An-Najah University Journal for Research – A
Natural Sciences

Certain Properties of Extended Mittag-Leffler-Type Function of Arbitrary Order

Maged G. Bin-Saad [1], Jihad Younis*, [2]

Type: Full Article. Received 14^{th} Nov. 2024, Accepted 24^{th} Jan. 2025, Published 1^{th} Feb. 2026,
https://doi.org/10.35552/anujr.a.40.1.2371

Abstract: In this paper, we introduce a new extension of Mittag-Leffler function. We investigate its basic properties, including recurrence relations, differential formulas, integral representations, Laplace transform and Mellin transform. Furthermore, we establish fractional integral and differential operators associated with this extended Mittag-Leffler type function. Several interesting special cases of our main results are derived.

Keywords: Mittag-Leffler function, Fox-Wright function, integral transforms; fractional calculus operators.

1 Introduction

The Mittag-Leffler function is highly useful and important due to its special properties and significant potential in solving applied problems. Therefore, it gained more attention in several areas of applied mathematics and engineering sciences, such as statistical distributions [12], stochastic processes [10], mechanical relaxation [11], modeling of processes (diffusion) [15], electrotechnics [19], complex network [2], etc. Moreover, the functions of the Mittag-Leffler type appear in many fields of science, such as physics [32], chemistry [14], biology [8], economics [30].

The real importance of the Mittag-Leffler function has been recognized when its special role in fractional calculus has been discovered. The fractional calculus extends the conventional integral and differential operators. The concept of fractional calculus emerged in History as a response to the query of whether derivatives of integer order can be expanded to fractional order. There are numerous definitions of fractional derivatives and integrals, each with distinct properties and capacities to model various behaviors.

In [3], Atangana and Baleanu introduced a new fractional derivative known as the ABC-fractional derivative, which utilizes the Mittag–Leffler function as its kernel. The ABC-fractional differential operator is more suitable for accurately describing real-world phenomena, as it is a nonlocal operator with a nonsingular kernel. The ABC-fractional derivative has gained popularity in recent years primarily due to its wide-ranging applications [4, 22, 31]. For further details on the fundamental advancements in ABC-type derivatives, readers can refer to recent works [1, 7, 9, 29].

Definition 1.1. *([16]). The Mittag-Leffler function $E_\alpha(z)$ is defined by*

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha,z\in\mathbb{C},\Re(\alpha)>0. \tag{1.1}$$

Definition 1.2. *([33]). The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0. \tag{1.2}$$

Definition 1.3. *([20]). The three-parameter Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ is defined by*

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},, \tag{1.3}$$

$$(\alpha,\beta,\gamma,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0),$$

where $(\gamma)_n$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function Γ by (see, e.g., [26])

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}:=\{1,2,...\}).\end{cases}$$

Definition 1.4. *([17]). The extended Mittag-Leffler function $E^{\gamma;c}_{\alpha,\beta}(z;p)$ is defined by*

$$E^{\gamma;c}_{\alpha,\beta}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.4}$$

[1]Department of Mathematics, College of Education, University of Aden, Aden, Yemen, mgbinsaad@yahoo.com
[2]Department of Mathematics, College of Education, University of Aden, Aden, Yemen, *Corresponding author: jihadalsaqqaf@gmail.com

where $\alpha, \beta, \gamma, c \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$, $p \in \mathbb{R}_0^+$ and $B_p(x,y)$ is an extension of beta function defined in [5, 21] as follows:

$$B_p(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt, , \tag{1.5}$$

$$\Re(x) > 0, \Re(y) > 0, \Re(p) > 0$$

and $B_0(x,y) = B(x,y)$ is the familiar beta function (see, [26]).

Definition 1.5. *([18]). The Mittag-Leffler-type function of arbitrary order $E_{\alpha,\beta}^{j,k}(z)$ is defined by*

$$E_{\alpha,\beta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma(\beta + \alpha(nj+k))}, \tag{1.6}$$

$$\alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, j \geq 1, k \geq 0.$$

Definition 1.6. *[27]. The Fox-Wright function is defined as*

$${}_p\Psi_q \left[\begin{array}{c} (d_1, D_1), ..., (d_p, D_p) \\ (e_1, E_1), ..., (e_q, E_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(d_i + D_i n)}{\prod_{j=1}^{q} \Gamma(e_j + E_j n)} \frac{z^n}{n!}, \tag{1.7}$$

where $d_i, D_i, e_j, E_j, z \in \mathbb{C}$, $\Re(d_i) > 0, \Re(D_i) > 0$, $i = 1, ..., p$, $\Re(e_i) > 0, \Re(E_i) > 0$,
$j = 1, ..., q$
and $1 + \Re\left(\sum_{j=1}^{q} E_j - \sum_{i=1}^{p} D_i\right) \geq 0$.

