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Center and Quasi Center on Banach Normal Hyperalgebra[1]

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Abstract

In this paper, we prove some results related to a normal hyperalgebra. Also, we prove that if X is a normed normal hyperalgebra with a property $z_{aox}.z_{boy} = z_{abcxy}$ and $|\lambda| > \|x\|$, then $(z_{\lambda oe} - x)$ is invertible. Moreover, we give a characterization of the center of a unital complex Banach normal hyperalgebra with the same property. Finally, we define the quasi-center, σ-quasi center and ρ-quasi center of Banach normal hyperalgebra as a generalization of the center and study some basic properties and relations between them.

[1] ***Keywords***— Banach hyperalgbera, normal hyperalgebra, Hahn Banach Theorem, center, quasi-center.

1 Preliminaries and Introduction

The theory of hyperalgebra starts in 1934 by F. Marty when he define the concept of hypergroup in his paper [4] at the eight congress of Scandinavian mathematician in Stockholm. After that many new definitions of the hyperstructure theory appear. The concept of hypervector space is introduced for the first time in 1988 by M. S. Tallini in [5]. Also, she defined the normed hypervector space in 1990 in [5] which forms a fundamental base of this paper. Recently, in [7]. Ali Taghavi and Roja Hosseinzadeh define the normal weakhyypervector space and study this properties. Morever, they prove the Hahn Banach Theorem and some of its results in hypervector spaces in [13]. On the other hand, A. Taghavi and R. Parviniazadeh define the Banach hyperalgebra in [6]. Also, in 2016 they prove the Gelfand theorem for Banach hyperalgebras in [7]. In this paper, we prove some results related to a normal hyperalgebra.Then, we a prove a generalization of Louivilles theorem on normal hyperalgebra as a result of the Hanhn Banach Theorem of functional on hypervector spaces. After that, we use this result on the proof of the characterization of the center of normal hyperalgebra. Finally, in the last two sections, we define the quasi center, σ-quasi center and ρ-quasi center on Banach normal hyperalgebra, study

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properties of them and the relations between them. Through out this paper, the field F that we use is either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. Also, $P^*(X)$ is refereed to the family of all non empty subsets of X. Let us begin with the definition of a normed hypervector space which introduced by M. S. Tallini [5].

Definition 1.1. *[8] Let F be a field. A hypervector space over F is a quadruplet $(X, +, \circ, F)$ such that $(X, +)$ is an abelian group and $\circ : F \times X \to P^*(X)$ such that for any $x, y \in X$ and $a, b \in F$ the following conditions hold:*

1. $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$
2. $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$
3. $a \circ (b \circ x) = (ab) \circ x$
4. $(-a) \circ x = a \circ (-x) = -(a \circ x)$
5. $x \in 1 \circ x$, *where* 1 *is the identity element of* F.

By [8] we note in the condition 1 of Definition 1.1, the sum of $(a \circ x) + (b \circ x)$ is meant in the sense of Frobenius, that is, $(a \circ x) + (b \circ x) = \{s + t : s \in (a \circ x), t \in (b \circ x)\}$. Also, a hypervector space is called anti-left distributive if the inverse inclusion in Definition 1.1 condition 1 holds, that is,

$$(a + b) \circ x \supseteq (a \circ x) + (b \circ x)$$

and strongly left distributive if equality in Definition 1.1 condition 1 hold, that is,

$$(a + b) \circ x = (a \circ x) + (b \circ x)$$

Similarly, a hypervector space is called an anti-right distributive and strongly right distributive hypervector spaces if the inverse inclusion and equality in Definition 1.1 condition 2 hold, respectively. Moreover, a hypervector space is called strongly distributive if it is both strongly left and strongly right distributive. Finally, in the condition 3 Definition 1.1, $a \circ (b \circ x) = \{t : t \in (a \circ y), \quad \text{such that} \quad y \in (b \circ x)\}$.

Definition 1.2. *[5] Let X be a hypervector space over a hyperfield F. A pseudonorm on X is a mapping $\|.\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $a \in F$ the following peoperties hold:.*

1. $\|0\| = 0,$
2. $\|x + y\| \leq \|x\| + \|y\|,$
3. $\sup \|a \circ x\| = |a|.\ \|x\|, \quad$ *where* $\sup \|a \circ x\| = \sup\{\|t\| : t \in a \circ x\}.$

A pseudonorm on X is called a norm if the following condition satisfied, $\|x\| = 0$ if and only if $x = 0$.

Remark 1.3. *In order to the third condition of a pseudonorm be well defined, we must assume that $a \circ x$ is a closed and bounded subset of X.*

A normed hypervector space ia a hypervector space with a norm $\|.\|$.
Let $(X, +, \circ, F)$ be a normed hypervector space. For $x \in X$ and $\epsilon > 0$, the open ball $B_\epsilon(x)$ is defined in [12], by $B_\epsilon(x) = \{y \in X : \|x - y\| < \epsilon\}$. The set of all open balls $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ form a basis for the topology on X which induced by this norm.

Definition 1.4. *[10] Let (x_n) be a sequence in a normed hypervector space $(X, +, \circ, \|.\|, F)$. This sequence converge to a point $x \in X$, if for every $\epsilon > 0$, there exists a positive number m such that $\|x_n - x\| < \epsilon$, for every $n \geq m$ and we write $\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0$.*

Definition 1.5. *[10] Let $(X, +, \circ, \|.\|, F)$ be a normed hypervector space. A sequence (x_n) in X is said to be a Cauchy sequence if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$, for every $m, n \geq N$.*

Definition 1.6. *[11] A hyperBanach space X is a complete normed hypervector space. That is, every Cauchy sequence in X is convergent.*

Definition 1.7. *[9] Let $(X, +_1, \circ_1, F), (Y, +_2, \circ_2, F)$ be two hypervector spaces over a field F. A homomorphism between X and Y is a mapping $f : X \to Y$ such that for all $a \in F$ and $x, y \in X$ f satisfies, $f(x +_1 y) = f(x) +_2 f(y)$ and $f(a \circ_1 x) \subseteq a \circ_2 f(x)$.*

Also, in [9] a strongly homomorphism is a homomorphism such that the equality in the second condition holds, that is: $f(a \circ_1 x) = a \circ_2 f(x)$.

Definition 1.8. *[10] Let $(X, +_1, \circ_1, \|.\|_1, F), (Y, +_2, \circ_2, \|.\|_2, F)$ be two normed hypervector spaces over F. A homomorphism or a strongly homomorphism $f : X \to Y$ is said to be bounded if there exists $K \geq 0$ such that $\|f(x)\|_2 \leq K. \|x\|_1$, for every $x \in X$.*

2 Normal hyperalgebra

Definition 2.1. *[6] Let $(X, +, \circ, F)$ be a hypervector space over a field F. Then X is called a hyperalgebra over the field F if there exist a mapping $X \times X \to X$ of (x, y) into $x.y \in X$ such that:*

1. $(x.y).z = x.(y.z)$
2. $(x + y).z = x.z + y.z,$
3. $x.(y + z) = x.y + x.z$
4. $(c \circ x).y = c \circ (x.y) = x.(c \circ y)\ \forall x, y, z \in X, \forall c \in F.$

Let X be a hyperalgebra and Y be a nonempty subset of X. Then Y is said to be a subhyperalgebra of X if Y is subhypervector space such that $x.y \in Y$ whenever $x, y \in Y$.

Definition 2.2. *[6] A normed hyperalgebra is a hyperalgebra which is normed as a hypervector space and in which $\|x.y\| \leq \|x\|\ \|y\|$.*

Definition 2.3. *[6] A Banach hyperalgebra is a complete normed hyperalgebra.*

A Banach hyperalgebra is called real or complex when the underlying hyperalgebra is real or complex.

