \frac{x^6}{30} + \left(-\frac{1}{210} + \frac{b}{1260}\right)x^7 - \frac{x^8}{720} + \left(-\frac{1}{6480} + \frac{b}{45360}\right)x^9 - \frac{x^{10}}{30240},$$

$$y_1 = \frac{-a}{120}x^5 + \left(\frac{-a}{2520} - \frac{b}{5040}\right)x^7 + \frac{x^8}{5040} + \left(\frac{1}{45360} - \frac{a}{90720} - \frac{b}{90720}\right)x^9 + \frac{x^{10}}{90720},$$

$$y_2 = \frac{a}{362880}x^9,$$

$$y_0 + y_1 + y_2 =$$

$$a\,x + \frac{b\,x^3}{6} - \frac{x^4}{3} - \frac{a\,x^5}{120} + \left(-\frac{1}{15} + \frac{b}{60}\right)x^5 - \frac{x^6}{30} + \left(\frac{-a}{2520} - \frac{b}{5040}\right)x^7 + \left(-\frac{1}{210} + \frac{b}{1260}\right)x^7 - \frac{x^8}{840} + \frac{a\,x^9}{362880} + \left(\frac{1}{45360} - \frac{a}{90720} - \frac{b}{90720}\right)x^9 + \left(-\frac{1}{6480} + \frac{b}{45360}\right)x^9 - \frac{x^{10}}{45360}.$$

Using boundary conditions in equation (27) in the above series solution we get $a = 1$, and $b = -3$.

$$y_0 + y_1 + y_2 = x - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360}.$$

The Solution by SADM

$$y_0 = a\,x + \frac{b\,x^3}{6} - \frac{x^4}{3} - \frac{x^5}{15} - \frac{x^6}{90} - \frac{x^7}{630} - \frac{x^8}{5040} - \frac{x^9}{45360} - \frac{x^{10}}{453600},$$

$$y_1 = \left(\frac{-a}{120} + \frac{b}{60}\right)x^5 - \frac{x^6}{45} + \left(-\frac{1}{315} - \frac{b}{5040}\right)x^7 - \frac{x^8}{5040} - \frac{x^9}{45360} - \frac{x^{10}}{453600},$$

$$y_2 = \left(\frac{-a}{2520} + \frac{b}{1260}\right)x^7 - \frac{x^8}{1260} + \left(-\frac{1}{11340} + \frac{a}{362880} - \frac{b}{90720}\right)x^9,$$

$$y_0 + y_1 + y_2 = a\,x + \frac{b\,x^3}{6} - \frac{x^4}{3} - \frac{x^5}{15} + \left(\frac{-a}{120} + \frac{b}{60}\right)x^5 - \frac{x^6}{30} - \frac{x^7}{630} + \left(-\frac{1}{315} - \frac{b}{5040}\right)x^7 + \left(\frac{-a}{2520} + \frac{b}{1260}\right)x^7 - \frac{x^8}{840} - \frac{x^9}{22680} + \left(-\frac{1}{11340} + \frac{a}{362880} - \frac{b}{90720}\right)x^9 - \frac{x^{10}}{226800}.$$

Using boundary conditions in equation (27) in the above series solution we get $a = 1$, and $b = -3$.

$$y_0 + y_1 + y_2 = x - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{10368} - \frac{x^{10}}{226800}.$$

The following is the Taylor series for the exact solution

$$x - \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{8} - \frac{x^6}{30} - \frac{x^7}{144} - \frac{x^8}{840} - \frac{x^9}{5760} - \frac{x^{10}}{45360} + O\left(x^{11}\right).$$

To clarify the convergence between the exact solution and the solutions obtained using the SADM and the NMADM in this example, we utilize a tenth-order Taylor series expansion. We find that the solution provided by the NMADM is identical to the exact solution and is closer to the exact solution than that of the SADM.

