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Generalizations of $2-$Absorbing Primary Hyperideals of Multiplicative Hyperrings

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Abstract: In this paper, we introduce the concept of $\phi-2-$absorbing primary hyperideals in multiplicative hyperrings. Several results concerning $\phi-2-$absorbing primary hyperideals are proved. We prove under certain conditions that the intersection of two $\phi-2-$absorbing primary hyperideals is also a $\phi-2-$absorbing primary hyperideal. Examples of $\phi-2-$absorbing primary hyperideals are also studied.

Keywords: Multiplicative hyperring, hyperideal, prime hyperideal.

Introduction

The hypergroup notion was introduced in 1934 by Marty (Marty, 1934) at the 8th Congress of Scandinavian Mathematicians. He defined the hypergroups as a generalization of groups. Later on, many researchers have done many papers in this field. They investigated that the theory of hyperstructures have many applications in pure and applied mathematics, for more details, see (Corsinig, 1993), Cristea & Jancic-Rasovic, 2013), (Davvaz & Leoreanu-Fotea, 2007) and (Omidi & Davvaz, 2017). Similar to hypergroups, hyperrings are algebraic structures more general than rings. Hyperrings were introduced and studied by many authors, see for example (Ameri & Norouzi, 2013), and (Asokkumar & Velrajan, 2012). There are many types of hyperrings. A well-known type of a hyperrig, called the Krasner hyperring, where the addition is a hyperoperation, while the multiplicative is an ordinary binary operation. For more study on this type of hyperrings, we refer to (Krasner, 1983), and (Rota, 1982). Another important type of a hyperring, called the multiplicative hyperring, obtained by considering the multiplication as a hyperoperation while the addition is an operation. This type of hyperring was introduced by Rota (Rota, 1982). A general type of hyperring, where both the addition and multiplication are hypeoperations can be found in (Davvaz & Leoreanu-Fotea, 2007). The notion of primeness of hyperideal in a multiplicative hyperring was conceptualized by Procesi and Rota (Procesi & Rota, 1999). Dasgupta introduced the concepts of prime and primary hyperideals in multiplicative hyrerrings (Dasgupta, 2012). The notion of $2-$absorbing and $2-$absorbing primary hyperideals in multiplicative hyperrings have been introduced and studied by Anbarloei (Anbarloei, 2017). The objective of this paper is to construct more accurate results and concepts regarding multiplicative hyperrings. In fact the motivation of writing this paper is two folded:

(1) To extend the concepts of prime, primary, $2-$absorbing and $2-$absorbing primary hyperideals in multiplicative hyperrings to the concepts of $\phi-$prime, $\phi-$primary, $\phi-2-$absorbing, and $\phi-2-$absorbing primary hyperideals respectively.

(2) To introduce the concepts of hyperideals of direct product of multiplicative hyperrings and how to classify them among absorbing hyperideals. The remains of this paper are organized as follows: Section 2 concerns some basic definitions and results in the sequel of this paper. In section 3, the main results concerning generalizations of $2-$absorbing primary hyperideals will be given. Section 4 concerns the conclusion.

Preliminary Notes

In this section we state some basic concepts and results related to hyperring theory. We hope that this will improve the readability and understanding of this paper.