Proposition 2.4. *[6] Let X be a normed hyperalgebra and $x \in X$, $a \in F$. Then there exists an essential point $z_{a\circ x} \in a \circ x$ has the property $\|z_{a\circ x}\| = \sup \|a \circ x\|$.*

Remark 2.5. *[6] We call the elements $z_{a\circ x} \in a \circ x$ which satisfy $\|z_{a\circ x}\| = \sup \|a \circ x\|$ the essential points of $a \circ x$. This essential points have property that $x \in a^{-1} \circ z_{a\circ x}$ when $a \neq 0$ and $z_{a\circ x} = 0$ when $a = 0$,.*

Remark 2.6. *The essential point $z_{a\circ x}$ for $a \circ x$ in a hyperalgebra is not unique. So let $Z_{a\circ x}$ be the set of all essential points for $a \circ x$. The next example shows that the essential point is not unique where we construct this example according to an example about weak hypervector space in [15] .*

Example 2.7. *Let $X = \mathbb{C}$ be the normed hyperalgebra of the set of complex numbers over the field of $\mathbb{R}$ with operations of usual sum, the follwing hyperscalar multiplication mapping $\circ : \mathbb{R} \times \mathbb{C} \to P^*(\mathbb{C})$ is defined by the following*

$$a \circ x = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a|\|x\|, 0 \leq \theta \leq 2\pi\} & \text{if} \quad a \neq 0 \quad \text{and} \quad x \neq 0 \\ \{0\} & \text{if} \quad a = 0 \quad \text{or} \quad x = 0 \end{cases}$$

Together with usual multiplication and norm defined by $\|x = (x_1, x_2)\| = \sqrt{(x_1^2 + x_2^2)}$. Then $(\mathbb{C}, +, \circ, ., \|.\|, \mathbb{R})$ is a normed hyperalgebra with the set of essential points given by

$$Z_{a\circ x} = \begin{cases} \{re^{i\theta} : r = |a|\|x\|, 0 \leq \theta \leq 2\pi\} & \text{if} \quad a \neq 0 \quad \text{and} \quad x \neq 0 \\ \{0\} & \text{if} \quad a = 0 \quad \text{or} \quad x = 0 \end{cases}$$

Definition 2.8. *[6] A hyperalgebra X is called a normal hyperalgebra if the following two conditions holds for all $a, b \in F$ and for any $x, y \in X$*

1. $z_{(a+b)\circ x} = z_{a\circ x} + z_{b\circ x}$,
2. $z_{a\circ(x+y)} = z_{a\circ x} + z_{a\circ y}$.

From the above definition we directly get the following Proposition.

Proposition 2.9. *Let X be normal hyperalgebra. Then following two conditions holds for all $a, b \in F$ and for any $x, y \in X$*

1. $Z_{(a+b)\circ x} \cap [Z_{a\circ x} + Z_{b\circ x}] \neq \phi$,
2. $Z_{a\circ(x+y)} \cap [Z_{a\circ x} + Z_{a\circ y}] \neq \phi$.

The following Lemma is similar to Lemma 2.15 in the case of weak hypervector space in [15] with a similar proof.

Lemma 2.10. *Let X be a hyperalgebra over F such that $x \in X$ and $a, b \in F$. The following properties hold:*

1. $x \in Z_{1\circ x}$,
2. *Let* $b \neq 0$, *then* $a \circ z_{b\circ x} = ab \circ x$,
3. $Z_{-a\circ x} = -Z_{a\circ x}$,

4. *If $a \neq 0$, then there is $y \in X$ such that $x \in Z_{a\circ y}$,*

5. *If X is normal, then $Z_{a\circ x}$ is a singleton.*

Proof. 1. Since $x \in 1 \circ x$, then $x \in Z_{1\circ x}$,

2. Let $b \neq 0$, then $z_{b\circ x} \in b \circ x$, which implies, $a \circ z_{b\circ x} \subseteq a \circ b \circ x = ab \circ x$. Conversely, since, $x \in b^{-1} \circ z_{b\circ x}$, then $ab \circ x \subseteq a \circ b \circ b^{-1} \circ z_{b\circ x} = a \circ z_{b\circ x}$. Thus, $a \circ z_{b\circ x} = ab \circ x$,

3.

$$\begin{aligned} Z_{-a\circ x} &= \{z : z \in -a \circ x, x \in (-a^{-1}) \circ z\}. \\ &= \{z : -z \in a \circ x, x \in (a^{-1}) \circ -z\}. \\ &= \{-z : z \in a \circ x, x \in (a^{-1}) \circ z\}. \\ &= -\{z : z \in a \circ x, x \in (a^{-1}) \circ z\} = -Z_{a\circ x}. \end{aligned}$$

4. Let $y = z_{a^{-1}\circ x}$, then by part 2, we have, $a \circ y = a \circ z_{a^{-1}\circ x} = aa^{-1} \circ x = 1 \circ x$ and so by part 1 , $x \in Z_{1\circ x} = Z_{a\circ y}$. Thus, $x \in Z_{a\circ y}$,

5. Let X be normal, then $[Z_{-a\circ x} + Z_{a\circ x}] \cap Z_{(-a+a)\circ x} \neq \phi$, but $Z_{(-a+a)\circ x} = Z_{0\circ x} = \{0\}$ by the Definition of the essential points. Therefore, $Z_{-a\circ x} + Z_{a\circ x} = \{0\}$ and by part 3, $Z_{-a\circ x} = -Z_{a\circ x}$. So $-Z_{a\circ x} + Z_{a\circ x} = \{0\}$ and this hold only when $Z_{a\circ x}$ is a singleton because if $Z_{a\circ x}$ has more than one essential point, then $-Z_{a\circ x} + Z_{a\circ x} = \{0\}$ is a set of at least three elements. Thus, $Z_{a\circ x}$ must be a singleton.

■

Remark 2.11. *The converse of Part 5 in Proposition 2.10, need not be true in general. That is there is a hyperalgebra with a a unique essential point but not normal. In the following example we show that the essential point is unique but the hyperalgebra is not normal which we construct it from a similar example about weak hypervector spaces in [15] .*

Example 2.12. *Let $X = \mathbb{R}^2$ over $\mathbb{R}$ with usual addition, usual product together with a hyperscalar multiplication $\circ : \mathbb{R} \times \mathbb{R}^2 \to P^*(X)$ defined by*

$$a \circ x = \begin{cases} \text{segment } \overline{-0x} & \text{if } a \neq 0 \text{ and } x \leq 0 \\ \text{segment } \overline{0x} & \text{if } a \neq 0 \text{ and } x \geq 0 \\ \{0\} & \text{if } a = 0 \text{ or } x = 0 \end{cases}$$

. Then $(\mathbb{R}^2, +, \circ, .)$ is a hyperalgebra with the set of essential points given by

$$Z_{a\circ x} = \begin{cases} \{-x\} & \text{if } a \neq 0 \text{ and } x \leq 0 \\ \{x\} & \text{if } a \neq 0 \text{ and } x \geq 0 \\ \{0\} & \text{if } a = 0 \text{ or } x = 0 \end{cases}$$

However, $(\mathbb{R}^2, +, \circ, .)$ is not normal because for any $a, b \in R$ such that $a, b \leq 0$ then $Z_{a\circ x} = Z_{b\circ x} = \{-x\}$ and so $Z_{a\circ x} + Z_{b\circ x} = \{-x\} + \{-x\} = \{-2x\}$ and $z_{(a+b)\circ x} = \{-x\}$. Thus, $z_{(a+b)\circ x} \cap z_{a\circ x} + z_{b\circ x} = \emptyset$. Hence, $(\mathbb{R}^2, +, \circ, .)$ is not normal.