Definition 1.7. *The H-function of two variables is defined as*

$$H_{C,D:C_1,D_1;C_2,D_2}^{0,B:A_1,B_1;A_2,B_2}$$

$$\left[\begin{array}{c|c} z_1 & (a_j; \alpha_j^{(1)}, \alpha_j^{(2)})_{1,C} : (c_j^{(1)}, \gamma_j^{(1)})_{1,C_1}; (c_j^{(2)}, \gamma_j^{(2)})_{1,C_2} \\ & \\ z_2 & (b_j; \beta_j^{(1)}, \beta_j^{(2)})_{1,D} : (d_j^{(1)}, \delta_j^{(1)})_{1,D_1}; (d_j^{(2)}, \delta_j^{(2)})_{1,D_2} \end{array} \right]$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(\eta_1, \eta_2) \prod_{i=1}^{2} (\theta_i(\eta_i) z_i^{\eta_i}) d\eta_1 d\eta_2, \tag{1.8}$$

where $\omega = \sqrt{-1}$, and

$$\phi(\eta_1, \eta_2) = \frac{\prod_{j=1}^{B} \Gamma\left(1 - a_j + \sum_{i=1}^{2} \alpha_j^{(i)} \eta_i\right)}{\prod_{j=1}^{D} \Gamma\left(1 - b_j + \sum_{i=1}^{2} \beta_j^{(i)} \eta_i\right)}$$

$$\times \frac{1}{\prod_{j=B+1}^{C} \Gamma\left(a_j - \sum_{i=1}^{2} \alpha_j^{(i)} \eta_i\right)}, \quad (i = 1, 2)$$

$$\theta_i(\eta_i) = \frac{\prod_{j=1}^{A_i} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \eta_i\right) \prod_{j=1}^{B_i} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} \eta_i\right)}{\prod_{j=A_i+1}^{D_i} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \eta_i\right)}$$

$$\frac{1}{\prod_{j=B_i+1}^{C_i} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} \eta_i\right)}, \quad (i = 1, 2).$$

For conditions of convergence for the H-function of two variables (1.8), one can refer to [28].

Definition 1.8. *([25]). The Laplace transform of the function $f(z)$ is defined as*

$$\mathcal{L}\{f(z); s\} = \int_0^{\infty} e^{-sz} f(z) dz, \quad \Re(s) > 0. \tag{1.9}$$

Definition 1.9. *([25]). The Mellin transform of the function $f(z)$ is given as*

$$\mathcal{M}\{f(z); s\} = \int_0^{\infty} z^{s-1} f(z) dz = f^*(s), \quad \Re(s) > 0, \tag{1.10}$$

and the inverse Mellin transform is defined as

$$f(z) = \mathcal{M}^{-1}\{f^*(s); z\} = \frac{1}{2\pi i} \int_L f^*(s) z^{-s} ds, \tag{1.11}$$

where L is a contour of integration that begins at $-i\infty$ and ends at $i\infty$.

Definition 1.10. *([23]). The left-sided Riemann-Liouville fractional integral operator I_{a+}^{ν} and the right-sided Riemann-Liouville fractional integral operator I_{b-}^{ν} are defined by*

$$\left(I_{a+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \quad (\Re(\nu) > 0, x > a), \tag{1.12}$$

$$\left(I_{b-}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t) dt, \quad (\Re(\nu) > 0, x < b). \tag{1.13}$$

Definition 1.11. *([23]). The left-sided Riemann-Liouville fractional derivative operator D_{a+}^{ν} and the right-sided Riemann-Liouville fractional derivative operator D_{b-}^{ν} are defined by*

$$\left(D_{a+}^{\nu} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I_{a+}^{m-\nu} f\right)(x), \quad (\Re(\nu) > 0, m = [\Re(\nu)] + 1), \tag{1.14}$$

$$\left(D_{b-}^{\nu} f\right)(x) = (-1)^m \left(\frac{d}{dx}\right)^m \left(I_{b-}^{m-\nu} f\right)(x), \tag{1.15}$$

$$(\Re(\nu) > 0, m = [\Re(\nu)] + 1),$$

where $\Re(\nu)$ denotes the real part of the complex number $\nu \in \mathbb{C}$ and $[\Re(\nu)]$ represents the integral part of $\Re(\nu)$.

Here, we recall the left and right-sided Riemann-Liouville fractional integrations of a power function are defined in [13] by

$$\left(I_{0+}^{\nu} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)} x^{\lambda+\nu-1}, \quad (\Re(\nu) > 0, \Re(\lambda) > 0), \tag{1.16}$$

$$\left(I_{-}^{\nu} t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda+\nu-1}, \quad (0 < \Re(\nu) < 1 - \Re(\lambda)), \tag{1.17}$$

respectively. The left and right-sided Riemann-Liouville fractional differentiations of a power function are defined, respectively, by (see [13])

$$\left(D_{0+}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\nu)} x^{\lambda-\nu-1}, \quad (\Re(\nu)>0, \Re(\lambda)>0) \tag{1.18}$$

and

$$\left(D_{-}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(1+\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda-\nu-1}, \tag{1.19}$$

$$(\Re(\nu)>0, \Re(\lambda)<\Re(\nu)-[\Re(\nu)]).$$

Next, motivated by the above such extensions of the Mittag-Leffler function, we define a new extension of the arbitrary order Mittag-Leffler-type function (1.6) and investigate its certain properties.

