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Types of subsequences of double sequences and their convergence properties

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Abstract: In this paper, we give some new formulas for the subsequences of double sequences by using matrices to represent these formulas. Also, we study some of their boundedness and convergence properties[2].

Keywords: Double Sequence; Subsequence; Matrix Form; Convergent, Bounded.

Introduction

In 1900, Pringsheim (Pringsheim,1900) introduced the concept of convergence of real double sequences. He stated that a double sequence $\{a_{nm}\}_{n,m=1}^{\infty}$ converges to $a \in \mathbb{R}$, if for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $|a_{m,n} - a| < \varepsilon$ for all $m, n > k$, see (Dumitru & Franco,2019). The limit a is called the Pringsheim limit of $\{a_{nm}\}_{n,m=1}^{\infty}$ and denotaed by $P - \lim a_{m,n} = a$.

If $P - \lim|a_{mn}| = \infty$, (equivalently, for every $M > 0$ there are $k_1, k_2 \in \mathbb{N}$ such that $|a_{mn}| > M$ whenever $m \geq k_1, n \geq k_2$), then (a_{mn}) is said to be definitely divergent.

Also, a double sequence (a_{mn}) is bounded if there is $M > 0$ such that $|a_{mn}| < M$ for all $m, n \in \mathbb{N}$. In contrast to the convergence for single sequences a P-convergent double sequence need not be bounded.

After 4 years, in 1904, Hardy (Hardy, 1904) introduced the notion of regular convergence for double sequences. He defined that a double sequence (a_{mn}) regularly converges to a point a if it has a limit in Pringsheim's sense and for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ there exist the following two limits:

$$\begin{aligned} \lim_{n\to\infty} a_{mn} &= R_m, \\ \lim_{m\to\infty} a_{mn} &= C_n. \end{aligned}$$

In 2000 , the notion of a subsequence of a double sequence was introduced by Patterson, see (Patterson, 2000), he stated that the double sequence B is a double subsequence of A provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $a_j = a_{n_j,k_j}$, then B is formed by

$$\begin{pmatrix} a_1 & a_2 & a_5 & a_{10} & \cdots \\ a_4 & a_3 & a_6 & \cdots & \cdots \\ a_9 & a_8 & a_7 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Later in 2019, Dumitru and Franco (Dumitru & Franco, 2019) used the definition of Patterson above to introduce another definition of a subsequence of double sequence named as β-subsequences of a double sequence and defined it in the following manner: If $A = [a_{k,l}]$ be a double sequence and let $\beta > 1$ be an extended real and Let $\psi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given in the following way

$$\begin{matrix} (1,1) \mapsto 1 & (1,2) \mapsto 2 & (2,2) \mapsto 3 & (2,1) \mapsto 4 \\ (1,3) \mapsto 5 & (2,3) \mapsto 6 & (3,3) \mapsto 7 & (3,2) \mapsto 8 \ldots \end{matrix}$$

The double sequence $y^{(\pi,\beta)}$ is called a β-subsequence of the double sequence A if and only if there exists a strictly increasing function $\pi: \psi(S_\beta) \to \psi(S_\beta)$ such that

$$y_{p,q}^{(\pi,\beta)} = \begin{cases} z_{\psi(p,q)}, & \text{if } \dfrac{1}{\beta} > \dfrac{p}{q} \text{ or } \dfrac{p}{q} > \beta, \\ z_{\pi(\psi(p,q))}, & \text{if } \dfrac{1}{\beta} \leqslant \dfrac{p}{q} \leqslant \beta, \end{cases}$$

where $z_i = a_{\psi^{-1}(i)}$

In recent years, several papers had appeared which study double subsequences from various points of view. See (Habil,2006), (Patterson, 2013), (Patterson & Cakalli, 2013), (Mursaleen,2014), and (Dumitru & Franco, 2019).

