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On the Numerical Treatment of Heat Conduction Problem by Boundary Element and Multigrid Methods

المعالجة العددية لمشكلة التوصيل الحراري بواسطة طريقة العناصر الحدية و الطرق متعددة المراحل

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Abstract

In this work we consider the boundary integral equation describing the steady state heat conduction taking place in three dimensional enclosure geometries. For the numerical realization of the Fredholm integral equation, we use the boundary element method based on the Galerkin weighted residuals method. Consequently, converting the original integral equation into a set of algebraic equations. We apply the multigrid iterations to solve the system of linear equations. To demonstrate the efficiency of this iterative scheme, we construct numerical example.

Keywords: Heat conduction, Fredholm integral equation, Boundary element method, Multigrid iterations.

ملخص

في هذا العمل ، نعتبر المعادلة التكاملية الحدودية التي تصف التوصيل الحراري الثابت للدولة الذي يحدث في هندسيات الضميمة ثلاثية الأبعاد. للتحقيق العددي لمعادلة فريدهولم التكاملية ، نستخدم طريقة العناصر الحدودية على أساس طريقة المتبقيات المرجحة لغالركن. وبالتالي ، تحويل المعادلة التكاملية الأصلية إلى مجموعة من

المعادلات الجبرية. نطبق التكرار المتعدد لحل نظام المعادلات الخطية. لشرح كفاءة هذا المخطط التكراري ، نبني مثالاً عددياً.

الكلمات المفتاحية : التوصيل الحراري ، معادلة فريدهولم التكاملية ، طريقة العناصر الحدودية ، التكرارات متعددة المراحل.

1 Introduction

The steady state heat conduction taking place in an enclosure $\Omega \in \mathbb{R}^3$ with boundary Γ (without the presence of internal heat source) can be described by the following boundary integral equation (see (R. A. Bialecki, 1993))

$$0.5T(\vec{p}) = \int_{\Gamma} [\overset{*}{T}(\vec{r},\vec{p})q(\vec{r}) - \overset{*}{q}(\vec{r},\vec{p})T(\vec{r})]d\Gamma(\vec{r}), \qquad \vec{r},\vec{p} \in \Gamma \tag{1.1}$$

with the temperature at internal points is expressed in terms of the boundary temperatures and boundary fluxes. r and p stand for the current and source points respectively. The temperature field $\overset{*}{T}(\vec{r},\vec{p})$ and the heat flux $\overset{*}{q}(\vec{r},\vec{p})$ known as kernels of the integral equation are given through

$$\overset{*}{T}(\vec{r},\vec{p}) = \frac{1}{4\pi}.\frac{1}{\left|\vec{r}-\vec{p}\right|} \tag{1.2}$$

$$\overset{*}{q}(\vec{r},\vec{p}) = \frac{1}{4\pi}.\frac{(\vec{p}-\vec{r})}{\left|\vec{p}-\vec{r}\right|} \tag{1.3}$$

In addition to some previous work (Atkinson, 2000; Atkinson & Chandler, 1998; R. A. Bialecki, 1993; Jamalabadi, Ghassemi, & Hamedi, 2013; Laitinen & Tiihonen, 2001; N. Qatanani & Schulz, 2004, 2006) involving the heat radiation integral equation we are aware of some other work (Alzeer & Qatanani, 2008; Amiri, Mansouri, & Coelho, 2012; R. Bialecki,

1992; Campo, J. Salazar, J. Celentano, & Raydan, 2014; Jamalabadi et al., 2013; Laitinen & Tiihonen, 2001; N. Qatanani, Barham, & Heeh, 2007; Yousefi, Barikbin, & Dehghan, 2012) on heat conduction together with other heat transfer modes. Our main concern in this work is to focus on the numerical solution of the boundary integral equation (1.1). This will be achieved by introducing the boundary element method based on the Galerkin weighted residual method that will convert the original boundary integral equation (1.1) to a system of linear equations. This linear system will be solved iteratively using multigrid methods. In fact, multigrid methods are among the most efficient methods for solving the linear system associated with the numerical solution of the integral equation. The characteristic feature of the multigrid method is its fast convergence in comparison to other iterative methods. Numerical example is considered to demonstrate the high performance of these iterations.