2 Basic properties

Definition 2.1. . *An extension of various functions described above can be given by the definition of extended Mittag-Leffler-tpe function of arbitrary order, represents as follows:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\, B(\gamma, c-\gamma)\, \Gamma(\beta+\alpha(nj+k))} z^{nj+k}, \tag{2.1}$$

where $\alpha,\beta,\gamma,c\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$; $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$.

We consider special cases enumerated by equation (2.1)

(i) Setting $p=0$ and $\gamma=1$ in (2.1) yields (1.6), as established by Pathan and Bin-Saad [18].

(ii) Setting $j=1$ and $k=0$ in (2.1) yields (1.4), as established by Özarslan and Yilmaz [17].

(iii) Setting $k=p=0$ and $j=1$ in (2.1) yields (1.3), as established by Prabhakar [20].

The following theorems list some basic properties of this function.

Theorem 2.1. *Let* $\alpha,\beta,\gamma,c,z\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, *then*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p). \tag{2.2}$$

Proof. Applying (2.1) to the right side of (2.2), we get

$$\begin{aligned}
&\beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p)\\
&=\beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p)+\alpha z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n\ z^{nj+k}}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)}\\
&=\beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p)+\alpha\sum_{n=0}^{\infty}\frac{(nj+k)B_p(\gamma+n,c-\gamma)(c)_n\ z^{nj+k}}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)}\\
&=\sum_{n=0}^{\infty}\frac{(\beta+\alpha(nj+k))B_p(\gamma+n,c-\gamma)(c)_n\ z^{nj+k}}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)}\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k))}z^{nj+k},
\end{aligned}$$

which, in terms of (2.1), yields the desired formula (2.2).
This result generalizes the Corollary 4 in [17], if we consider $j=1$ and $k=0$ in (2.2), we have the well-known result of [17].

Theorem 2.2. *Let* $\alpha,\beta,\gamma,c,z\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, $m\in\mathbb{N}$, *then*

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega z^{\alpha};p)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}^{\gamma,c;j,k}(\omega z^{\alpha};p). \tag{2.3}$$

Proof. From (2.1), we find that

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega z^{\alpha};p)\right]\\
&=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k))}z^{\alpha(nj+k)+\beta-1}\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma,c-\gamma)\Gamma(\beta-m+\alpha(nj+k))}z^{\alpha(nj+k)+\beta-m-1}\\
&=z^{\beta-m-1}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{n!\ B(\gamma,c-\gamma)\Gamma(\beta-m+\alpha(nj+k))}(\omega z^{\alpha})^{nj+k}.
\end{aligned} \tag{2.4}$$

Now, using (2.1) in (2.4) we obtain the desired formula.

This result generalizes the Theorem 11 in [17], if we consider $j=1$ and $k=0$ in (2.3), we have the well-known result of [17].

Theorem 2.3. *Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(\nu)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, *then*

$$\frac{1}{\Gamma(\nu)}\int_0^1 t^{\beta-1}(1-t)^{\nu-1}E_{\alpha,\beta}^{\gamma,c;j,k}(zt^{\alpha};p)dt=E_{\alpha,\beta+\nu}^{\gamma,c;j,k}(z;p), \tag{2.5}$$

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega t^{\alpha};p)dt=z^{\beta}E_{\alpha,\beta+1}^{\gamma,c;j,k}(\omega z^{\alpha};p). \tag{2.6}$$

Proof. We have

$$\begin{aligned}
&\int_0^1 t^{\beta-1}(1-t)^{\nu-1}E_{\alpha,\beta}^{\gamma,c;j,k}(zt^{\alpha};p)dt\\
&=\int_0^1 t^{\beta-1}(1-t)^{\nu-1}\\
&\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k))}(zt^{\alpha})^{nj+k}dt\\
&=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\alpha(nj+k))}z^{nj+k}\\
&\int_0^1 t^{\beta+\alpha(nj+k)-1}(1-t)^{\nu-1}dt\\
&=\Gamma(\nu)\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{n!\ B(\gamma,c-\gamma)\Gamma(\beta+\nu+\alpha(nj+k))}z^{nj+k}\\
&=\Gamma(\nu)E_{\alpha,\beta+\nu}^{\gamma,c;j,k}(z;p),
\end{aligned}$$

hence (2.5) is proved.

Next, we have

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega t^{\alpha};p)dt$$
$$=\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n\ \omega^{nj+k}}{n!\ B\left(\gamma,c-\gamma\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^z t^{\beta+\alpha(nj+k)-1}dt$$
$$=\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n\ \omega^{nj+k}}{n!\ B\left(\gamma,c-\gamma\right)\left(\beta+\alpha(nj+k)\right)\Gamma\left(\beta+\alpha(nj+k)\right)}z^{\beta+\alpha(nj+k)}$$
$$=z^{\beta}\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n}{n!\ B\left(\gamma,c-\gamma\right)\Gamma\left(\beta+\alpha(nj+k)+1\right)}(\omega z^{\alpha})^{nj+k}$$
$$=z^{\beta}E_{\alpha,\beta+1}^{\gamma,c;j,k}(\omega z^{\alpha};p),$$

which is the result (2.6).
When $p=0$ and $\gamma=1$, Theorem 2.3 reduces to the following corollary.