However, these definitions impose restrictive conditions on the entries of the double sequence that can be considered for the subsequence of double sequences. The aim of this paper is to introduce less restrictive new definitions of a subsequence of double sequence. We establish the definitions of single subsequences of double sequences, double subsequences of double sequences, Block subsequence of double sequences, upper triangle subsequence and lower triangle subsequence of double sequences, followed by studying their convergence properties.

Matrix representation of the double sequence

One of the most important points that must be discussed and presented before defining the subsequence of the double sequence is how to express the elements of the double sequence and how to arrange its elements. In this part of the paper, we will focus on three types of double sequences, we will present the features of these types and the most important limits that can be found for these double sequence and the consequences of that, for this we will use Matrix representation. It is known that a Matrix transformations for double sequences are considered by various authors, (see (Hamilton,1936),(Mursaleen &Mohiuddine, 2009) and (Robison, 1926).

The first type is the double sequence $\{a_{nm}\}_{n,m=1}^{\infty}$. We can write the entries of this sequence by using a matrix and any entry (a_{nm}) set, in the position (n, m), in the following form

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$$a_{nm}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Using matrices, it becomes clear to us that we can calculate the limits either for rows or columns or for both together. In other words, we can calculate the limits

$$\lim_{n,m\to\infty} a_{nm},\ \lim_{n\to\infty} a_{nm} \text{ and } \lim_{m\to\infty} a_{nm}$$

Thus, we can also calculate the iterated limits

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{nm} \text{ and } \lim_{n\to\infty}\lim_{m\to\infty} a_{nm}$$

The second type of double sequences is the double sequence of the form $\{a_{nm}\}_{n\geq m}^{\infty}$. We can write the entries of this sequences by using lower triangle matrix with no elements in the upper triangle entries. For this type, the matrix representation of the double sequence has the form

$$a_{nm}=\begin{pmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ a_{41} & a_{42} & a_{43} & a_{44} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In contrast to the first type, we find that the rows in this type contain a limited number of elements, may be the number of elements is large, but it remains limited, so this prevents the possibility of calculating the limit for the rows $\lim_{n\to\infty} a_{nm}$. Thus we can only calculate the limits

$$\lim_{n,m\to\infty} a_{nm} \text{ and } \lim_{m\to\infty} a_{nm}$$

So, we can only calculate the iterated limit

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{nm}$$

The third type of double sequences is the double sequence of the form $\{a_{nm}\}_{n\leq m}^{\infty}$. For this type, we use the upper triangle matrix with no elements in the lower triangle entries, and the matrix representation of the double sequence has the form

$$a_{nm}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\ & & a_{33} & a_{34} & a_{35} & \cdots \\ & & & a_{44} & a_{45} & \cdots \\ & & & & a_{45} & \cdots \\ & & & & & \ddots \end{pmatrix}$$

According to the matrix representation we find that the columns in this type contain a limited number of elements, may be the number of elements is large, but it remains limited, so this prevents the possibility of calculating the limit for the columns $\lim_{m\to\infty} a_{nm}$. Thus we can only calculate the limits

$$\lim_{n,m\to\infty} a_{nm} \text{ and } \lim_{n\to\infty} a_{nm}$$

So, we can only calculate the iterated limit

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{nm}$$

Single subsequence

In this section, we introduce the definition of a single subsequence of a double sequence. The advantage of this definition is its independence from the properties of the original double sequence.

Definition 3.1. Let $a,b,c,d\in\mathbb{R}$. We say

- $(a,b)\preceq(a,c)$ if and only if $b\leq c$.
- $(a,b)\preceq(c,b)$ if and only if $a\leq c$.
- $(a,b)\prec(c,d)$ if and only if $a\leq c$ and $b<d$ or $a<c$ and $b\leq d$.

Definition 3.2. Let (a_{nm}) be a double sequence and let

$$(n_1,m_1)\prec(n_2,m_2)\prec\cdots\prec(n_k,m_k)\prec\cdots$$

be an increasing sequence of $\mathbb{N}\times\mathbb{N}$, then we call the sequence $b_k=a_{n_k m_k}$ a single subsequence of the double sequence (a_{nm}).