Example 3 *Consider a special sixth-order boundary value problem with parameter c [42–44]*

$$y^{(6)}(x) - (1+c)y^{(4)}(x) = cx - cy^{(2)}(x), \tag{31}$$

with boundary conditions

$$y(0) = y'(0) = 1, y''(0) = 0,$$
$$y(1) = \frac{7}{6} + sinh(1), y'(1) = \frac{1}{2}cosh(1), y''(1) = 1 + sinh(1). \tag{32}$$

$y(x) = 1 + \frac{1}{6}x^3 + sinh(x)$ *is the exact solution. Equation (31) in an operator form is*

$$Ly = cx - cy^{(2)}(x), \tag{33}$$

L *obtains when* $m = 3$, $n = 1$ *and* $\nu = -(1+c)$ *in equation (15)*

$$Ly = \frac{d^3}{dx^3} e^{\pm i\sqrt{-(1+c)}\,x} \frac{d}{dx} e^{\mp 2i\sqrt{-(1+c)}\,x} \frac{d}{dx} e^{\pm i\sqrt{-(1+c)}\,x} \frac{d}{dx}(y), \tag{34}$$

and L^{-1} *of equation (34)*

$$L^{-1} = \int_0^x e^{\mp i\sqrt{-(1+c)}\,x} \int_0^x e^{\pm 2i\sqrt{-(1+c)}\,x} \int_0^x e^{\mp i\sqrt{-(1+c)}\,x} \int_0^x \int_0^x \int_0^x dxdxdxdxdxdx(y). \tag{35}$$

Applying (35) on (33) we get

$$y_0 = 1 + x + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{24} + \frac{x^5 a_3}{120} + \frac{x^6 a_2}{720} + \frac{cx^6 a_2}{720} + \frac{cx^7}{5040} + \frac{x^7 a_3}{5040} + \frac{cx^7 a_3}{5040} + \frac{x^8 a_2}{40320} + \frac{cx^8 a_2}{20160} + \frac{c^2 x^8 a_2}{40320} + \frac{cx^9}{362880} + \frac{x^9 a_3}{362880} + \frac{cx^9 a_3}{181440} + \frac{c^2 x^9 a_3}{362880} + \frac{c^2 x^9}{362880} + \frac{x^{10} a_2}{3628800} + \frac{cx^{10} a_2}{1209600} + \frac{c^2 x^{10} a_2}{1209600} + \frac{c^3 x^{10} a_2}{3628800},$$

$$y_1 = -\frac{cx^7 a_1}{5040} - \frac{cx^8 a_2}{40320} - \frac{cx^9 a_1}{362880} - \frac{c^2 x^9 a_1}{362880} - \frac{cx^9 a_3}{362880} - \frac{cx^{10} a_2}{1814400} - \frac{c^2 x^{10} a_2}{1814400},$$

the series solution by NMADM

$$y(x) = y_0 + y_1$$
$$= 1 + x + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{24} + \frac{x^5 a_3}{120} + \frac{x^6 a_2}{720} + \frac{cx^6 a_2}{720} + \frac{cx^7}{5040} - \frac{cx^7 a_1}{5040} + \frac{x^7 a_3}{5040} + \frac{cx^7 a_3}{5040} + \frac{x^8 a_2}{40320} + \frac{cx^8 a_2}{40320} + \frac{c^2 x^8 a_2}{40320} + \frac{cx^9}{362880} + \frac{c^2 x^9}{362880} - \frac{cx^9 a_1}{362880} - \frac{c^2 x^9 a_1}{362880} + \frac{x^9 a_3}{362880} + \frac{cx^9 a_3}{362880} + \frac{c^2 x^9 a_3}{362880} + \frac{x^{10} a_2}{3628800} + \frac{cx^{10} a_2}{3628800} + \frac{c^2 x^{10} a_2}{3628800} + \frac{c^3 x^{10} a_2}{3628800}.$$