Corollary 2.1. *The following results hold:*

$$\frac{1}{\Gamma(\nu)}\int_0^1 t^{\beta-1}\left(1-t\right)^{\nu-1}E_{\alpha,\beta}^{j,k}(zt^{\alpha})dt=E_{\alpha,\beta+\nu}^{j,k}(z) \tag{2.7}$$

and

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}^{j,k}(\omega t^{\alpha})dt=z^{\beta}E_{\alpha,\beta+1}^{j,k}(\omega z^{\alpha}). \tag{2.8}$$

3 Integral representations

Theorem 3.1. *Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^{+}$, *then*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\frac{z^k}{B\left(\gamma,c-\gamma\right)}\int_0^1 t^{\gamma-1}\left(1-t\right)^{c-\gamma-1}e^{-\frac{p}{t(1-t)}}E_{\alpha j,\beta+\alpha k}^{c}\left(tz^j\right)dt. \tag{3.1}$$

Proof. Using (1.5) in (3.1), we obtain

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\sum_{n=0}^{\infty}\frac{z^k}{B\left(\gamma,c-\gamma\right)}\left\{\int_0^1 t^{\gamma+n-1}\left(1-t\right)^{c-\gamma-1}e^{-\frac{p}{t(1-t)}}dt\right\}\frac{(c)_n\ (z^j)^n}{n!\ \Gamma\left(\alpha nj+\beta+\alpha k\right)},$$

which can be written as

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\frac{z^k}{B\left(\gamma,c-\gamma\right)}\int_0^1 t^{\gamma-1}\left(1-t\right)^{c-\gamma-1}e^{-\frac{p}{t(1-t)}}\sum_{n=0}^{\infty}\frac{(c)_n\ (tz^j)^n}{\Gamma\left(\alpha nj+\beta+\alpha k\right)n!}dt.$$

Applying (1.3) in above equation, we obtain the required result.
Setting $t=sin^2\theta$ and $t=\frac{u}{1+u}$ in Theorem 3.1 yields Corollaries 3.1 and 3.2, respectively.

Corollary 3.1. *The following result holds true:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\frac{2z^k}{B\left(\gamma,c-\gamma\right)}\int_0^{\frac{\pi}{2}} sin^{2\gamma-1}\theta\ cos^{2c-2\gamma-1}\theta\ e^{-\frac{p}{sin^2\theta cos^2\theta}}\ E_{\alpha j,\beta+\alpha k}^{c}\left(z^j sin^2\theta\right)d\theta. \tag{3.2}$$

Corollary 3.2. *The following result holds true:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)=\frac{z^k}{B\left(\gamma,c-\gamma\right)}\int_0^{\infty}\frac{u^{\gamma-1}}{(1+u)^c}e^{-\frac{p(1+u)^2}{u}}E_{\alpha j,\beta+\alpha k}^{c}\left(\frac{uz^j}{1+u}\right)du. \tag{3.3}$$

These results generalize Corollaries 2 and 3 in [17], if we consider $j=1$ and $k=0$ in (3.2) and (3.3), we obtain the well-known results of [17].

4 Integral transforms

Here, we evaluate the Laplace and Mellin transforms of the extended Mittag-Leffler function (2.1).

Theorem 4.1. *(Laplace transform): Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(\sigma)>0$, $\Re(q)>0$, $\Re(s)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^{+}$, *then*

$$\int_0^{\infty} z^{q-1}e^{-sz}E_{\alpha,\beta}^{\gamma,c;j,k}\left(xz^{\sigma};p\right)dz=\frac{s^{-(q+\sigma k)}x^k}{\Gamma\left(\gamma\right)\Gamma\left(c-\gamma\right)}\times H_{1,1:0,2;2,2}^{0,1:1,0;1,2}\left[\begin{array}{c|l} p & (1-\gamma;1,1):\qquad -;\ (1-q-\sigma k,\sigma j),(1-c,1) \\ -\left(\frac{x}{s^{\sigma}}\right)^j & (1-c;1,2):\quad (0,1),(c-\gamma,1);\ (0,1),(c-\gamma,1); \end{array}\right]. \tag{4.1}$$

Proof. We find from (2.1) that

$$\int_0^{\infty} z^{p-1}e^{-sz}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(xz^{\sigma}\right)dz$$
$$=\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n\ x^{nj+k}}{n!\ B\left(\gamma,c-\gamma\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^{\infty} z^{q+\sigma nj+\sigma k-1}e^{-sz}dz$$
$$=\frac{s^{-(q+\sigma k)}x^k}{\Gamma\left(\gamma\right)\Gamma\left(c-\gamma\right)}\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)\Gamma\left(q+\sigma k+\sigma nj\right)\Gamma\left(c-n\right)x^{nj}}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)s^{\sigma nj}}. \tag{4.2}$$