Definition 2.13. *[6] Let X be a hyperalgebra. An element $e \in X$ is called an identity or a unit if for every $x \in X$, $e.x = x.e = x$. In this case, we say that X is a unital hyperalgebra.*

Definition 2.14. *[6] Let X be a unital hyperalgebra. An element $x \in X$ is said to be invertible if it has an inverse in X, that is, there exists an element $x^{-1} \in X$ such that, $x.x^{-1} = x^{-1}.x = e$, where e is the identity element in X.*

Remark 2.15. *[6] In a unital hyperalgebra X,*

1. *Any nonzero element x in X, has at most one inverse,*
2. *The set of all invertible elements is denoted by $Inv(X)$. The complement of $Inv(X)$ in X is the set of all non invertible elements in X and it's denoted by $Sing(X)$.*

Lemma 2.16. *[6] Let $(X, \|.\|)$ be a Banach hyperalgebra. If $x \in X$ with $\|x\| < 1$, then $(e - x) \in Inv(X)$.*

Lemma 2.17. *[6] Let $(X, \|.\|)$ be a Banach hyperalgebra. Then, $Inv(X)$ the set of all invertible elements in X, is an open set in X.*

Definition 2.18. *[6] Let X be a hyperalgebra and $x \in X$. The spectrum of x, is denoted by $\sigma_X(x)$ or simply $\sigma(x)$, is the set of all complex numbers λ such that $(z_{\lambda oe} - x) \in Sing(X)$. That is,*

$$\sigma(x) = \{\lambda \in \mathbb{C} : (z_{\lambda oe} - x) \in Sing(X)\}$$

The complement of $\sigma(x)$ in $\mathbb{C}$ is called the resolvent of x and it's denoted by $\rho(x)$.

Theorem 2.19. *[6] Let $(X, \|.\|)$ be a Banach normal hyperalgebra and $x \in X$. Then $\sigma(x)$ is nonempty.*

Theorem 2.20. *[6] Let $(X, \|.\|)$ be a Banach normal hyperalgebra and $x \in X$. Then $\sigma(x)$ is bounded in $\mathbb{C}$ and is contained in the closed disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.*

Definition 2.21. *[2] Let $(X, \|.\|)$ be a normed hyperalgebra with $x \in X$. The spectral radius of x, denoted by $r(x)$, is defined by*

$$r(x) = \inf\{\|x^n\|^{\frac{1}{n}} : n = 1, 2, ...\}.$$

Theorem 2.22. *[2] Let $(X, \|.\|)$ be a Banach hyperalgebra with a unit element and $x \in X$ such that $r(x) < 1$. Then, $(e - x)$ is invertible and $(e - x)^{-1} = e + \sum_{n=1}^{\infty} x^n$*

Theorem 2.23. *Let X be a complex Banach normal hyperalgebra and $x \in X$, then $\sigma(x)$ is a compact subset of $\mathbb{C}$.*

Proof. By Theorem 2.20 , $\sigma(x)$ is bounded. So, it remains to prove that $\sigma(x)$ is closed. let $\lambda_0 \notin \sigma(x)$, then $(z_{\lambda_0 oe} - x) \in Inv(X)$. Define $f : \mathbb{C} \to X$ by $f(\lambda) = z_{\lambda oe} - x$, then f is a continuous function and since by Lemma 2.17, $Inv(X)$ is an open subset of X containing $f(\lambda_0)$, then there exists an open subset $S(f(\lambda_0))$ such that $S(f(\lambda_0)) \subseteq Inv(X)$. Since f is continuous, then as $\lambda \to \lambda_0$ we have $f(\lambda) \to f(\lambda_0)$ and so there exists a neighborhood $N(\lambda_0)$ such that for $\lambda \in N(\lambda_0)$, we have $f(\lambda) = (z_{\lambda oe} - x) \in S(f(\lambda_0)) \subseteq Inv(X)$. So, $\lambda \notin \sigma(x)$. Therefore, there exists $N(\lambda_0)$ such that $N(\lambda_0) \subseteq \mathbb{C} - \sigma(x)$. Thus $\rho(x) = \mathbb{C} - \sigma(x)$ is an open set and thus, $\sigma(x)$ is a closed. Hence, $\sigma(x)$ is a compact set. ■

Proposition 2.24. *Let X be normed normal hyperalgebra such that $z_{aox}.z_{boy} = z_{aboxy}$ for all $a, b \in \mathbb{C}, x, y \in X$. If $\lambda \in \mathbb{C}$ such that $|\lambda| > \|x\|$, then $z_{\lambda oe} - x$ is invertible.*

Proof. Let $|\lambda| > \|x\|$ then $|\lambda|^{-1}\|x\| < 1$ but $|\lambda|^{-1}\|x\| = \sup \|\lambda^{-1} \circ x\| < 1$ which implies for the essential point $z_{\lambda^{-1}ox}$ of $\lambda^{-1} \circ x$, $(e - z_{\lambda^{-1}ox})$ is invertible but $z_{\lambda oe}$ is invertible (it's inverse is $z_{\lambda^{-1}oe}$) and so $z_{\lambda oe}.(e - z_{\lambda^{-1}ox}) = e.z_{\lambda oe} - z_{\lambda oe}.z_{\lambda^{-1}ox} = z_{\lambda oe} - z_{\lambda\lambda^{-1}oe.x} = z_{\lambda oe} - x$ is inverible. ■

3 Center in Hyperalgebra

In order to measure how close any Banach hyperalgebra X to be a commutative hyperalgebra we define the center Z(X) by

$$Z(X) = \{x \in X : xy = yx \quad \text{for all} \quad y \in X\}.$$

It's easy to see that $Z(x)$ is a commutitve subhyperalgebra of X that contain the identity. Moreover, If X is commutative, then $Z(X) = X$. In what follows we will give a characterization of the center of a special class of a unital Banach hyperalgebra namely, a unital Banach normal hyperalgebra. This characterization is done by way of slight modification of the characterization of the center of a classical unital Banach algebra.

Firstly, we need to define the exponential function in a way similar to the definition of it on a unital complex Banach algebra and stay the Hahn Banach Theorem for functionals on hypervector spaces, then we use it to prove the generalization of Liouville's Theorem.

Definition 3.1. *Let X be a unital complex Banach normal hyperalgebra. For an element $x \in X$, the $\exp(x)$ is defined by*

$$\exp(x) = e + \sum_{n=1}^{\infty} z_{\frac{1}{n!} \circ x^n}$$

where $z_{\frac{1}{n!}\circ x^n}$ is the essential point of $\frac{1}{n!} \circ x^n$.

The proof of the following Lemma is similar to the case in Banach algebra as in [14] with a slight modification as follows.

Lemma 3.2. *Let X be a unital complex Banach normal hyperalgebra with a property $z_{a\circ x}.z_{b\circ y} = z_{ab\circ xy}$ for all $x, y \in X$ and $a, b \in F$. Then*

1. $\exp(x + y) = \exp(x)\exp(y)$, *where $xy = yx$,*
2. $(\exp(x))^{-1} = \exp(-x)$,
3. $\exp(x) = \lim_{n\to\infty}(e + z_{\frac{1}{n}\circ x})^n$.