In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. *Let H be a non empty set and $\mathbb{P}^*(H)$ be the family of all nonempty subsets of H.* As in (Davvaz & Leoreanu-Fotea, 2007), *a hyperoperation $\bullet$ on H is a mapping $\bullet : H \times H \longrightarrow \mathbb{P}^*(H)$. The couple $(H,\bullet)$ is called a hypergroupoid. If $A, B \in \mathbb{P}^*(H)$ and $x \in H$, then we define $A \bullet B = \bigcup_{a\in A, b\in B} a \bullet b, A \bullet x = A \bullet \{x\}$ and $x \bullet B = \{x\} \bullet B$.* The notions of semihypergroups, quasihypergroups and hypergroups are defined in (Davvaz & Leoreanu-Fotea, 2007) as follows. *A hypergroupoid $(H,\bullet)$ is called a semihypergroup if for all a, b, c of H we have $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, which means that $\bigcup_{u\in a\bullet b} u \bullet c = \bigcup_{v\in b\bullet c} a \bullet v$. A hypergroupoid $(H,\bullet)$ is called a quasihypergroup if for all a of H we have $a \bullet H = H = H \bullet a$.*(It is also called the reproduction axiom). *A hypergroupoid $(H,\bullet)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup.* Recall from (Davvaz & Leoreanu-Fotea, 2007) that a *triple $(R, +, \bullet)$ is called a multiplicative hyperring if*

(1) $(R, +)$ is an abelian group;

(2) $(R, \bullet)$ is a semihypergroup;

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(3) for all $a, b, c \in R$, we have $a \bullet (b+c) \subseteq a \bullet b + a \bullet c$ and $(b+c) \bullet a \subseteq b \bullet a + c \bullet a$;

(4) for all $a, b, c \in R$, we have $a \bullet (-b) = (-a) \bullet b = -(a \bullet b)$.

If in (3) we have equalities instead of inclusions, then we say that the multiplicative hyperring is *strongly distributive*.

A multiplicative hyperring $(R, +, \bullet)$ is said to be *commutative* if R is commutative with respect to hyperoperation $\bullet$. Throughout this paper all multiplicative hyperrings are assumed to be commutative with absorbing zero; i.e., there exists $0 \in R$ such that $x = 0 + x$ and $0 \in x \bullet 0$ for all $x \in R$. Recall from (Ameri et al., 2017), that if *$(R, +, \bullet)$ is a multiplicative hyperring, then an element $e \in R$ is called a left (resp. right) identity if $a \in e \bullet a$ (resp. $a \in a \bullet e$) for $a \in R$. e is called an identity element if it is both left and right identity element.* An element $e \in R$ is called *a left (resp. right) scalar identity* if $a = e \bullet a$ (resp. $a = a \bullet e$) for $a \in R$. e is called *a scalar identity element* if it is both left and right scalar identity element. If $(R, +, \bullet)$ is a multiplicative hyperring with identity e, then $a \in R$ is called *a left (resp. right) invertible* (with respect to e) if there exists $x \in R$ such that $e \in x \bullet a$ (resp. $e \in a \bullet x$). a is *called invertible* if it is both left and right invertible. *A multiplicative hyperring $(R, +, \bullet)$ is called unitary if it contains an element u such that $a \bullet u = u \bullet a = \{a\}$ for all $a \in R$. A nonempty subset H of a multiplicative hyperring $(R, +, \bullet)$ is called a subhyperring of $(R, +, \bullet)$ if $(H, +, \bullet)$ is a multiplicative hyperring.* In other words, H is *a subhyperring* of $(R, +, \bullet)$ if $H - H \subseteq H$ and $x \bullet y \subseteq H$ for any $x, y \in H$. A nonempty subset I of a multiplicative hyperring $(R, +, \bullet)$ *is called a hyperideal of $(R, +, \bullet)$ if $I - I \subseteq I$ and $x \bullet r \cup r \bullet x \subseteq I$ for any $x \in I$ and $r \in R$.* The intersection of two subhyperrings of a multiplicative hyperring $(R, +, \bullet)$ is a subhyperring of R. The intersection of two hyperideals of a multiplicative hyperring $(R, +, \bullet)$ is a hyperideal of R. Moreover any intersection of subhyperrings of a multiplicative hyperring is a subhyperring, and any intersection of hyperideals of a multiplicative hyperring is a hyperideal. The hyperideal generated by any subset S of $(R, +, \bullet)$ is the intersection of all hyperideals of R containing S. From (Davvaz & Leoreanu-Fotea, 2007), and (Dasgupta, 2012), *let A and B be non empty hyperideals of a multiplicative hyperring $(R, +, \bullet)$.*

(1) The sum $A + B$ is the hyperideal defined by

$A + B = \{a + b : a \in A, b \in B\}$.