Evaluating $B_p\left(\gamma+n,c-\gamma\right)$ in (4.2) using the series form of (1.5), we obtain

$$\int_0^{\infty} z^{p-1}e^{-sz}E_{\alpha,\beta,\gamma,\delta}^{j,k}\left(xz^{\sigma}\right)dz$$
$$=\frac{s^{-(q+\sigma k)}x^k}{\Gamma\left(\gamma\right)\Gamma\left(c-\gamma\right)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma\left(q+\sigma k+\sigma nj\right)\Gamma\left(c-n\right)}{\Gamma\left(\beta+\alpha(nj+k)\right)}\times\frac{\Gamma\left(\gamma+n-m\right)\Gamma\left(c-\gamma-m\right)}{\Gamma\left(c+n-2m\right)m!\ n!}(-p)^m\left(\frac{x}{s^{\sigma}}\right)^{nj}.$$

In view of (1.8), we get the desired result.
When $j=1$ and $k=1$, Theorem 4.1 reduces to the following corollary.

Corollary 4.1. *The following result holds true:*

$$\int_0^\infty z^{q-1}e^{-sz}E^{\gamma,c}_{\alpha,\beta}(xz^\sigma;p)\,dz=\frac{s^{-(q)}}{\Gamma(\gamma)\Gamma(c-\gamma)}$$
$$\times H^{0,1:1,0;1,2}_{1,1:0,2;2,2}\left[\begin{array}{c} p \\ -\left(\frac{x}{s^\sigma}\right)\end{array}\middle|\begin{array}{lll}(1-\gamma;1,1): & & -; \\ (1-c;1,2): & (0,1),(c-\gamma,1); & \end{array}\right.$$
$$\left.\begin{array}{c}(1-q,\sigma),(1-c,1)\\ (0,1),(c-\gamma,1);\end{array}\right]. \tag{4.3}$$

Theorem 4.2. *(Mellin transform): Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(s)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, *then*

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\frac{z^k\Gamma(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)}{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+s,1)\\(\beta+\alpha k,\alpha j),(c+2s,1)\end{array}\middle|z^j\right]. \tag{4.4}$$

Proof. Applying the Mellin transform to equation (2.1), we get

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\int_0^\infty p^{s-1}E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)\,dp. \tag{4.5}$$

Now,using (3.1) in (4.5), we have

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\frac{z^k}{B(\gamma,c-\gamma)}\int_0^\infty p^{s-1}$$
$$\left[\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}e^{\frac{-p}{t(1-t)}}\right]E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)dtdp. \tag{4.6}$$

Changing the order of integrations in (4.6), we obtain

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\frac{z^k}{B(\gamma,c-\gamma)}$$
$$\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)\left[\int_0^\infty p^{s-1}e^{\frac{-p}{t(1-t)}}dp\right]dt. \tag{4.7}$$

Substituting $u=\frac{p}{t(1-t)}$ in (4.7), we have

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]$$
$$=\frac{z^k}{B(\gamma,c-\gamma)}$$
$$\int_0^1 t^{\gamma+s-1}(1-t)^{c-\gamma+s-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)\left[\int_0^\infty u^{s-1}e^{-u}du\right]dt$$
$$=\frac{z^k\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{c-\gamma+s-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)dt. \tag{4.8}$$

Further, using (1.3) in (4.8), we find

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\frac{z^k\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+n+s-1}(1-t)^{c-\gamma+s-1}$$
$$\sum_{n=0}^\infty\frac{(c)_n\,z^{nj}}{\Gamma(\alpha nj+\beta+\alpha k)\,n!}dt,$$

which gives,

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right]=\frac{z^k\Gamma(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)}$$
$$\sum_{n=0}^\infty\frac{\Gamma(c+n)}{\Gamma(\beta+\alpha k+\alpha nj)}\frac{\Gamma(\gamma+s+n)}{\Gamma(c+2s+n)}\frac{(z^j)^n}{n!}.$$

Finally, using (1.7) in above equation, we obtain the required result. When $s=1$, Theorem 4.2 reduces to the following corollary.

Corollary 4.2. *The following result holds true:*

$$\int_0^\infty E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)\,dp=\frac{z^k\Gamma(c-\gamma+1)}{\Gamma(\gamma)\Gamma(c-\gamma)}{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+1,1)\\(\beta+\alpha k,\alpha j),(c+2,1)\end{array}\middle|z^j\right]. \tag{4.9}$$

This result generalizes the Corollary 6 in [17], if $j=1$ and $k=0$, equation (4.9) coincides with the result in [17].