And the solution in SADM

$$y_0 = 1 + x + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{24} + \frac{x^5 a_3}{120} + \frac{cx^7}{5040},$$

$$y_1 = \frac{x^6 a_2}{720} + \frac{cx^6 a_2}{720} - \frac{cx^7 a_1}{5040} + \frac{x^7 a_3}{5040} + \frac{cx^7 a_3}{5040} + \frac{cx^9}{362880} + \frac{c^2 x^9}{362880} - \frac{cx^8 a_2}{40320} - \frac{cx^9 a_3}{362880} - \frac{c^2 x^{11}}{39916800},$$

$$y_0 + y_1 = 1 + x + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{24} + \frac{x^5 a_3}{120} + \frac{x^6 a_2}{720} + \frac{cx^6 a_2}{720} + \frac{cx^7}{5040} - \frac{cx^7 a_1}{5040} + \frac{cx^7 a_3}{5040} - \frac{cx^8 a_2}{40320} + \frac{cx^9}{362880} + \frac{c^2 x^9}{362880} - \frac{cx^9 a_3}{362880} - \frac{c^2 x^{11}}{39916800}.$$

The value of constants a_1, a_2 *and* a_3 *can be get by using boundary conditions in (32).*

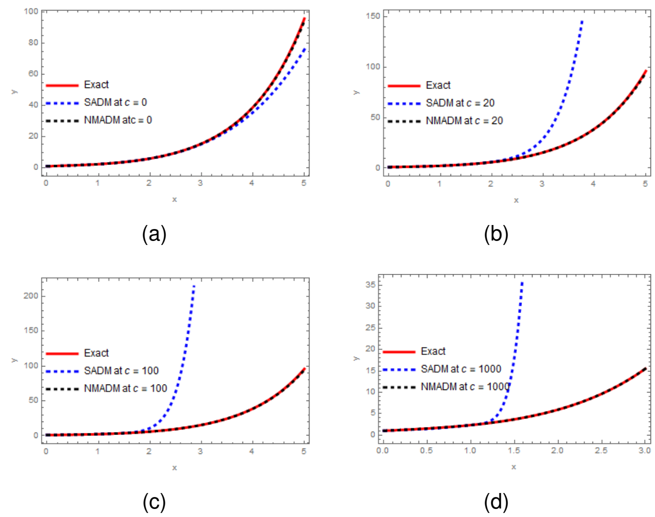


Figure 4: Comparing between Exact, SADM and NMADM solution when $y(x) = y_0 + y_1$) for different value of $c = 0, 20, 100, 1000$

The performance of the new modified Adomian decomposition method (NMADM) and the Standard Adomian Decomposition Method (SADM) reveals significant differences as the parameter (c) increases. From our analysis, particularly illustrated in Figure 4, it is evident that NMADM consistently provides solutions that closely approximate the exact solution, irrespective of how large the parameter (c) becomes. This robustness indicates the effectiveness of NMADM in handling higher values of (c) without significant deviation from the true solution. In stark contrast, the SADM exhibits a noticeable divergence from the exact solution as the parameter (c) increases. This indicates that while SADM may perform adequately for lower values of (c), its reliability diminishes significantly as the parameter grows. Furthermore, our findings align with previous literature [43], which has stated that the approximate solutions derived from the Adomian Decomposition Method are quite accurate for small values of (c) only. As (c) increases, the accuracy of the SADM solutions deteriorates, highlighting a limitation in this method, which NMADM effectively overcomes.

Remark 4 *While the focus of this study has been on higher-order ordinary differential equations, the structure of the modified operator introduced in the NMADM suggests potential applicability to certain classes of partial differential equations (PDEs). In particular, extension to nonlinear diffusionreaction PDEs may be feasible, given the operator's capability to handle nonlinearity and higher-order derivatives. This opens a possible direction for future investigation.*

Conclusion

The study presented a new modified Adomian decomposition method (NMADM) tailored for solving higher-order ordinary differential equations, particularly in the context of magnetohydrodynamic (MHD) flows, elastic beam dynamics, and sixth-order boundary value problems in physics. The key findings of this research indicate that the modified approach demonstrates significant advantages over traditional methods in terms of convergence speed and accuracy. By applying the NMADM, the researchers were able to achieve rapid convergence for various complex problems, providing explicit analytical solutions that can effectively describe physical phenomena in MHD flows and the behavior of elastic beams. The method's versatility also extends to accommodating sixth-order boundary value problems, which are often encountered in engineering and applied physics.