(2) The product $A \bullet B$ is the hyperideal defined by

$A \bullet B = \bigcup \{\sum_{i=1}^{n} a_i \bullet b_i : a_i \in A, b_i \in B, n \in \mathbb{N}\}$.

(3) The principal hyperideal of R generated by an element a is given by

$$< a >= \{pa : p \in \mathbb{Z}\} + \{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{l} z_k : \forall\, i, j, k, \exists\, r_i, s_j, u_k, t_k \in R, x_i \in r_i \bullet a, y_j \in a \bullet s_j, z_k \in t_k \bullet a \bullet u_k\}.$$

(4) The zero hyperideal is the hyperideal generated by the additive identity zero denoted by $< 0 >$ and we have $< 0 >= \{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j + \sum_{k=1}^{l} z_k : \forall\, i, j, k, \exists\, r_i, s_j, u_k, t_k \in R, x_i \in r_i \bullet 0, y_j \in 0 \bullet s_j, z_k \in t_k \bullet 0 \bullet u_k\}$.

Recall from (Procesi & Rota, 1999) that *a prober hyperideal I of a multiplicative hyperring $(R, +, \bullet)$ is called a prime hyperideal of R if for any $a, b \in R$, $a \bullet b \subseteq I$, then $a \in I$ or $b \in I$. From* (Dasgupta, 2012), *let C be the class of all finite products of elements of a multiplicative hyperring $(R, +, \bullet)$, i.e., $C = \{r_1 \bullet r_2 \bullet \ldots \bullet r_n : r_i \in R, n \in \mathbb{N}\} \subseteq \mathbb{P}^*(R)$. A hyperideal I of R is called a C –ideal of R if for any $A \in C, A \cap I \neq \emptyset \Longrightarrow A \subseteq I$. The radical of I denoted by $Rad(I)$ is the intersection of all prime hyperideals of R containing I. If R does not have any prime hyperideal containing I, then $Rad(I) = R$.*

Let $\sqrt{I}$ defined as $\sqrt{I} = \{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$, where $r^n = \underbrace{r \bullet r \bullet \ldots \bullet r}_{n-times}$ for any positive integer $n > 1$ and $r^1 = \{r\}$, then by Proposition 3.2. in (Dasgupta, 2012), *$\sqrt{I} \subseteq Rad(I)$. The equality holds when I is a C –ideal of R. A hyperideal $I \neq R$ of a multiplicative hyperring $(R, +, \bullet)$ is called a primary hyperideal of R if for any $a, b \in R$, $a \bullet b \subseteq I$, then $a \in I$ or $b \in \sqrt{I}$.*

Recall from (Davvaz & Leoreanu-Fotea, 2007) that *a homomorphism (resp. good homomorphism) between two multiplicative hyperrings $(R, +, \bullet)$ and $(R', +', \bullet')$ is a map $f : R \longrightarrow R'$ such that for all x, y of R, we have $f(x + y) = f(x) +' f(y)$ and $f(x \bullet y) \subseteq f(x) \bullet' f(y)$ (resp. $f(x \bullet y) = f(x) \bullet' f(y)$).*

The kernel of f is the inverse image of $< 0 >$, the hyperideal generated by the zero in R' and is denoted by $Ker(f)$. From (Davvaz & Leoreanu-Fotea, 2007), *let $(R, +, \bullet)$ be a multiplicative hyperring and I be a hyperideal of R. The usual addition of cosets and multiplication defined as:*

$(a + I) \star (b + I) = \{c + I : c \in a \bullet b\}$ on the set $R/I = \{a + I : a \in R\}$ of all cosets of I. Then, $(R/I, +, \star)$ is a multiplicative hyperring and it is strongly distributive if R is so.