Proof. 1. Let $x_n, y_n, z_n, \xi_n, \eta_n, \zeta_n$ be defined by:

$$x_n = e + \sum_{k=1}^{n} z_{\frac{1}{k!}\circ x^k}, \qquad y_n = e + \sum_{k=1}^{n} z_{\frac{1}{k!}\circ y^k}, \qquad z_n = e + \sum_{k=1}^{n} z_{\frac{1}{k!}\circ (x+y)^k},$$

$$\xi_n = 1 + \sum_{k=1}^{n} \frac{1}{n!}\|x\|^k, \qquad \eta_n = 1 + \sum_{k=1}^{n} \frac{1}{k!}\|y\|^k, \qquad \zeta_n = 1 + \sum_{k=1}^{n} \frac{1}{k!}(\|x\| + \|y\|)^k.$$

Then, $x_n y_n - z_n = \sum_{j,k=1}^{n} z_{\alpha_{jk}\circ x^j y^k}$ where $\alpha_{jk} \geq 0$ for all j, k and so $\|x_n y_n - z_n\| = \|\sum_{j,k=1}^{n} z_{\alpha_{jk}\circ x^j y^k}\| \leq \sum_{j,k=1}^{n} \|z_{\alpha_{jk}\circ x^j y^k}\| \leq \sum_{j,k=1}^{n} |\alpha_{jk}|\|x^j\|\|y^k\| \leq \sum_{j,k=1}^{n} \alpha_{jk}\|x\|^j\|y\|^k = \xi_n\eta_n - \zeta_n$.
Since $\lim_{n\to\infty}(\xi_n\eta_n - \zeta_n) = \exp(\|x\|)\exp(\|y\|) - \exp(\|x\| + \|y\|) = 0$ Since the norm is continuous we have,

$$0 = \lim_{n\to\infty}(x_n y_n - z_n) = (e + \lim_{n\to\infty}\sum_{k=1}^{n} z_{\frac{1}{k!}\circ x^k})(e + \lim_{n\to\infty}\sum_{k=1}^{n} z_{\frac{1}{k!}\circ y^k})$$

$$= (e + \lim_{n\to\infty} \sum_{k=1}^{n} z_{\frac{1}{k!} \circ (x+y)^k}) = \exp(x)\exp(y) - \exp(x+y).$$

Thus, $\exp(x)\exp(y) = \exp(x+y)$

2. Since $\exp(x)\exp(y) = \exp(x+y)$. Let $y = -x$, then $\exp(x)\exp(-x) = \exp(x + -x) = e$. Thus, $(\exp(x))^{-1} = \exp(-x)$.

3. Let x_n, y_n, ξ_n, η_n be defined by:

$$x_n = e + \sum_{k=1}^{n} z_{\frac{1}{k!} \circ x^k}, \qquad y_n = (e + z_{\frac{1}{n} \circ x})^n, \qquad \xi_n = 1 + \sum_{k=1}^{n} \frac{1}{k!}\|x\|^k, \qquad \eta_n = (1 + \frac{1}{n}\|x\|)^n$$

Then, $x_n - y_n = \sum_{k=2}^{n} z_{\alpha_k \circ x^k}$ where $\alpha_k \geq 0$ for all k and so we have, $\|x_n - y_n\| = \|\sum_{k=2}^{n} z_{\alpha_k \circ x^k}\| \leq \sum_{k=2}^{n} \|z_{\alpha_k \circ x^k}\| = \sum_{k=2}^{n} |\alpha_k| \|x^k\| \leq \sum_{k=2}^{n} \alpha_k \|x\|^k = xi_n - \eta_n$. Since $\lim_{n\to\infty}(\xi_n - \eta_n) = 1 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!}\|x\|^k - \lim_{n\to\infty}(1 + \frac{1}{n}\|x\|)^n = \exp(\|x\|) - \exp(\|x\|) = 0$. By the continuity of norm, $0 = \lim_{n\to\infty}(x_n - y_n) = e + \lim_{n\to\infty} \sum_{k=1}^{n} z_{\frac{1}{k!} \circ x^k} - \lim_{n\to\infty}(e + z_{\frac{1}{n} \circ x})^n = \exp(x) - \lim_{n\to\infty}(e + z_{\frac{1}{n} \circ x})^n$. Thus, $\exp(x) = \lim_{n\to\infty}(e + z_{\frac{1}{n} \circ x})^n$

■

Definition 3.3. *[13] Let X be a hypervector space and Y be a nonempty subset of X. Then, Y is a weak subhypervector space of X if the following two conditions holds:*

1. $y_1 + y_2 \in Y$
2. $z_{a \circ y} \in Y$, *for all $y, y_1, y_2 \in Y$ and $a \in F$.*

Definition 3.4. *[13] Let X be a hypervector space over F. A mapping $f : X \to F$ is said to be a weak linear functional if and only if f is additive and $f(z_{a \circ x}) = af(x)$ holds for all $x \in X$ and $a \in F$.*

We note that if X is a normed hypervector space over F, we will detone to the set of all bounded weak linear functionals on X by X_h^*.

Definition 3.5. *[13] Let X be a hypervector space. A sublinear functional is a real valued function $g : X \to \mathbb{R}$ which is*

1. $g(x+y) \leq g(x) + g(y)$ *for all $x, y \in X$*
2. $\sup g(a \circ x) = ag(x) = g(z_{a \circ x})$ *for all $x \in X$ and all $a \geq 0$.*

Theorem 3.6. *[13] Let X be a real normal hypervector space and p is a sublinear functional on X. Furthermore, Let f be a weak linear functional which is defined on a weak subhypervector space M of X and satisfies $f(x) \leq p(x)$ for all $x \in M$. Then there exists a weak linear functional $g : X \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in M$ and $g(x) \leq p(x)$ for all $x \in X$.*

Corollary 3.7. *[13] Let x be in a normed normal hypervector space X over F.Then we have $\|x\| = \sup\{\frac{|f(x)|}{\|f\|} : f \in X_h^* : f \neq 0\}$. Hence, if x_0 is such that $f(x_0) = 0$ for all $f \in X_h^*$, then $x_0 = 0$.*

Definition 3.8. *Let X be a normal hyperBanach space and $f : \mathbb{D} \to X$ is a mapping on a domain $\mathbb{D}$ in the complex plane $\mathbb{C}$ into X. Then f is called analytic on $\mathbb{D}$ if and only if*

$$\lim_{h\to 0}[z_{h^{-1}\circ f(x+h)} - z_{h^{-1}\circ f(x)}]$$

exists for every $x \in \mathbb{D}$

In the usual way we say that f is an integral function if it is analytic on $\mathbb{C}$. Moreover, we define it to be a bounded if $\|f(x)\| \leq M$ for all $x \in \mathbb{C}$. Now we are ready to prove the generalized Liouville theorem on hypervector spaces.

Theorem 3.9. *Let X be a normal hyperBanach space and $f : \mathbb{C} \to X$ be a bounded integral function, then f is constant.*

Proof. Let $g \in X_h^*$ and $f : \mathbb{C} \to X$ be a bonded integral function, Then there is M such that $\|f(x)\| \leq M$ and so, $|g(f(x))| \leq \|g\|\|f(x)\| \leq \|g\|M$ on $\mathbb{C}$ and gf is bounded. Since gf is also integral function in a complex variable sense, the ordinary Liouville's theorem yields $g(f(x)) = g(f(\acute{x}))$, for any $x, \acute{x} \in \mathbb{C}$ and so $g(f(x) - f(\acute{x})) = 0$ for any $x, \acute{x} \in \mathbb{C}$. Thus, by Corollary3.7, we have $f(x) - f(\acute{x}) = 0$ and so $f(x) = f(\acute{x})$ for any $x, \acute{x} \in \mathbb{C}$. Therefore, f is a constant. ■

In what follows we will prove the main Proposition in this section which determines the characterization of the center of a unital complex Banach normal hyperalgebra in a way similar to the same characterization of the center of a unital complex Banach algebra with slight modifications.