Example 3.3. Consider the double sequence

$$a_{nm}=\begin{cases}1, & \text{if } n=m;\\ 0, & \text{if } n\neq m.\end{cases}$$

The subsequence $a_{nn}=1,\forall n\in\mathbb{N}$ is a single subsequence, and $a_{n(n+1)}=0,\forall n\in\mathbb{N}$ is also a single subsequence of (a_{nm})

Remark 3.4. Let (a_{nm}) ba a double sequence and $(a_{n_k m_k})$ be a single subsequence of (a_{nm}), then the limit $\lim_{k\to\infty} a_{n_k m_k}$ need not be equal to the double limit $\lim_{n,m\to\infty} a_{nm}$ or the iterated limits $\lim_{n\to\infty}\lim_{m\to\infty} a_{nm}$ or $\lim_{m\to\infty}\lim_{n\to\infty} a_{nm}$ of the original sequence. Moreover, there is no double limit or iterated limits for the single subsequence.

Example 3.5. Consider the double sequence

$$a_{nm}=\frac{1}{n}+\frac{1}{m}$$

It is clear that the double limit $\lim_{n,m\to\infty} a_{nm}=0$. Take the subsequence $a_{1m}=1+\frac{1}{m}$, then $\lim_{m\to\infty} 1+\frac{1}{m}=1$ and so

$$\lim_{n,m\to\infty} a_{nm}=0\neq 1=\lim_{m\to\infty} a_{1m}$$

Moreover, the limits of the single subsequences need not be equal or exist. for example the sequence $a_{nm}=\frac{1}{n}+\frac{1}{m}$ has the single subsequences $a_{1m}=1+\frac{1}{m}$ and $a_{2m}=\frac{1}{2}+\frac{1}{m}$ both have different limits, i.e

$$\lim_{m\to\infty} 1+\frac{1}{m}=1 \text{ and } \lim_{m\to\infty}\frac{1}{2}+\frac{1}{m}=\frac{1}{2}$$

Remark 3.6. In general, a single subsequence of a convergent sequence need not be convergent. For instance, take the double sequence

$$a_{nm}=\begin{pmatrix} 1 & 2 & 3 & \cdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is a convergent double sequence with $\lim_{n,m\to\infty} a_{nm}$, but the single subsequence a_{1m} diverges

Double subsequence

In this section, we will form a subsequence that has the same matrix representation as the original double sequence, and we will study the extent to which the new double sequence maintains the same properties as the original double sequence or not.

Definition 4.1. Let (a_{nm}) be a double sequence and defined two increasing subsequence n_p and m_q of natural numbers. Then the sequence $(a_{n_p m_q})$ is called a double subsequence of a_{nm}.

By applying the definition to the types of double sequences that were defined in section 2, we find that the matrix representation of the double subsequence has the forms

$$a_{n_p m_q}=\begin{pmatrix} a_{n_1m_1} & a_{n_1m_2} & a_{n_1m_3} & a_{n_1m_4} & a_{n_1m_5} & \cdots \\ a_{n_2m_1} & a_{n_2m_2} & a_{n_2m_3} & a_{n_2m_4} & a_{n_2m_5} & \cdots \\ a_{n_3m_1} & a_{n_3m_2} & a_{n_3m_3} & a_{n_3m_4} & a_{n_3m_5} & \cdots \\ a_{n_4m_1} & a_{n_4m_2} & a_{n_4m_3} & a_{n_4m_4} & a_{n_4m_5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or has the form,

$$a_{n_p m_q}=\begin{pmatrix} a_{n_1m_1} & \cdots & a_{n_1m_{r_1}} & & & & & \\ a_{n_2m_1} & \cdots & \cdots & \cdots & a_{n_2m_{r_2}} & & & \\ a_{n_3m_1} & \cdots & \cdots & \cdots & \dots & \dots & a_{n_3m_{r_3}} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

or has the form,

$$a_{nm} = \begin{pmatrix} a_{n_1m_1} & a_{n_1m_2} & a_{n_1m_3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ a_{n_{r_1}m_1} & \vdots & \vdots & \cdots \\ & \vdots & \vdots & \cdots \\ & a_{n_1m_{r_2}} & \vdots & \cdots \\ & & \vdots & \cdots \\ & & a_{n_1m_{r_3}} & \cdots \\ & & & \ddots \end{pmatrix}$$

In this way we remove some columns and some rows from the original sequence to get the subsequence. So the subsequence still has the double sequence properties and matrix form and called the double subsequence.