In addition to the analytical benefits, the structure of NMADM suggests favorable scalability to more complex or higher-dimensional problems. The incorporation of weighted exponential operators enhances the convergence behavior, which may reduce computational costs in multidimensional applications. Furthermore, its semi-analytical nature avoids the intensive discretization required by many numerical methods. Future studies could explore its integration with domain decomposition or spectral methods to improve performance in large-scale simulations. The results validate the efficiency of the modified Adomian decomposition method as a robust computational tool, suggesting its potential for broader applications in tackling nonlinear dynamic systems and enhancing the understanding of diverse physical mechanisms. Further studies could explore its applicability to other types of differential equations and complex physical models, paving the way for advancements in both theoretical and practical realms of physics.

Ethics approval and consent to participate

Not applicable

Consent for publication

Not applicable

Availability of data and materials

The raw data required to reproduce these findings are available in the body and illustrations of this manuscript.

Author's contribution

The authors confirm contribution to the paper as follows: Study conception and design: Nuha Mohammed Dabwan; Theoretical calculations and modeling: Nuha Mohammed Dabwan; Data analysis and validation: Nuha Mohammed Dabwan, Yahya Qaid Hasan; Draft manuscript preparation: Nuha Mohammed Dabwan. Both authors reviewed the results and approved the final version of the manuscript.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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Feature	NMADM	Duan-Rach Modification [36]	Wazwaz Modification [37]
Linear Operator (L)	Redefined linear operator $L(y) = y^{(r+2)} + \nu y^{(r)}$, reformulated using integral-exponential operators for improved spectral condition and stability.	Traditional operator $L(y) = y^{(r+2)}$ with adjustments for boundary conditions.	Second-order operator designed to simplify nonlinear expressions.
Key Innovation	Redesign of the operator structure to enhance spectral properties and convergence in stiff or oscillatory problems.	Improved inversion matching specific boundary conditions.	Utilization of Adomian polynomials for nonlinear term approximation via source term partitioning.
Convergence Enhancement	Faster convergence due to better operator conditioning and exponential-integral formulation.	Improved convergence for problems with structured boundaries.	Improved convergence by matching structure of nonlinear terms.
Typical Applications	Stiff and oscillatory nonlinear differential equations.	Nonlinear boundary value problems and weak nonlinear systems.	PDEs and solitary wave solutions.
Distinct Advantage	Improved conditioning and accelerated convergence.	Flexible inversion options.	Significant reduction in computational work and fast convergence to the exact solution.
Proposed Improvement	Customized operator based on the problem's structure, controlling higher-order term growth.	Focus on operator inversion only, no structural changes to the equation.	Alteration of nonlinear terms' form, base equation remains unchanged.

Table 1: Comparison between NMADM and other Modified ADM Variants

	$c = f''(0)$			
	$\alpha = -5^\circ$		$\alpha = 5^\circ$	
Ha (Hartmann number)	NMADM	SADM	NMADM	SADM
0	-1.78545	-1.783225	-2.25333	-2.25429
200	-1.57552	-1.572131	-1.98479	-1.98461
400	-1.39632	-1.38969	-1.75465	-1.75529
600	-1.23815	-1.23185	-1.55609	-1.55813
800	-1.10261	-1.09500	-1.38393	-1.38726
1000	-0.98481	-0.976044	-1.23409	-1.23828
2000	-0.583545	-187.87	-1.72186	-0.72328

Table 2: **(Example 1)** $c = f''(0)$ at different Ha, Re=10, $\alpha = 5^\circ$ and $\alpha = -5^\circ$.

	SADM			NMADM		
	c=0	c=20	c=100	c=0	c=20	c=100
a_1	2.00411	2.00293	1.84544	2.00000313	2.0000061	1.99999991
a_2	-0.0441241	0.237589	1.01826	-0.00000288	-0.0006482	0.0000002
a_3	1.16996	-0.12382	-5.70828	1.00009	1.00033	0.999997

Table 3: **(Example 3)** Comparing value of $y(1) = a_1$, $y'(1) = a_2$ and $y''(1) = a_3$ at different c using SADM and NMADM when $y(x) = y_0 + y_1$