2 Numerical realization of (1.1)

2.1 Construction of the system of equations

For the numerical realization of equation (1.1) we use the boundary element method based on the Galerkin weighted residuals method. The temperature T and the heat flux q within each boundary element are approximated as follows:

$$T(r) = \sum_{i=1}^{N} T_i \varphi_i(r) \tag{2.1}$$

$$q(r) = \sum_{i=1}^{N} q_i \varphi_i(r) \tag{2.2}$$

where T_i and q_i are the values of the nodal point r_i of the temperature and the heat flux respectively. φ_i are basis functions of the nodal point r_i. Inserting (2.1) and (2.2) into equation (1.1) and collocation at one nodal point p_k yields a linear equation linking nodal temperatures and heat fluxes (R. A. Bialecki, 1993),

$$\sum_{j=1}^{N} H_{kj} T_j = \sum_{j=1}^{N} G_{kj} q_j \tag{2.3}$$

where

$$H_{kj} = \sum_{j=1}^{N} (\int_{\Gamma} \overset{*}{q}(r, p_k)\varphi_j(r)d\Gamma(r) + 0.5\delta_{kj}) \tag{2.4}$$

and

$$G_{kj} = \sum_{j=1}^{N} (\int_{\Gamma} \overset{*}{T}(r, p_k)\varphi_j(r)d\Gamma(r)) \tag{2.5}$$

Equation (2.3) can be written for a sequence of collocation points $p_k, k = 1,2,...,N$ yields a set of linear equations

$$HT = Gq \tag{2.6}$$

The vectors T and q contain values of temperatures and heat fluxes of collocation points. H is called the temperature influence matrix and G is called the heat flux influence matrix. The above integrals in (2.4) and (2.5) can be evaluated numerically using numerical quadrature (R. A. Bialecki, 1993; N. Qatanani & Schulz, 2004, 2006). This can be accomplished by transforming the integral over an arbitrary shaped boundary element into an integral over a unit square. Once the transformation is carried out the value of the appropriate integral is computed using numerical quadratures. Using shape functions ψ_i, one can approximate the global coordinate of the current point r laying within a boundary element as follows :

$$x = \sum_{i=1}^{L} x_i \psi_i(\xi, \zeta) \tag{2.7}$$

$$y = \sum_{i=1}^{L} y_i \psi_i(\xi, \zeta) \tag{2.8}$$

$$z = \sum_{i=1}^{L} z_i \psi_i(\xi, \zeta) \tag{2.9}$$

where x_i, y_i and z_i are the Cartesian coordinates of the nodal points defining the geometry of the element. By virtue of (2.7)-(2.9), it can be

readily shown (R. A. Bialecki, 1993) that the integrals in (2.4) and (2.5) take the following form :

$$H_{kj} = \int_{-1}^{1}\int_{-1}^{1} \overset{*}{q}\,[r(\xi,\zeta),p_k]\varphi_j(\xi,\zeta)\,|N_r(\xi,\zeta)|\,d\xi d\zeta + 0.5\delta_{kj} \qquad (2.10)$$

and

$$G_{kj} = \int_{-1}^{1}\int_{-1}^{1} \overset{*}{T}\,[r(\xi,\zeta),p_k]\varphi_j(\xi,\zeta)\,|N_r(\xi,\zeta)|\,d\xi d\zeta. \qquad (2.11)$$

To solve equation (2.6) the values of the temperatures and heat fluxes prescribed as

$$T(r) = \widetilde{T} \qquad \text{and} \qquad q(r) = \widetilde{q} \qquad (2.12)$$

are inserted into equation (2.6). Consequently, we arrive at the set of linear equations having the form

$$Au = f \qquad (2.13)$$

where the entries of matrix A are defined as

$$A = \begin{cases} H_{k_j} & \text{if at point } r_j \text{ the heat flux is known} \\ -G_{k_j} & \text{if at point } r_j \text{ the temperature is known} \end{cases} \qquad (2.14)$$

the coefficients of the vector of unknowns u are

$$u = \begin{cases} q_k & \text{if at point } r_j \text{ the temperature is given} \\ T_k & \text{if at point } r_j \text{ the heat flux is given} \end{cases} \qquad (2.15)$$

and the coefficients of the right-hand vectors are computed as

$$f_k = -\sum_{i_T} H_{ki_T}\widetilde{T}_{i_T} + \sum_{i_q} G_{ki_q}\widetilde{q}_{i_q} \tag{2.16}$$

where $\widetilde{T}_{i_T}$ and $\widetilde{q}_{i_q}$ denote the prescribed values of temperature and heat flux respectively.