It must be noted that the double sequence can be a double subsequence of itself just by taking $n_p = p$ and $m_q = q$, which is in contrast to the patterson subsequence definition where the double subsequence can't be a double subsequence of itself in the patterson sense. (see (Hamilton,1936))

Example 4.2. Let a_{nm} be a double sequence with the matric form

$$a_{nm} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We can define the subsequence $\left(a_{(2n)(2m-1)}\right)$ which has the matrix form

$$a_{(2n)(2m-1)} = \begin{pmatrix} a_{21} & a_{23} & a_{25} & a_{27} & \cdots \\ a_{41} & a_{43} & a_{45} & a_{47} & \cdots \\ a_{61} & a_{63} & a_{65} & a_{67} & \cdots \\ a_{81} & a_{83} & a_{85} & a_{87} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 4.3. We can restrict the definition of double subsequence on only one of the indexes, i.e. $\left(a_{n_pm}\right)$ and $\left(a_{nm_q}\right)$ have the matrix forms

$$a_{n_pm} = \begin{pmatrix} a_{n_11} & a_{n_12} & a_{n_13} & a_{n_14} & a_{n_15} & \cdots \\ a_{n_21} & a_{n_22} & a_{n_23} & a_{n_24} & a_{n_25} & \cdots \\ a_{n_31} & a_{n_32} & a_{n_33} & a_{n_34} & a_{n_35} & \cdots \\ a_{n_41} & a_{n_42} & a_{n_43} & a_{n_44} & a_{n_45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$a_{nm_q} = \begin{pmatrix} a_{1m_1} & a_{1m_2} & a_{1m_3} & a_{1m_4} & a_{1m_5} & \cdots \\ a_{2m_1} & a_{2m_2} & a_{2m_3} & a_{2m_4} & a_{2m_5} & \cdots \\ a_{3m_1} & a_{3m_2} & a_{3m_3} & a_{3m_4} & a_{3m_5} & \cdots \\ a_{4m_1} & a_{4m_2} & a_{4m_3} & a_{4m_4} & a_{4m_5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is clear that the matrix form of $\left(a_{n_pm}\right)$ is obtained by removing some rows, while the matrix form of $\left(a_{nm_q}\right)$ is obtained by removing some columns.

Theorem 4.4. If a_{nm} is bounded sequence, then the double subsequence is bounded.

It is trivial from the definition that if $|a_{nm}| \le K,\ \forall n, m \in \mathbb{N}$, then $|a_{nm}| \le K,\ \forall n \ge N$ and $m \ge M$

Remark 4.5. The converse of the above theorem is not correct. For example, consider the double sequence

$$a_{nm} = \begin{cases} nm, & \text{if } m \text{ or } n \text{ is even ;} \\ \dfrac{1}{nm}, & \text{if } m \text{ and } n \text{ is odd .} \end{cases}$$

The double sequence a_{nm} is unbounded, while the double subsequence

$$\left(a_{(2n+1)(2m+1)}\right) = \frac{1}{(2n+1)(2m+1)}$$

is bounded

Theorem 4.6. Any double subsequence of a convergent double sequence converges to the same limit.

Proof. Assume that (a_{nm}) is a double sequence converges to a and $\left(a_{n_pm_q}\right)$ is a double subsequence of (a_{nm}).