Proposition 3.10. *Let X be a unital complex Banach normal hyperalgebra with a property $z_{\lambda_1\circ x}.z_{\lambda_2\circ y} = z_{\lambda_1\lambda_2\circ xy}$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. If $\|a(z_{\lambda\circ e} - x)\| \leq \|(z_{\lambda\circ e} - x)a\|$ for all $a \in X$ and $\lambda \in \mathbb{C}$, then $x \in Z(X)$.*

Proof. Let $|\lambda| > \|x\|$ then by Proposition 2.24, $(z_{\lambda\circ e} - x)^{-1}$ exists and so $(z_{\lambda\circ} - x)^{-1}y \in X$ for any $y \in X$. Moreover, by assumption, $\|a(z_{\lambda\circ e} - x)\| \leq \|(z_{\lambda\circ e} - x)a\|$ for all $a \in X$ and $\lambda \in \mathbb{C}$. Choose $a = (z_{\lambda\circ e} - x)^{-1}y \in X$. Then,

$$\|(z_{\lambda\circ e} - x)^{-1}y(z_{\lambda\circ e} - x)\| \leq \|(z_{\lambda\circ e} - x)(z_{\lambda\circ e} - x)^{-1}y\| = \|y\| \quad \text{for all} \quad a \in X \quad \text{and} \quad \lambda \in \mathbb{C}.$$

Fix $u \in \mathbb{C}$ such that $u \neq 0$ and let $n \in \mathbb{Z}^+$ such that $\frac{n}{|u|} > \|x\|$. Also, let $\lambda = \frac{n}{u} \in \mathbb{C}$. Then, for any $y \in X$ we have, $\|(z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x)\| \leq \|y\|$ and since $\sup \|1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x)\| = |1|.\|(z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x)\| \leq \|y\|$. So ,

$$\sup \|1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x)\| \leq \|y\|.$$

Since $1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x) = 1.1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}y(z_{\frac{n}{u}\circ e} - x) = (1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}).y.(1 \circ (z_{\frac{n}{u}\circ e} - x))$ we have,

$$\sup \|(1 \circ (z_{\frac{n}{u}\circ e} - x)^{-1}).y.(1 \circ (z_{\frac{n}{u}\circ e} - x))\| \leq \|y\|$$

Writing the first one by $1 = (\frac{n}{u})^{-1}.\frac{n}{u}$ and the second one by $1 = \frac{n}{u}.\frac{u}{n}$ so we have $(1\circ(z_{\frac{n}{u}\circ e} - x)^{-1}).y.(1\circ(z_{\frac{n}{u}\circ e} - x)) = ((\frac{n}{u})^{-1}.\frac{n}{u}\circ(z_{\frac{n}{u}\circ e} - x)^{-1}).y.(\frac{n}{u}.\frac{u}{n}\circ(z_{\frac{n}{u}\circ e} - x)) = (\frac{n}{u})^{-1}\circ(\frac{n}{u}\circ(z_{\frac{n}{u}\circ e} - x)^{-1}).y.(\frac{n}{u}\circ(\frac{u}{n}\circ(z_{\frac{n}{u}\circ e} - x))) = (\frac{n}{u})^{-1}\circ(1 \circ e - \frac{u}{n}\circ x)^{-1}.y.\frac{n}{u}\circ(1 \circ e - \frac{u}{n}\circ x)$. So we have,

$$\sup \|((\frac{n}{u})^{-1} \circ (1 \circ e - \frac{u}{n} \circ x)^{-1}).y.(\frac{n}{u} \circ (1 \circ e - \frac{u}{n} \circ x))\| \leq \|y\|$$

and hence,

$$\sup \|((\frac{n}{u})^{-1}.\frac{n}{u})\circ(1\circ e-\frac{u}{n}\circ x)^{-1}.y.(1\circ e-\frac{u}{n}\circ x))\| \leq \|y\| = \sup \|(1\circ e-\frac{u}{n}\circ x)^{-1}.y.(1\circ e-\frac{u}{n}\circ x)\| \leq \|y\|$$

But $(e - z_{\frac{u}{n}\circ x})^{-1}.y.(e - z_{\frac{u}{n}\circ x}) \in (1\circ e - \frac{u}{n}\circ x)^{-1}.y.(1\circ e - \frac{u}{n}\circ x)$

Hence,

$$\|(e - z_{\frac{u}{n}\circ x})^{-1}.y.(e - z_{\frac{u}{n}\circ x})\| \leq \|y\|$$

By induction, we have

$$\|(e - z_{\frac{u}{n}\circ x})^{-m}.y.(e - z_{\frac{u}{n}\circ x})^{m}\| \leq \|y\| \qquad \text{for all} \quad m \in \mathbb{N}$$

. Then,

$$\|((e - z_{\frac{u}{n}\circ x})^{-n}.y.(e - z_{\frac{u}{n}\circ x})^{n}\| \leq \|y\|$$

By taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \|((e - z_{\frac{u}{n}\circ x})^{-n}.y.(e - z_{\frac{u}{n}\circ x})^{n}\| \leq \|y\|$$

By continuity of the norm we have

$$\| \lim_{n\to\infty} (e - z_{\frac{u}{n}\circ x})^{-n}.y \lim_{n\to\infty} (e - z_{\frac{u}{n}\circ x})^{n}\| \leq \|y\|$$

Which is equivalent to

$$\| \lim_{n\to\infty} (e - z_{\frac{1}{n}\circ x})^{-un}.y \lim_{n\to\infty} (e - z_{\frac{1}{n}\circ x})^{un}\| \leq \|y\|$$

Hence, by Lemma 3.2, we have $\|(\exp x)^{u}.y.(\exp -x)^{u}\| \leq \|y\|$ for all $y \in X$ and $u \in \mathbb{C}/\{0\}$. Also this inequality is true for $u = 0$ and so it's true for all $u \in \mathbb{C}$.
Let y be any fixed element in X and let $f : \mathbb{C} \to X$ be defined by $f(u) = \exp(x)^{u} y \exp(-x^{u})$, then $\|f(u)\| = \|\exp(x)^{u} y \exp(-x)^{u}\| \leq \|y\|$ for all $u \in \mathbb{C}$. Hence, f is entire bounded function on a Banach hyperalgebra X which implies, by Theorem 3.9, f must be a constant function and so $f(u) = \exp(x)^{u} y \exp(-x)^{u} = y$. Since $y \in X$ is arbitrary in X, then, $\exp(x)^{u} y \exp(-x)^{u} = y$ for all $y \in X$ and so by lemma3.2, $\exp(x)^{u} y = y \exp(x)^{u}$. Thus,

$$(\sum_{n=0}^{\infty} z_{\frac{1}{n!}\circ x})^{u} y = y(\sum_{n=0}^{\infty} z_{\frac{1}{n!}\circ x})^{u} \qquad \text{for all} \quad y \in X \quad \text{and} u \in \mathbb{C}$$

and so, for $u = 1$ we have

$$(e + z_{1\circ x} + z_{\frac{1}{2!}\circ x} + ...)y = y(e + z_{1\circ x} + z_{\frac{1}{2!}\circ x} + ...) \qquad \text{for all} \quad y \in X$$

But X is normal so,

$$(e + x + z_{\frac{1}{2!}\circ x} + ...)y = y(e + x + z_{\frac{1}{2!}\circ x} + ...) \qquad \text{for all} \quad y \in X$$

Therefore, $xy = yx$ for all $y \in X$. Thus, $x \in Z(X)$. ■

4 σ-Quasi Center in a Banach Normal Hyperalgebra

In this section we will generalize the definition of center of a unital complex Banach normal hyperalgebra with property $z_{aox}.z_{boy} = z_{aboxy}$ to what will be called quasi-center and σ-quasi center of a unital complex Banach normal hyperalgebra. Recall in Definition 2.18, that in a Banach normal hyperalgebra with a unit element over the complex field $\mathbb{C}$ and $x \in X$, the spectrum set of x is $\sigma_X(x) = \{\lambda \in \mathbb{C} : (x - z_{\lambda oe}) \in Sing(X)\}$ and the resolvent of x is the set $\rho_X(x) = \mathbb{C} - \sigma_X(x)$.