2.2 Iterative methods for (2.13)

Iterative methods for solving this problem are formulated as follows:

$$u^{(i+1)} = Mu^{(i)} + Nf \tag{2.17}$$

where M and N are constructed in such a way that given an arbitrary initial vector $u^{(0)}$, the sequences $u^{(i)}$, $i = 0,1,...,$ converges to the solution $u = A^{-1}f$. This method is called the Picard iteration. The method (2.17) converges if and only if $\rho(M) < 1$.

2.2.1 Two-grid scheme

The two-grid scheme starts at the line level with pre-smoothing, performs a coarse-grid correction to solve the coarse-grid auxiliary problem, and ends with post-smoothing. A pictorial representation of this process where " fine " is a high level and " coarse" is a low level looks like a " V " workflow. This is called " V " cycle. To solve the problem to a given tolerance, we have to apply the two-grid V-cycle iteratively. To emphasize that the iteration (2.17) is a smoothing procedure, we denote it by

$$u_h^{(\ell)} = S_h(u_h^{(\ell-1)}, f_h) \tag{2.18}$$

where we use S to denote the iteration matrix M. We summarize the two-grid (TG) procedure with the following algorithm 2.1 (Borzì, 2003; Hackbusch, 1995):

Lemma 2.1. *(Borzì, 2003) The iteration matrix of the two grid scheme is*

$$M_{TG} = S_h^{\nu_2}(I_h - I_H^h(A_H)^{-1}I_h^H A_h)S_h^{\nu_1} \tag{2.19}$$

Algorithm (2.1): TG scheme

1. Pre-smoothing steps on the fine grid:

$$u_h^{(\ell)} = S_h(u_h^{(\ell-1)}, f_h), \ell = 1, ..., \nu_1$$

2. Computation of the residual :

$$r_h = f_h - A_h u_h^{(\nu_1)}$$

3. Restriction of the residual :

$$r_H = I_h^H r_h$$

4. Solution of the coarse-grid problem :

$$e_H = (A_H)^{-1} r_H$$

5. Coarse-grid correction :

$$u_h^{(\nu_1+1)} = u^{(\nu_1)} + I_H^h e_H$$

6. Post-smoothing steps on the fine grid :

$$u_h^{(\ell)} = S(u_h^{(\ell-1)}, f_h), \ell = \nu_1 + 2, ..., \nu_1 + \nu_2 + 1.$$

where I_h^H is the restriction matrix and I_H^h is the prolongation matrix. The H and h are used for the coarse grid size and fine grid size respectively with $H = 2h$.

where I_h is the identity and S_h is the smoothing iteration matrix.

Theorem 2.1. *(Borzì, 2003) Let the two-grid scheme algorithm 2.1 with $\nu = \nu_1 + \nu_2 \geq 1$.The spectral radius of the iteration matrix M_{TG} given by (2.19) is bounded by*

$$\rho(M_{TG}) \leq max\left\{\chi(1-\chi)^{\nu} + (1-\chi)\chi^{\nu} : 0 \leq \chi \leq \frac{1}{2}\right\} = \rho_{\nu} < 1$$

uniformly with respect to mesh size H. Hence (2.19) is a convergent iteration.

2.2.2 Multigrid scheme

In the two-grid scheme the size of the coarse grid is twice larger than the fine one, thus the coarse problem may be very large. However, the coarse problem has the same form as the residual problem on the fine level.Therefore one can use the two-grid method to determine $\widetilde{e}_H$. This process can be repeated recursively until a coarsest grid is reached where the corresponding residual equation is inexpensive to solve. For a more detailed description, we introduce a sequence of grids with mesh size $h_1 > h_2 > ... > h_L > 0$ so that $h_{k-1} = 2h_k$. Here $k = 1,2,...,L$, is called the level or level number. On each level k we define the problem $A_k u_k = f_k$ where A_k is a $n_k \times n_k$ matrix and u_k and f_k are vectors of size n_k. The multigrid scheme can be illustrated by the following algorithm 2.2 (Borzì, 2003; Hackbusch, 1995).