Since (a_{nm}) converges to a, then for every $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{N}$ such that

$$n, m \ge K \ \Rightarrow\ |a_{nm} - a| < \epsilon.$$

Since $n_1 \le n_2 \le \cdots \le n_p \le \cdots$ and $m_1 \le m_2 \le \cdots \le m_q \le \cdots$, we have

$$n_p \ge p, m_q \ge q\ \forall p, q \in \mathbb{N}$$

Hence, it follows that if $p, q \ge K$ then $n_p, m_q \ge K$, which implies that

$$\left|a_{n_pm_q} - a\right| < \epsilon,$$

and therefore $\lim_{n,m\to\infty} a_{n_pm_q} = a$.

Note that the double subsequence has the same matrix representation of the original sequence, but it is not the only type that achieves this. In the same sense of tail subsequence of single sequence, we can define block subsequence of a double sequence.

Definition 4.7. Let (a_{nm}) be a double sequence and let $M, N \in \mathbb{N}$, then the sequence $(a_{nm}), \forall n \ge N, m \ge M$ is called a block subsequence. It is clear that we obtained the block subsequences by removing the first N rows and the first M columns from the matrix form of the double sequence (a_{nm})

$$a_{nm} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{NM} & a_{(N)(M+1)} & \cdots \\ \cdot & a_{(N+1)(M)} & \cdots & \cdots \\ \cdot & \vdots & \vdots & \ddots \end{pmatrix}$$

Definition 4.8. Let (a_{nm}) be a double sequence. If $N \in \mathbb{N}$, then the sequence $(a_{nm}), \forall n \ge N$ is called R-block subsequence.

$$a_{nm} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdots \\ a_{N1} & a_{(N)(2)} & \cdots & \cdots \\ a_{(N+1)(1)} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

If $M \in \mathbb{N}$, then the sequence $(a_{nm}), \forall m \ge M$ is called C-block subsequence.

$$a_{nm} = \begin{pmatrix} \cdot & a_{(1)(M)} & a_{(1)(M+1)} & \cdots \\ \cdot & a_{(2)(M)} & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The block subsequence is also a double subsequence. Because of this, it has the same theorems about limits as similar as double subsequence.

Theorem 4.9. If a_{nm} is bounded sequence, then the block subsequence is bounded.

Remark 4.10. The converse of the above theorem is not correct. For example, fix $m_k, n_k \in \mathbb{N}$ and define the double sequence

$$a_{nm} = \begin{cases} nm, & \text{if } m < m_k \text{ or } n < n_k; \\ \dfrac{1}{nm}, & \text{if } m \ge m_k \text{ and } n \ge n_k \end{cases}$$

The double sequence a_{nm} is unbounded, while the block subsequence

$$(a_{nm})_{n \ge n_k, m \ge m_k} = \frac{1}{nm}$$

is bounded

Theorem 4.11. If a double sequence a_{nm} converges to a, then any block subsequence of a_{nm} also converges to a.

In contrast to the previous types, we can define a subsequence that violates the original sequence shape, which is in the form of a square matrix, while only maintaining the shape of the upper triangle matrix or the lower triangle matrix

Definition 4.12. The upper triangle subsequence of the double sequence (a_{nm}) is the subsequence $(a_{nk})_{k\geq n}$

$$a_{nk} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ & a_{22} & a_{23} & \cdots \\ & & a_{33} & \cdots \\ & & & \ddots \end{pmatrix}$$

Definition 4.13. The lower triangle subsequence of the double sequence (a_{nm}) is the subsequence $(a_{nk})_{k\leq n}$

$$a_{nk} = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Remark 4.14. We can define upper triangle double subsequence (lower triangle) by take $(a_{nm})_{m\geq f(n)}$ where $f(n)$ is increasing sequence and lower triangle double subsequence by take $(a_{nm})_{g(m)\leq n}$ where $g(n)$ is increasing sequence.