Recall from (Anbarloei, 2017) that *a prober hyperideal I of a multiplicative hyperring $(R, +, \bullet)$ is called a 2 –absorbing (resp. 2 –absorbing primary) hyperideal of R if $a \bullet b \bullet c \subseteq I$, then $a \bullet b \subseteq I$ or $b \bullet c \subseteq I$ or $a \bullet c \subseteq I$ (resp. $a \bullet b \subseteq I$ or $b \bullet c \subseteq \sqrt{I}$ or $a \bullet c \subseteq \sqrt{I}$) for any $a, b, c \in R$.*

Results and Discussion

We start by the following definitions.

Definition 3.1 *Let $(R, +, \bullet)$ be a multiplicative hyperring , $L(R)$ be the lattice of all hyperideals of R and $\phi : L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function.*

(1) A proper hyperideal I of R is called a *ϕ –prime* hyperideal of R if whenever $a, b \in R$ with $a \bullet b \subseteq I - \phi(I)$, then $a \in I$ or $b \in I$.

(2) A proper hyperideal I of R is called a *ϕ –primary* hyperideal of R if whenever $a, b \in R$ with $a \bullet b \subseteq I - \phi(I)$, then $a \in I$ or $b \in \sqrt{I}$.

(3) A proper hyperideal I of R is called a *$\phi - 2$ –absorbing hyperideal* of R if whenever $a, b, c \in R$ with $\bullet\, b \bullet c \subseteq I - \phi(I)$, then $a \bullet b \subseteq I$ or $a \bullet c \subseteq I$ or $b \bullet c \subseteq I$.

(4) A proper hyperideal I of R is called a *$\phi - 2$ –absorbing primary hyperideal* of R if whenever $a, b, c \in R$ with $a \bullet b \bullet c \subseteq I - \phi(I)$, then $a \bullet b \subseteq I$ or $a \bullet c \subseteq \sqrt{I}$ or $b \bullet c \subseteq \sqrt{I}$.

Definition 3.2 *Let $(R, +, \bullet)$ be a multiplicative hyperring, $\phi : L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function, and I be a $\phi - 2$ –absorbing primary hyperideal of R. Then,*

(1) If $\phi(P) = \emptyset$ for every $P \in L(R)$, then we say that $\phi = \phi_\emptyset$, and I is called a *$\phi_\emptyset - 2$ –absorbing primary hyperideal* of R, and hence I is a 2 –absorbing primary hyperideal of R.

(2) If $\phi(P) = P$ for every $P \in L(R)$, then we say that $\phi = \phi_1$ and I is called a *$\phi_1 - 2$ –absorbing primary hyperideal* of R.

(3) If $n \geq 2$ a positive integer and $\phi(P) = P^n$ for every $P \in L(R)$, then we say that $\phi = \phi_n$ and I is called a $\phi_n -$

$2-$*absorbing primary hyperideal* of R. In the case that $n=2$, we say that $\phi=\phi_2$ and I is called *an almost*$-2-$*absorbing primary hyperideal* of R.

(4) If $\phi(P)=\bigcap_{n=1}^{\infty} P^n$ for every $P\in L(R)$, then we say that $\phi=\phi_w$ and I is called a ϕ_w-2-*absorbing primary hyperideal* of R.

Remark 3.3

(1) As $I-\phi(I)=I-(I\cap \phi(I))$, so we may assume that $\phi(I)\subseteq I$.

(2) Given two functions $\psi_1,\psi_2: L(R)\longrightarrow L(R)\cup \{\emptyset\}$, we say that $\psi_1\leq\psi_2$ if $\psi_1(P)\subseteq\psi_2(P)$ for each $P\in L(R)$. Thus, one can be easily seen that $\phi_\emptyset\leq\phi_w\leq\ldots\leq\phi_{n+1}\leq\phi_n\leq\ldots\leq\phi_2\leq\phi_1$.