The multigrid algorithm involves a new parameter (cycle index) γ which is the number of times the MG procedure is applied to the coarse level problem. Since this procedure converges very fast, $\gamma = 1$ or $\gamma = 2$ are the typical values used.

Lemma 2.2. *(Borzì, 2003) The iteration matrix of the multigrid is given in recursive form by the following*
For $k = 1$ let $M_1 = 0$. For $k = 2,...,L$:

$$M_k^{MG} = S_k^{\nu_2}(I_k - I_{k-1}^k(I_{k-1} - M_{k-1}^{\gamma})(A_k - 1)^{-1} I_k^{k-1} A_k) S_k^{\nu_1}$$

Algorithm (2.2): MG scheme

1. If $k = 1$ solve $A_k u_k = f_k$ directly
2. Pre-smoothing steps on the fine grid :

$$u_k^{(\ell)} = S(u_k^{(\ell-1)}, f_k), \ell = 1, ..., \nu_1$$

3. Computation of the residual :

$$r_k = f_k - A_k u_k^{(\nu_1)}$$

4. Restriction of the residual :

$$r_{k-1} = I_k^{k-1} r_k$$

5. Set

$$u_{k-1} = 0$$

6. Call γ times the MG scheme to solve

$$A_{k-1} u_{k-1} = r_{k-1}$$

7. Coarse-grid correction :

$$u_k^{(\nu_1+1)} = u_k^{(\nu_1)} + I_{k-1}^k u_{k-1}$$

8. Post-smoothing steps on the fine grid :

$$u_k^{(\ell)} = S(u_k^{(\ell-1)}, f_k), \ell = \nu_1 + 2, ..., \nu_1 + \nu_2 + 1.$$

where I_k is the identity, S_k is the smoothing iteration matrix and M_k is the multigrid iteration matrix for the level k.

A sufficient convergence condition is the matrix norm estimate $||M_k^{MG}|| < 1$.

Table (3.1): The exact and numerical solutions of the primary velocity U.

	Two-grid Scheme		Multigrid Scheme	
n_k	No. of iteration steps	second	No. of iteration steps	second
32	5	< 1	3	< 1
64	5	< 1	3	< 1
128	5	< 1	3	< 1
256	5	2.01	3	1.02
512	5	8.02	3	4.2
1024	5	31.08	3	16.4

3 Numerical example and results

Since the convergence requirements (regularity , consistency and stability) for the two-grid and multigrid iterations are satisfied (N. A. Qatanani, 2003), we can now apply these algorithms to solve the linear system (2.13). For the numerical applications, we consider a cylindrical enclosure geometry whose boundary Γ has the following parametric representation:

$$\Gamma = \left\{ \vec{r} \in \mathbb{R}^3, \vec{r} = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \\ 4z \end{pmatrix}, (t,z) \in (0,1)^2 \right\}$$

The entries of the matrix A as well as the entries of the vector f have been computed numerically. To keep the numerical integration error small, we handle the singularity of the integral kernels by employing double partial derivatives, see (Hansen, 2003). Note that the step size h_k is associated with the dimension parameter n_k, where $h_k = \frac{1}{n_k}$ with $n_k = 2^k$ and k is called the level number. Table (3.1) shows the numerical results for this case. It contains both the number of iteration steps and the CPU-time in seconds required by each iteration.

4 Conclusions

The numerical results for the two-grid and multigrid iterations shown in Table(3.1) illustrate clearly that both these schemes require less number

of iteration steps and CPU-time in comparison to other iterations (Alzeer & Qatanani, 2008; N. Qatanani & Schulz, 2004). This demonstrates that one of the characteristic features of the multigrid schemes is its fast convergence. The convergence speed does not deteriorate when the democratization is refined, where as other classical iterative methods slow down for decreasing grid size. As consequence one obtains an acceptable approximation of the discrete problem at the expense of the computational work proportional to the number of unknowns, which is also the number of equations of the system. It is not only complexity which is optimal, also the constant of proportionality is so small that other methods can hardly surpass the multigrid efficiency.I

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