Corollary 4.3. *(Inverse Mellin transform): Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(s)>0$ *and* $j\geq 1$, $k\geq 0$, $\mu>0$, $p\in\mathbb{R}_0^+$, *then*

$$E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)=\frac{z^k}{2\pi i\,\Gamma(\gamma)\Gamma(c-\gamma)}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)\Gamma(c-\gamma+s)$$
$$\times{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+s,1)\\(\beta+\alpha k,\alpha j),(c+2s,1)\end{array}\middle|z^j\right]p^{-s}ds. \tag{4.10}$$

Proof. Applying the inverse Mellin transform to both sides of equation (4.4), we get the result.

This result generalizes the Corollary 7 in [17], if $j=1$ and $k=0$, equation (4.10) coincides with the result in [17].

5 Fractional calculus operators

In this section, we derive certain interesting properties of $E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)$ related to the operators of Riemann-Liouville fractional integral and derivative.

Theorem 5.1. *Let* $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ *with* $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, $x>a$. *Let* I^ν_{a+} *be the left-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I^\nu_{a+}\left[(t-a)^{\beta-1}E^{\gamma,c;j,k}_{\alpha,\beta}\left(\omega(t-a)^\alpha;p\right)\right]\right)(x)$$
$$=(x-a)^{\beta+\nu-1}E^{\gamma,c;j,k}_{\alpha,\beta+\nu}\left(\omega(x-a)^\alpha;p\right). \tag{5.1}$$

Proof. Using definitions (2.1) and (1.12), then after some simplification, we obtain

$$\left(I_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega\,(t-a)^{\alpha};p)\right]\right)(x)$$
$$=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)\;\omega^{nj+k}\;(c)_n}{n!\;\Gamma(\beta+\alpha(nj+k))}$$
$$\left(I_{a+}^{\nu}\left[(t-a)^{\beta+\alpha(nj+k)-1}\right]\right)(x)$$
$$=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p\,(\gamma+n,c-\gamma)\,\omega^{nj+k}(c)_n}{n!\;\Gamma(\beta+\alpha(nj+k))}$$
$$(x-a)^{\beta+\nu+\alpha(nj+k)-1}$$
$$=(x-a)^{\beta+\nu-1}\sum_{n=0}^{\infty}\frac{B_p\,(\gamma+n,c-\gamma)\,(c)_n}{n!\;B\,(\gamma,c-\gamma)\,\Gamma\,(\beta+\alpha(nj+k))}$$
$$\frac{\Gamma(\beta+\alpha(nj+k))}{\Gamma\,(\beta+\nu+\alpha(nj+k))}(\omega(x-a)^{\alpha})^{nj+k}$$
$$=(x-a)^{\beta+\nu-1}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma\,(\beta+\nu+\alpha(nj+k))}$$
$$(\omega(x-a)^{\alpha})^{nj+k}$$
$$=(x-a)^{\beta+\nu-1}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\,(\omega(x-a)^{\alpha};p)\,.$$

This result generalizes the Theorem 2.2 in [21].

Corollary 5.1. *Take $j=1$ and $k=0$ in (5.1), we get the result of Rahman et al. [21]*

$$\left(I_{a+}^{\nu}\left[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\gamma;c}\,(\omega\,(t-a)^{\alpha}\,;p)\right]\right)(x)$$
$$=(x-a)^{\beta+\nu-1}\,E_{\alpha,\beta+\nu}^{\gamma;c}\,(\omega(x-a)^{\alpha};p)\,. \tag{5.2}$$

Theorem 5.2. *Let $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ and $j\geq1$, $k\geq0$, $p\in\mathbb{R}_0^+$. Let I_{-}^{ν} be the right-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I_{-}^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right). \tag{5.3}$$

Proof. Using definitions (2.1) and (1.17), then after some simplification, we obtain

$$\left(I_{-}^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)$$
$$=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)\;\omega^{nj+k}(c)_n}{n!\;\Gamma\,(\beta+\alpha(nj+k))}$$
$$\left(I_{-}^{\nu}t^{-\nu-\beta-\alpha(nj+k)}\right)(x)$$
$$=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p\,(\gamma+n,c-\gamma)\,\omega^{nj+k}(c)_n}{n!\;\Gamma\,(\beta+\alpha(nj+k))}x^{-\beta-\alpha(nj+k)}$$
$$=x^{-\beta}\sum_{n=0}^{\infty}\frac{B_p\,(\gamma+n,c-\gamma)\,(c)_n}{n!\;B\,(\gamma,c-\gamma)\,\Gamma\,(\beta+\alpha(nj+k))}$$
$$\frac{\Gamma(\beta+\alpha(nj+k))}{\Gamma\,(\beta+\nu+\alpha(nj+k))}(\omega x^{-\alpha})^{nj+k}$$
$$=x^{-\beta}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)}{n!\;B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma\,(\beta+\nu+\alpha(nj+k))}(\omega x^{-\alpha})^{nj+k}$$
$$=x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right).$$

This completes the desired proof.
Setting $p=0$ and $\gamma=1$ in the result in Theorem 5.2, we get an interesting formula involving the Mittag-Leffler function of arbitrary order (1.6), which is asserted by the following corollary:

Corollary 5.2. *Let $\nu,\alpha,\beta,\omega\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $j\geq1$, $k\geq0$. Then the following right fractional integral formula holds true:*

$$\left(I_{-}^{\nu}\left[t^{-\nu-\beta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\alpha}\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta+\nu}^{j,k}(\omega x^{-\alpha}). \tag{5.4}$$

Theorem 5.3. *Let $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ and $j\geq1$, $k\geq0$, $p\in\mathbb{R}_0^+$, $x>a$. Let D_{a+}^{ν} be the left-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}\,(\omega(t-a)^{\alpha};p)\right]\right)(x)$$
$$=(x-a)^{\beta-\nu-1}E_{\alpha,\beta-\nu}^{\gamma,c;j,k}\,(\omega(x-a)^{\alpha};p)\,. \tag{5.5}$$

Proof. For the proof of assertion (5.5), we use (1.14) and (5.1) in the left hand side of (5.5)

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}\,(\omega(t-a)^{\alpha};p)\right]\right)(x)$$
$$=\left(\frac{d}{dx}\right)^{m}\left[(x-a)^{\beta+m-\nu-1}E_{\alpha,\beta+m-\nu}^{\gamma,c;j,k}(\omega(x-a)^{\alpha},p)\right]$$
$$=(x-a)^{\beta-\nu-1}\,E_{\alpha,\beta+m-\nu}^{\gamma,c;j,k}\,(\omega\,(x-a)^{\alpha}\,;p)\,,$$

This completes the proof of (5.5).
This result generalizes the Theorem 2.2 in [21].

Corollary 5.3. *Take $j=1$ and $k=0$ in (5.5), we get the result of Rahman et al. [21]*

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma;c}(\omega(t-a)^{\alpha};p)\right]\right)(x)$$
$$=(x-a)^{\beta-\nu-1}E_{\alpha,\beta-\nu}^{\gamma,c}\,(\omega(x-a)^{\alpha};p)\,. \tag{5.6}$$

Theorem 5.4. *Let $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(\nu)>0$, $\Re(\beta)>[\Re(\nu)]+1$ and $j\geq1$, $k\geq0$, $p\in\mathbb{R}_0^+$. Let D_{-}^{ν} be the right-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{-}^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta-\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right). \tag{5.7}$$

Proof. In order to the prove (5.7), we use (1.15) and (5.3) in the left hand side of (5.7)

$$\left(D_{-}^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)$$
$$=\left(-\frac{d}{dx}\right)^{m}\left[x^{\beta+m-\nu-1}E_{\alpha,\beta+m-\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right)\right]$$
$$=x^{-\beta}E_{\alpha,\beta-\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right),$$

which completes the desired proof.

Corollary 5.4. *If we take $p=0$ and $\gamma=1$ in equation (5.7), we obtain*

$$\left(D_{-}^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\alpha}\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta-\nu}^{j,k}\left(\omega x^{-\alpha}\right). \tag{5.8}$$

6 Conclusion

In this paper, we derived a new extension of the Mittag-Leffler function and investigated several of its properties. The special cases of the main result for $j = 1$ and $k = 0$ are detailed in [17]. Therefore, the results introduced in this present article are new and an extension of the related outcomes in the current literature (see, e.g., [18, 21]). In addition, it is important to note that the function $E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)$ is highly compatible with fractional calculus. The newly defined Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)$ discussed in this article will have applications across various fields of applied sciences.

Disclosure statement

– Ethics approval and consent to participate:

Not applicable.

– Consent for publication:

Not applicable

– Availability of data and materials:

Not applicable.

– Author's contribution:

The authors confirm contribution to the paper as follows: study conception and design: Maged Bin-Saad, theoretical calculations and modeling: Jihad Younis, data analysis and validation: Maged Bin-Saad, Jihad Younis, manuscript preparation: Maged Bin-Saad, Jihad Younis. All authors reviewed the results and approved the final version of the manuscript.

– Funding

Not applicable.

– Conflicts of interest

No potential conflict of interest was reported by the author(s).

Acknowledgements

The authors wish to thank the refrees for valuable suggestions and comments.

Open Access

References

[1] Ali SM, Abdo MS, Sontakke B, Shah K, Abdeljawad T. New results on a coupled system for second-order pantograph equations with ABC fractional derivatives. AIMS Mathematics. 2022; 7(10):19520–19538.

[2] Arrigo F, Durastante F. Mittag-Leffler functions and their applications in network science. SIAM Journal on Matrix Analysis and Applications. 2021; 42(4):1581–1601.

[3] Atangana A, Baleanu D. New fractional derivative with non-local and non-singular kernel: theory and application to heat transfer model. Thermal Science. 2016; 20(2):763–769.

[4] Bansal J, Kumar A, Kumar A, Khan A, Abdeljawad T. Investigation of monkeypox disease transmission with vaccination effects using fractional order mathematical model under Atangana-Baleanu Caputo derivative. Modeling Earth Systems and Environment. 2025; 11(40):1–20.