Definition 4.1. *Let X be a unital Banach normal hyperalgebra with a unit element e over the complex field $\mathbb{C}$ such that $z_{aox}.z_{boy} = z_{aboxy}$. An element $x \in X$ is called quasi central if there exists $K \geq 1$ such that*

$$\|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$$

for all $y \in X$ and $\lambda \in \mathbb{C}$.

For any $K \geq 1$ we can write

$$Q(K, X) = \{x \in X : \|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\| \quad \text{for all} \quad y \in X \quad \text{and} \quad \lambda \in \mathbb{C}\}.$$

Moreover, the collection of all quasi central elements in a unital complex Banach normal hyperalgebra X will be denoted by $Q(X)$ which defined by

$$Q(X) = \bigcup_{K \geq 1} Q(K, X).$$

Remark 4.2. *For $K = 1$, we have $Z(X) = Q(1, X) \subseteq Q(X)$ and hence $Q(X)$ is not empty since the zero element and the unit element e are always in $Z(X)$.*

Note that in the definition of quasi central, λ is chosen from the whole of the complex numbers $\mathbb{C}$ but in the following definition the choice of λ will be restricted to the resolvent set $\rho(x)$ only which lead us to a useful definition σ-quasi central.

Definition 4.3. *Let X be a unital complex Banach normal hyperalgebra with a property $z_{aox}.z_{boy} = z_{aboxy}$. An element $x \in X$ is called σ-quasi central if there exists $K \geq 1$ such that*

$$\|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$$

for all $y \in X$ and $\lambda \in \rho(x)$.

So, for $K \geq 1$, we have

$$Q_\sigma(K, X) = \{x \in X : \|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\| \quad \text{for all} \quad y \in X \quad \text{and} \quad \lambda \in \rho(x)\}.$$

Also, the collection of all σ-quasi central elements in X will be denoted by $Q_\sigma(X)$ and is defined by

$$Q_\sigma(X) = \bigcup_{K \geq 1} Q_\sigma(K, X)$$

Moreover, we have, $Q(X) \subseteq Q_\sigma(X)$ and by Remark 4.2, $Z(X) \subseteq Q(X) \subseteq Q_\sigma(X)$

Proposition 4.4. *Let X be a unital complex Banach normal hyperalgebra with a property $z_{aox}.z_{boy} = z_{aboxy}$. Then $x \in Q_\sigma(K,X)$ if and only if $\|(z_{\lambda oe} - x)^{-1} y (z_{\lambda oe} - x)\| \leq K\|y\|$ for all $y \in X$ and $\lambda \in \rho(x)$.*

Proof. Assume that $x \in Q_\sigma(K,X)$, then by Definition 4.3, $\|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$ for all $y \in X$ and $\lambda \in \rho(x)$. Since $\lambda \in \rho(x)$, $(z_{\lambda oe} - x)^{-1}$ exists and so $(z_{\lambda oe} - x)^{-1} y \in X$. So, $\|(z_{\lambda oe} - x)^{-1} y (z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)(z_{\lambda oe} - x)^{-1} y\| = K\|y\|$.
Conversely, assume $\|(z_{\lambda oe} - x)^{-1} y (z_{\lambda oe} - x)\| \leq K\|y\|$ for all $y \in X$ and $\lambda \in \rho(x)$. Then, $\|y(z_{\lambda oe} - x)\| = \|(z_{\lambda oe} - x)^{-1}((z_{\lambda oe} - x)y)(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$. Hence, $x \in Q_\sigma(K,X)$. ■

Definition 4.5. *Let X be a Banach hyperalgebra and $x \in X$. The inner derivation with respect to x is denoted by $D_x(y)$ and is defined as by*

$$D_x(y) = xy - yx \qquad \text{for all} \quad y \in X.$$

Lemma 4.6. *Let X be a complex Banach hyper algebra and $x \in X$. Then $L_x = xy$ is the left multiplication operator and $R_x = yx$ is the right multiplication operator on X corresponding to x for all $y \in X$. Then the following properties hold where $B_h(X)$ is the set of all bounded strongly homomorphisms on X as in [2]:*

- $L_x, R_x \in B_h(X)$
- $L_x R_x = R_x L_x$
- $D_x = L_x - R_x$

Proposition 4.7. *Let X be a unital complex Banach normal hyperalgebra with a property $z_{aox}.z_{box} = z_{abox}$ and $x \in Q_\sigma(K,X)$. Then if D_x is the inner derivation corresponding to x. then*

$$\|(z_{\lambda_1 oe} - x)^{-1}(z_{\lambda_2 oe} - x)^{-1} ... (z_{\lambda_n oe} - x)^{-1} D_x^n y\| \leq (K+1)^n \|y\|.$$

for all $y \in X$ and $\lambda_i \in \rho(x)$ where $1 \leq i \leq n$.

Proof. The proof is by the induction over n. For $n = 1$, let $y \in X$ and $\lambda_1 \in \rho(x)$, we have

$$D_x y = xy - yx = -(z_{\lambda_1 oe} - x)y + y(z_{\lambda_1 oe} - x)$$

which implies,

$$(z_{\lambda_1 oe} - x)^{-1} D_x y = (z_{\lambda_1 oe} - x)^{-1} y (z_{\lambda_1 oe} - x) - y$$

By Proposition 5.4,

$$\|(z_{\lambda_1 oe} - x)^{-1} D_x y\| \leq \|(z_{\lambda_1 oe} - x)^{-1} y (z_{\lambda_1 oe} - x)\| + \|y\| \leq K\|y\| + \|y\| = (K+1)\|y\|.$$

Assume the inequality hold for some $n \geq 1$,we will show that it holds for $n+1$. Let $y \in X$ and $\lambda_i \in \rho(x)$ for $1 \leq i \leq n$ then we have $(z_{\lambda_i oe} - x)^{-1} x = x(z_{\lambda_i oe} - x)^{-1}$ because $(z_{\lambda_i oe} - x)$ commutes with x. Also,

$$D_x(z_{\lambda_i oe} - x)^{-1} y = x(z_{\lambda_i oe} - x)^{-1} y - (z_{\lambda_i oe} - x)^{-1} yx. \tag{1}$$
$$= (z_{\lambda_i oe} - x)^{-1} xy - (z_{\lambda_i oe} - x)^{-1} yx. \tag{2}$$
$$= (z_{\lambda_i oe} - x)^{-1} D_x y. \tag{3}$$