(3) Every $\phi-$prime hyperideal of a multiplicative hyperring R is $\phi-$primary hyperideal of R.

(4) It is clear that every $\phi-2-$absorbing hyperideal is a $\phi-2-$absorbing primary hyperideal.

(5) It is clear that every $\phi-$primary hyperideal is a $\phi-2-$absorbing primary hyperideal.

The following example shows that a $\phi-2-$absorbing primary hyperideal need not be $\phi-2-$absorbing hyperideal.

Example 3.4 *Let R be the ring Z under ordinary addition and multiplication. For any $a,b\in R$, we define the hyperoperation $a\bullet b=\{2ab,3ab\}$. Then, $R=(\mathbb{Z},+,\bullet)$ is a multiplicative hyperring. Let $H=12\mathbb{Z}=\{12n: n\in\mathbb{Z}\}$ be a subset of R. Then, H is a ϕ_n-2-absorbing primary hyperideal of R that is not ϕ_n-2-absorbing hyperideal of $\forall\, n\geq 2$.*

The following example shows that a $\phi-2-$absorbing primary hyperideal need not be $\phi-$primary hyperideal.

Example 3.5 *Consider the multiplicative hyperring $\mathbb{Z}$ define in Example 3.4.The hyperideal $12\mathbb{Z}=\{12n: n\in\mathbb{Z}\}$ is a ϕ_n-2-absorbing primary hyperideal of $\mathbb{Z}$ $\forall\, n\geq 2$. However, $12\mathbb{Z}$ is not ϕ_n-primary hyperideal of $\mathbb{Z}$. $4\bullet 3\subseteq 12\mathbb{Z}-\phi_\emptyset(12\mathbb{Z})=12\mathbb{Z}$ and $4\not\subseteq 12\mathbb{Z}$, $3^n\nsubseteq 12\mathbb{Z}\forall\, n\geq 2$. Also, $3\not\subseteq 12\mathbb{Z}$, $4^n\not\subseteq 12\mathbb{Z}\forall\, n\geq 2$. Therefore, $12\mathbb{Z}$ is not ϕ_n-primary hyperideal of $\mathbb{Z}$.*

Now, we give the following diagram which clarifies the place of $\phi-2-$absorbing primary hyperideal in the lattice of all hyperideals $L(R)$ of R.

prime hyperideal $\Rightarrow$ $\phi-$prime hyperideal $\Rightarrow$ $\phi-2-$absorbing hyperideal $\Rightarrow$ $\phi-2-$absorbing primary hyperideal.

Proposition 3.6 *Let I be a proper hyperideal of a multiplicative hyperring $(R,+,\bullet)$ and let $\psi_1,\psi_2: L(R)\longrightarrow L(R)\cup \{\emptyset\}$ with $\psi_1\leq\psi_2$. If I is a ψ_1-2-absorbing primary hyperideal of R, then I is a ψ_2-2-absorbing primary hyperideal of R.*

Proof. Assume that I is a ψ_1-2-absorbing primary hyperideal of R and let $a,b,c\in R$ with $a\bullet b\bullet c\subseteq I-\psi_2(I)$. Now, $a\bullet b\bullet c\subseteq I-\psi_2(I)\subseteq I-\psi_1(I)$. Therefore, I is a ψ_2-2-absorbing primary hyperideal of R.

Now, we need the following definition.

Definition 3.7 *A proper hyperideal I of a multiplicative hyperring $(R,+,\bullet)$ is called an idempotent if $I=I^2$.*

Theorem 3.8 *Let I be a proper hyperideal of a multiplicative hyperring $(R,+,\bullet)$. Then the following assertions hold.*

(1) If I is a $2-$*absorbing primary hyperideal* of R, then I is a ϕ_w-2-absorbing primary hyperideal of R.

(2) If I is a ϕ_w-2-*absorbing primary hyperideal* of R, then I is a $\phi_{n+1}-2-$*absorbing primary hyperideal* of R for every positive integer $n\geq 2$.