Remark 4.15. For the upper triangle subsequence only $\lim_{n\to\infty}\lim_{k\to\infty}a_{nk}$ defined, and for the lower triangle subsequence only $\lim_{k\to\infty}\lim_{n\to\infty}a_{nk}$ defined. Example 4.16. Consider the double sequence $a_{nm} = \frac{(-1)^n}{m}$. Note, first, that

$$\lim_{n,m\to\infty} a_{nm} = \lim_{n,m\to\infty} \frac{(-1)^n}{m} = 0$$

For the upper triangle subsequence $(a_{nk})_{k\geq n}$ only $\lim_{n\to\infty}\lim_{k\to\infty}a_{nk}$ defined, and $\lim_{n\to\infty}\lim_{k\to\infty}\frac{(-1)^n}{k} = 0$ On the other hand, for the Lower triangle subsequence $(a_{nk})_{k\leq n}$ only $\lim_{k\to\infty}\lim_{n\to\infty}a_{nk}$ defined, but for this sequence $\lim_{k\to\infty}\lim_{n\to\infty}a_{nk}$ dose not exist.

Results and discussion

It is clear from the previous sections that any subsequence of a bounded double sequence is bounded, because any element in the subsequence is also an element in the original double sequence and satisfied the same condition of boundedness. On the other hand, Pringsheim (Pringsheim, 1900) has proven that there is no relationship between boundedness and convergence for the double sequences. However, we can prove that we can find a bounded convergent subsequence of a convergent double sequence.

Theorem 5.1. Every convergent double sequence has a bounded convergent subsequence.

Proof. Let (a_{nm}) be a complex double sequence converges to $a \in \mathbb{C}$, then

$$\forall\epsilon > 0,\ \exists K \in \mathbb{N}, \text{ such that } |a_{nm} - L| < \epsilon$$

this implies that $|a_{nm}| < \max\{|\epsilon + L|, |-\epsilon + L|\}$ for all $n, m > K$.

Therefore the block subsequence $(a_{nm})_{n,m>K}$ is a bounded subsequence of the double sequence (a_{nm}), and since each Block subsequence of convergent double sequence is convergent then the block subsequence $(a_{nm})_{n,m>K}$ is a bounded convergent subsequence.

On the other hand, the convergent condition dose not satisfied for the single subsequence of the double subsequence, we can find a divergent single subsequence of a convergent double sequence, In contrast to the single subsequence, the other types of subsequences of a convergent double sequence are all convergent and via versa.

Theorem 5.2. Let (a_{nm}) be a bounded double sequence of complex numbers and let a $\in \mathbb{C}$ have the property that every block subsequence of a_{nm} converges to a. Then the double sequence (a_{nm}) converges to a.

Also the same theorem satisfied in the case of double subsequences, upper triangle subsequences or lower triangle subsequences.

Theorem 5.3. Let (a_{nm}) be a bounded double sequence of complex numbers and let a $\in \mathbb{C}$ have the property that every double subsequence of a_{nm} converges to a. Then the double sequence (a_{nm}) converges to a.

Note that if we have at least one convergent block subsequence, then the original double sequence is convergent also. The same can't happen for the double subsequence. To see this, consider the double sequence $a_{nm} = (-1)^{n+m}$, this sequence is not convergent but it has many double subsequences which is convergent as $a_{2n,2m} = 1$.

In addition, for the single subsequence $(a_{n_k m_k})$ we can calculate only single limit $\lim_{k\to\infty} a_{n_k m_k}$, but there is no itreated limits. Also, in case of upper and lower triangle subsequences we can calculate only one of the itreated limits As we explained in the previous section.

For the other types of subsequences we have the following theorem,

Theorem 5.4. If the iterated limits of a double sequence a_{nm} exist and satisfy

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{nm} = \lim_{n\to\infty}\lim_{m\to\infty} a_{nm} = a$$

then the iterated limits of a block subsequence $(a_{nm})_{n\geq n_k, m\geq m_k}$ exist and satisfy

$$\lim_{m\to\infty}\lim_{n\to\infty} (a_{nm})_{n\geq n_k, m\geq m_k} = \lim_{n\to\infty}\lim_{m\to\infty} (a_{nm})_{n\geq n_k, m\geq m_k} = a$$

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