Moreover, $(z_{\lambda_i oe}-x)^{-1}(z_{\lambda_j oe}-x)^{-1}=(z_{\lambda_j oe}-x)^{-1}(z_{\lambda_i oe}-x)^{-1}$ for all $1\leq i,j\leq n+1$. Thus, $\|(z_{\lambda_1 oe}-x)^{-1}(z_{\lambda_2 oe}-x)^{-1}...(z_{\lambda_{n+1} oe}-x)^{-1}D_x^{n+1}y\|=\|(z_{\lambda_{n+1} oe}-x)^{-1}(z_{\lambda_1 oe}-x)^{-1}...(z_{\lambda_n oe}-x)^{-1}D_x^{n+1}y\|$ $=\|(z_{\lambda_{n+1} oe}-x)^{-1}D_x(z_{\lambda_1 oe}-x)^{-1}...(z_{\lambda_n oe}-x)^{-1}D_x^n y\|\leq(K+1)\|(z_{\lambda_1 oe}-x)^{-1}(z_{\lambda_2 oe}-x)^{-1}...(z_{\lambda_n oe}-x)^{-1}D_x^n y\|$. By the induction hypothesis, we have

$$\|(z_{\lambda_1 oe}-x)^{-1}(z_{\lambda_2 oe}-x)^{-1}...(z_{\lambda_{n+1} oe}-x)^{-1}D_x^{n+1}y\|\leq(K+1)^{n+1}\|y\|$$

. Thus the Proposition holds for all $n\in\mathbb{N}$. ■

Remember that a topological space is called connected if it is can't be written as the union of two disjoint open subsets and a component is a maximal connected of it.

Definition 4.8. *[1] Let K be a compact subset of $\mathbb{C}$, then $\omega(K)=\hat{K}^c$ where $\hat{K}^c$ is the unbounded componet of K^c*

Definition 4.9. *[1] Let K be a compact subset of $\mathbb{C}$. A function f is called admissible for K if f is defined and analytic on $\omega(K)$ and satisfies $\|f\|\leq 1$ and $f(\infty)=0$. A function f is a admissible for some compact subset of E.*

The set of all admissible functions for a set E will be denoted by $F(E)$.

Definition 4.10. *[1] Let E be a subset of $\mathbb{C}$. The analytic capacity of E is defined by*

$$\gamma(E)=\sup\{\lim_{z\to\infty} zf(z):f\in F(E)\}.$$

Definition 4.11. *[1] Let K be a compact set in $\mathbb{C}$ and let T be the class of all complex valued functions which are bounded and analytic on the unbounded component of the complement of K. Then if each element in T is constant, K is called a painleve null set.*

Theorem 4.12. *[1] The painleve null set coincide with the compact sets of zero analytic capacity.*

Theorem 4.13. *Let X be a complex Banach normal hyperalgebra with a unit element. If $x\in Q_\sigma(X)$ and $\sigma(x)$ has zero analytic capacity, then $D_x=0$.*

Proof. Let $x\in Q_\sigma(X)$, then there is $K\geq 1$ such that $x\in Q_\sigma(K,X)$. Fix a bounded weak linear functional ϕ on X and consider the function $f:\rho_X(x)\to\mathbb{C}$ by $f(\lambda)=\phi((z_{\lambda oe}-x)^{-1}D_x y)$. Then f is bounded since $|f(\lambda)|=|\phi((z_{\lambda oe}-x)^{-1}D_x y)|\leq\|\phi\|\|(z_{\lambda oe}-x)^{-1}D_x y\|\leq\|\phi\|(K+1)\|y\|$ the last two inqualties hold since ϕ is bounded and by Proposition 4.3, respectively. Thus f is bounded on $\rho_X(x)$. Also,

$$\frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0}=(\lambda-\lambda_0)^{-1}\phi(((z_{\lambda oe}-x)^{-1}-(z_{\lambda_0 oe}-x)^{-1})D_x y)=\phi(z_{(\lambda-\lambda_0)^{-1}.((z_{\lambda oe}-x)^{-1}-(z_{\lambda_0 oe}-x)^{-1})}D_x y)$$

Since

$$(z_{\lambda oe}-x)^{-1}-(z_{\lambda_0 oe}-x)^{-1}=(z_{\lambda oe}-x)^{-1}(z_{\lambda_0 oe}-z_{\lambda oe})(z_{\lambda_0 oe}-x)^{-1}$$

Then,

$$\phi((z_{(\lambda-\lambda_0)^{-1}}(z_{\lambda oe}-x)^{-1}-(z_{\lambda_0 oe}-x)^{-1})D_x y)=\phi(z_{(\lambda-\lambda_0)^{-1}(z_{\lambda oe}-x)^{-1}(z_{\lambda oe}-z_{\lambda_0 oe})(z_{\lambda_0 oe}-x)^{-1}}D_x y)$$

So

$$\frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} = -\phi(z_{(z_{\lambda oe}-x)^{-1}(z_{\lambda_0 oe}-x)^{-1}}D_x y)$$

By continuity of ϕ and inversion, we obtain

$$\acute{f}(\lambda_0) = \lim_{\lambda\to\lambda_0}\frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} = -\phi((z_{\lambda_0 oe}-x)^{-2}D_x y)$$

Hence, $\acute{f}(\lambda_0)$ exists for arbitrary $\lambda_0 \in \rho_X(x)$ and so f is analytic on $\rho_X(x)$. Thus, f is bounded and analytic complex valued function on unbounded component of $\rho_X(x)$ the complement of the compact subset $\sigma_X(x)$. By assumption $\sigma_X(x)$ has zero analytic capacity and since it compact then by Theorem 4.12, $\sigma_X(x)$ is a painleve null set. Moreover, Let T be the class of all bounded and analytic complex valued functions on the unbounded component of the complement of $\sigma_X(x)$, then $f \in T$ and so by Definition 4.11, f must be a constat. Finally, let $\lambda \in \{\lambda \in \mathbb{C} : |\lambda| > \|x\|\} \subseteq \rho_X(x)$, then $\|\frac{x}{\lambda}\| < 1$ and by Theorem 2.24, $(z_{\lambda oe} - x)$ is invertible and so $(z_{\lambda oe} - x)^{-1} = (z_{\lambda oe}.(e - z_{\lambda^{-1}ox}))^{-1} = z_{\lambda oe}^{-1}.(e + \sum_{n=1}^{\infty}(z_{\lambda^{-1}ox})^n) = z_{\lambda oe}^{-1} + z_{\lambda oe}^{-1}.\sum_{n=1}^{\infty}(z_{\lambda^{-1}ox})^n$ and so

$$f(\lambda) = \phi((z_{\lambda oe}^{-1} + z_{\lambda oe}^{-1}.\sum_{n=1}^{\infty}(z_{\lambda^{-1}ox})^n).D_x y) = z_{\lambda oe}^{-1}.\phi(D_x y) + z_{\lambda oe}^{-1}\phi(\sum_{n=1}^{\infty}(z_{\lambda^{-1}ox})^n).D_x y)$$

But f is a constat so $\phi(D_x y) = 0$ which implies $D_x y = 0$ where ϕ is an arbitrary bounded linear functional on X and x is an arbitrary element in X. Thus, $D_x = 0$ ■

Corollary 4.14. *Let X be a complex Banach normal hyperalgebra with a unit element. If $x \in Q_\sigma(X)$ and $\sigma(x)$ has zero analytic capacity, then $Q_\sigma(X) = Z(X)$.*

Proof. Let $x \in Q_\sigma(X)$, then by Theorem 4.13, $D_x y = 0$ which implies that $xy - yx = 0$ and so $xy = yx$ for all $y \in X$. Thus, $x \in Z(X)$. Therefore, $Q_\sigma(X) \subseteq Z(X)$, however, $Z(X) \subseteq Q_\sigma(X)$. Hence, $Q_\sigma(X) = Z(X)$. ■

5 ρ-Quasi Center in a Banach Hyperalgebra

In the previous section we generalize the concept of the quasi center $Q(X)$ which is given by $x \in Q(X)$ if there is $K \geq 1$ such that $\|(z_{\lambda oe} - x)y\| \leq K\|y(z_{\lambda oe} - x)\|$ for all $y \in X$ and $\lambda \in \mathbb{C}$ by reduce the choosing of λ only in $\rho(x)$ instead of the whole of the complex numbers $\mathbb{C}$ and this generalization produce the definition of σ- quasi center. In this section we will define a new generalization of the quasi center called ρ-quasi center. This generalization hold by choosing λ only in $\sigma(x)$. Consider the following definition.