(3) If I is a $\phi_{n+1}-2-$*absorbing primary hyperideal* of R, then I is a ϕ_n-2-*absorbing primary hyperideal* of R for every positive integer $n\geq 2$.

(4) If I is a ϕ_n-2-*absorbing primary hyperideal* of R for every positive integer $n\geq 2$, then I is *almost* $2-$*absorbing primary hyperideal* of R.

(5) If I is an *idempotent hyperideal* of R, *then* I is a ϕ_w-2-*absorbing primary hyperideal* of R and I is a ϕ_n-2-absorbing primary hyperideal of R for every positive integer $n\geq 1$.

(6) If $I=\sqrt{I}$, then I is a ϕ_n-2-*absorbing primary hyperideal* of R if and only if I is a ϕ_n-2-*absorbing hyperideal* of R.

(7) I is a ϕ_n-2-*absorbing primary hyperideal* of R for every positive integer $n\geq 2$ if and only if I is a ϕ_w-2-*absorbing primary hyperideal* of R.

(8) If I is a $\phi-$ *primary hyperideal* of R and $\phi(\sqrt{I})=\sqrt{\phi(I)}$, then $\sqrt{I}$ is a $\phi-$prime hyperideal of R.

Proof.

(1-4) Follow directly from Proposition 3.6.

(5) Assume that I is an idempotent hyperideal of R. Then, $I=I^n$ for every positive integer $n\geq 1$. Thus, $\phi_w(I)=\bigcap_{n=1}^{\infty} I^n=I$. Thus, we are done.

(6) Since, $\sqrt{\sqrt{I}}=\sqrt{I}$, we are done.

(7) Assume that I is a ϕ_n-2-absorbing hyperideal of R and let $a,b,c\in R$ with $a\bullet b\bullet c\subseteq I-\bigcap_{n=1}^{\infty} I^n$. Thus, $a\bullet b\bullet c\subseteq I-I^n$ for some positive integer $n\geq 2$. Now, I is a ϕ_n-2-absorbing primary hyperideal of R implies that $a\bullet b\subseteq I$ or $b\bullet c\subseteq\sqrt{I}$ or $a\bullet c\subseteq\sqrt{I}$. The converse is clear from parts $(1),(2)$.

(8) Proceed similar as (6).

Lemma 3.9 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup \{\emptyset\}$ be a function, I be a $\phi-$prime hyperideal of R and J be a subset of R. For any $a\in R$, $a\bullet J\subseteq I-\phi(I)$ and $a\not\subseteq I$, implies that $J\subseteq I$.*

Proof. Let $a\in R$, $a\bullet J\subseteq I-\phi(I)$ and $a\not\subseteq I$. Thus, $a\bullet J=\bigcup a\bullet j_i\subseteq I-\phi(I)$ for all $j_i\in J$. Then, $a\bullet j_i\subseteq I-\phi(I)$ for all $j_i\in J$. Since I is a $\phi-$prime hyperideal of R and $a\not\subseteq I$, we conclude that $j_i\in I$ for all $j_i\in J$. Therefore, $J\subseteq I$.

Now, we extend Lemma 3.9 to $\phi-$primary case.

Lemma 3.10 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup \{\emptyset\}$ be a function, I be a $\phi-$primary hyperideal of R and P be a subset of R. For any $a\in R$, $a\bullet P\subseteq I-\phi(I)$, then either $a\in I$ or $P\subseteq\sqrt{I}$.*

Proof. Assume that $a\in R$, $a\bullet P\subseteq I-\phi(I)$ and $a\not\subseteq I$. Then, $a\bullet P=\bigcup a\bullet J_\alpha\subseteq I-\phi(I)$ for all $J_\alpha\in P$. Thus, $a\bullet J_\alpha\subseteq I-\phi(I)$ for all $J_\alpha\in P$. Since I is a $\phi-$primary hyperideal of R and $a\not\subseteq I$, we conclude that $J_\alpha\in\sqrt{I}$ for all $J_\alpha\in P$. Therefore, $P\subseteq I$.