[5] Chaudhry MA, Qadir A, Srivastava HM, Paris RB. Extended hypergeometric and confluent hypergeometric functions. Applied Mathematics and Computation. 2004; 159(2):589–602.

[6] Capelas de Oliveira E, Mainardi F, Vaz Jr J. Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. The European Physical Journal Special Topics. 2011; 193:161–171.

[7] Fernandez A, Abdeljawad T, Baleanu D. Relations between fractional models with three-parameter Mittag-Leffler kernels. Advances in Difference Equations. 2020; 186:1–13.

[8] Estrada E. Fractional diffusion on the human proteome as an alternative to the multi-organ damage of SARS-CoV-2. Chaos. 2020; 30:1–14.

[9] Ghafoor A, Fiaz M, Shah K, Abdeljawad T. Analysis of nonlinear Burgers equation with time fractional Atangana-Baleanu-Caputo derivative. Heliyon. 2024; 10(13):1–25.

[10] Gorenflo R, Kilbas AA, Mainardi F, Rogosin SV. Mittag-Leffler functions, related topics and applications. Berlin (Germany): Springer-Verlag; 2014.

[11] Gross B. On creep and relaxation. Journal of Applied Physics. 1957; 28(8):906–909.

[12] Haubold HJ, Mathai AM, Saxena RK. Mittag-Leffler functions and their applications. Journal of Applied Mathematics. 2011; 2011:1–51.

[13] Kilbas AA, Sebastian N. Generalized fractional integration of Bessel function of the first kind. Integral Transforms and Special Functions. 2008; 19(12):869–883.

[14] Lemes NHT, dos Santos JPC, Braga JP. A generalized Mittag-Leffler function to describe nonexponential chemical effects. Applied Mathematical Modelling. 2016; 40(17-18):7971–7976.

[15] Mainardi F. Fractional calculus and waves in linear vescoelasticity: an introduction to mathematical models. London (Uk): Imperial College Press; 2018.

[16] Mittag-Leffler MG. Sur la nouvelle fonction $E_\alpha(x)$. Comptes Rendus de l'Academie des Sciences Paris. 1903; 137:554–558.

[17] Özarslan MA, Yilmaz B. The extended Mittag-Leffler function and its properties. Journal of Inequalities and Applications. 2014; 2014(85):1–10.

[18] Pathan MA, Bin-Saad M. Mittag-leffler-type function of arbitrary order and their application in the fractional kinetic equation. Partial Differential Equations and Applications. 2023; 4(15):1–25.

[19] Petras I, Sierociuk D, Podlubny I. Identification of parameters of a half-order system. IEEE Transactions on Signal Processing. 2012; 60(10):5561–5566.

[20] Prabhakar TR. A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Mathematical Journal. 1970; 19:7–15.

[21] Rahman G, Agarwal P, Mubeen S, Arshad M. Fractional integral operators involving extended Mittag-Leffler function as its kernel. Boletín de la Sociedad Matemática Mexicana. 2018; 24:381–392.

[22] Sado AE, Kotola BS. A mathematical model based on ABC fractional order for TB transmission with treatment interruptions in case of Bule Hora town, Ethiopia. Informatics in Medicine Unlocked. 2024; 47(3):1–13.

[23] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: theory and applications. Philadelphia (USA): Gordon and Breach; 1993.

[24] Talib I, Bilal Riaz M, Batool A, Tunç C. Exploring the lower and upper solutions approach for ABC-fractional derivative differential equations. International Journal of Applied and Computational Mathematics. 2024; 10(170):1–14.

[25] Sneddon I.N. The use of integral transforms. New York (USA): McGraw-Hill; 1972.

[26] Srivastava HM, Choi J. Zeta and q-Zeta functions and associated series and integrals. Amsterdam (Netherlands): Elsevier Science Publishers; 2012.

[27] Srivastava HM, Manocha HL. A treatise on generating functions. Chichester (UK): John Wiley and Sons; 1984.

[28] Srivastava HM, Panda R. Some bilateral generating functions for a class of generalized hypergeometric polynomials. Journal für die reine und angewandte Mathematik. 1976; 283/284:265–274.

[29] Talib I, Bilal Riaz M, Batool A, Tunç C. Exploring the lower and upper solutions approach for ABC-fractional derivative differential equations. International Journal of Applied and Computational Mathematics. 2024; 10(170):1–14.

[30] Tarasov VE, Tarasova VV. Dynamic Keynesian model of economic growth with memory and lag. Mathematics. 2019; 7(2):1–17.

[31] Uçar S, Uçar E, Özdemir N, Hammouch Z. Mathematical analysis and numerical simulation for a smoking model with Atangana-Baleanu derivative. Chaos Solitons & Fractals. 2019; 118:300–306.

[32] Uchaikin VV. Fractional derivatives for physicists and engineers. Berlin (Germany): Springer; 2013.

[33] Wiman A. Über den fundamentalsatz in der theorie der funktionen $E_\alpha(x)$. Acta Mathematica. 1905; 29:191–201.