Definition 5.1. *Let X be a Banach normal hyperalgebra with a unit element e over the complex field $\mathbb{C}$. An element $x \in X$ is called ρ-quasi central if there exists $K \geq 1$ such that*

$$\|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$$

for all $y \in X$ and $\lambda \in \sigma(x)$.

So, for $K \geq 1$, we have

$$Q_\rho(K,X) = \{x \in X : \|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\| \quad \text{for all} \quad y \in X \quad \text{and} \quad \lambda \in \sigma(x)\}.$$

Also, the collection of all ρ-quasi central elements in X will be denoted by $Q_\rho(X)$ and is defined by $Q_\rho(X) = \bigcup_{K \geq 1} Q_\rho(K,X)$. The following Proposition shows the relations between $Z(X), Q(X)$ and $Q_\rho(X)$. The prove is directly form the these definitions so we omit the proof.

Proposition 5.2. *Let X be a Banach normal hyperalgebra with a unit element. Then, $Z(X) \subseteq Q(X) \subseteq Q_\rho(X)$.*

Proposition 5.3. *Let X be a Banach normal hyperalgebra. Then $x \in Q_\rho(X)$ if and only if there is a constant L such that $\|xy - yx\| \leq L\|(z_{\lambda oe} - x)y\|$ for all $y \in X$ and $\lambda \in \sigma(x)$.*

Proof. Let $x \in Q_\rho(X)$ then there exists $K \geq 1$ such that $\|y(z_{\lambda oe} - x)\| \leq K\|(z_{\lambda oe} - x)y\|$ for all $y \in X$ and $\lambda \in \sigma(x)$. So, $\|xy - yx\| = \|y(z_{\lambda oe} - x) - (z_{\lambda oe} - x)y\| \leq \|y(z_{\lambda oe} - x)\| + \|(z_{\lambda oe} - x)y\| \leq K\|(z_{\lambda oe} - x)y\| + \|(z_{\lambda oe} - x)y\| = L\|(z_{\lambda oe} - x)y\|$ where $L = K + 1$.
Conversely, suppose that there is L such that $\|xy - yx\| \leq L\|(z_{\lambda oe} - x)y\|$ for all $y \in X$ and $\lambda \in \sigma(x)$. Then, $\|y(z_{\lambda oe} - x)\| = \|(z_{\lambda oe} - x)y + xy - yx\| \leq \|(z_{\lambda oe} - x)y\| + \|xy - yx\| \leq |(z_{\lambda oe} - x)y\| + L\|(z_{\lambda oe} - x)y\| = K\|(z_{\lambda oe} - x)y\|$ where $K = L + 1$. Hence, by Definition 5.1 , $x \in Q_\rho(X)$. ∎

Proposition 5.4. *Let X be a Banach normal hyperalgebra. Then $Q(X) = Q_\sigma(X) \cap Q_\rho(X)$.*

Proof. Since $Q(X) \subseteq Q_\sigma(X)$ and $Q(X) \subseteq Q_\rho(X)$, then $Q(X) \subseteq Q_\sigma(X) \cap Q_\rho(X)$. Conversely, Let $x \in Q_\sigma(X) \cap Q_\rho(X)$, then there is $K \geq 1$ such that $x \in Q_\sigma(K,X) \cap Q_\rho(K,X)$ which implies that $x \in Q_\sigma(K,X)$ and so $\|(z_{\lambda oe} - x)y\| \leq K\|y(z_{\lambda oe} - x)\|$ for all $y \in X$ and $\lambda \in \rho(x)$. Also, we have $x \in Q_\rho(K,X)$ and so $\|(z_{\lambda oe} - x)y\| \leq K\|y(z_{\lambda oe} - x)\|$ for all $y \in X$ and $\lambda \in \sigma(x)$. Thus, $\|(z_{\lambda oe} - x)y\| \leq K\|y(z_{\lambda oe} - x)\|$ for all $y \in X$ and $\lambda \in \mathbb{C}$. Hence, $x \in Q(K,X) \subseteq Q(X)$. Thus, $Q(X) = Q_\sigma(X) \cap Q_\rho(X)$. ∎

Proposition 5.5. *Let X be a complex Banach normal hyperalgebra with unity. If $x \in Q_\sigma(X)$ such that $\sigma(x)$ has zero analytic capacity, then $Q_\sigma(X) \subseteq Q_\rho(X)$.*

Proof. By Corollary 4.14, $Q_\sigma(X) = Z(X)$ and since $Z(X) \subseteq Q(X) \subseteq Q_\sigma(X)$ so we have $Z(X) = Q(X) = Q_\sigma(X)$, and by Proposition 5.2, $Q(X) \subseteq Q_\rho(X)$. Thus, $Q_\sigma(X) \subseteq Q_\rho(X)$. ∎

Proposition 5.6. *Let X be a complex Banach normal hyperalgebra with unity and $K \geq 1$. If Y is a closed commutative subhyperalgebra of $B_h(X)$ that containing L_x, R_x and the identity operator I. Then, for $x \in Q_\rho(X)$, we have, $\|D_xT\| \leq (K+1)\|(z_{\lambda oe} - L_x)T\|$ for all $T \in Y$ and $\lambda \in \sigma(x)$.*

Proof. Let $x \in Q_\rho(X)$, then for all $y \in X$ and $\lambda \in \sigma(x)$, we have, $\|y(z_{\lambda oe} - x)\| \leq \|(z_{\lambda oe} - x)y\|$. However, $\|y(z_{\lambda oe} - x)\| = \|(z_{\lambda oe} - R_x)y\|$ and $\|(z_{\lambda oe} - x)y\| = \|(z_{\lambda oe} - L_x)y\|$. Then $\|(z_{\lambda oe} - R_x)y\| \leq K\|(z_{\lambda oe} - L_x)y\|$. So that $\|D_xy\| = \|(z_{\lambda oe} - R_x)y - (z_{\lambda oe} - L_x)y\| \leq (K+1)\|(z_{\lambda oe} - L_x)y\|$. By replacing y by Ty where $T \in M$ in the above inequality and taking the supremum over all y such that $\|y\| = 1$ we have $\|D_xT\| =\leq (K+1)\|(z_{\lambda oe} - L_x)T\|$. ∎

6 Conclusion

In this article we use the Hahn Banach Theorem for functionals on hypervector space to prove the generalization of Liouvell's Theorem on hyperBanach spaces. Then we prove a characterization of the center of a unital complex Banach normal hyperalgebra with property $z_{aox}.z_{boy} = z_{aboxy}$. Moreover, we define the quasi center $Q(X)$, σ-quasi center $Q_\sigma(X)$ and ρ-quasi center $Q_\rho(X)$ of Banach hyperalgebra X and prove their properties and relations between them. In additions we show that for a unital complex Banach normal hyperalgebra X with the above property, if $x \in Q_\sigma(X)$ and $\sigma(x)$ has zero analytic capacity, then $Q_\sigma(X) = Q(X) = Z(X)$ and $Q_\sigma(X) \subseteq Q_\rho(X)$. Finally, we prove that $Q(X) = Q_\sigma(X) \cap Q_\rho(X)$.

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