Theorem 3.11 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup \{\emptyset\}$ be a function, I be a $\phi-$prime hyperideal of R, and A and B are subsets of R. If $A\bullet B\subseteq I-\phi(I)$, then $A\subseteq I$ or $B\subseteq I$.*

Proof. Assume that $A \bullet B \subseteq I-\phi(I)$, $A \nsubseteq I$, and $B \nsubseteq I$. Since, $A \bullet B = \bigcup x_i \bullet y_i \subseteq I-\phi(I)$, then $x_i \bullet y_i \subseteq I-\phi(I)$ for all $x_i \in A$ and $y_i \in B$. Since, $A \nsubseteq I$ and $B \nsubseteq I$, then there exist $a,b \notin I$ for some $a \in A$ and $b \in B$. Thus, $a \bullet b \subseteq A \bullet B \subseteq I-\phi(I)$. Since $a,b \notin I$ and I is a ϕ −prime hyperideal of R, then $a \bullet b \nsubseteq I$, a contradiction. Therefore, $A \subseteq I$ or $B \subseteq I$.

Theorem 3.12 *Let I be a proper hyperideal of a multiplicative hyperring $(R,+,\bullet)$ and let $\phi: L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function. If I is a ϕ −prime hyperideal that is not prime, then $I^2 \subseteq \phi(I)$. Hence, a ϕ −prime hyperideal I with $I^2 \nsubseteq \phi(I)$ is prime.*

Proof. Assume that $I^2 \nsubseteq \phi(I)$, we show that I is prime hyperideal of R. Let $a,b \in R$ such that $a \bullet b \subseteq I$. If $a \bullet b \nsubseteq \phi(I)$, and since I is a ϕ −prime hyperideal, we have $a \in I$ or $b \in I$. So, assume that $a \bullet b \subseteq \phi(I)$. First, assume that $a \bullet I = \bigcup a \bullet j_\alpha \nsubseteq \phi(I)$ for all $j_\alpha \in I$. Thus, there exists $j_\alpha \in I$ such that $a \bullet j_\alpha \nsubseteq \phi(I)$. Then, $a \bullet (b+j_\alpha) \subseteq I-\phi(I)$. So, $a \in I$ or $b+j_\alpha \in I$ and hence $a \in I$ or $b \in I$. So, we can assume that $a \bullet I = \bigcup a \bullet j_\alpha \subseteq \phi(I)$ for all $j_\alpha \in I$. Then, $a \bullet j_\alpha \subseteq \phi(I)$ for all $j_\alpha \in I$. Likewise, we can assume that $b \bullet I = \bigcup b \bullet j_\beta \subseteq \phi(I)$ for all $j_\beta \in I$. Then, $b \bullet j_\beta \subseteq \phi(I)$ for all $j_\beta \in I$. Since $I^2 = \bigcup j_\alpha \bullet j_\beta \nsubseteq \phi(I)$ for all $j_\alpha, j_\beta \in I$, then there exist $j_\alpha, j_\beta \in I$ with $j_\alpha \bullet j_\beta \nsubseteq \phi(I)$. Then, $(a+j_\alpha) \bullet (b+j_\beta) \subseteq I-\phi(I)$. Since I is a ϕ −prime hyperideal of R, we conclude that $a+j_\alpha \in I$ or $b+j_\beta \in I$, and hence $a \in I$ or $b \in I$. Thus, I is prime hyperideal of R.

Theorem 3.13 *Let $(R,+,\bullet)$ be a multiplicative hyperring, I be a ϕ −prime hyperideal of R for some ϕ, and $\phi(I) \subseteq \phi(J)$ for some hyperideal J of R such that $J=\sqrt{J}$ and $J \subset I$. Then, I is a prime hyperideal of R.*

Proof. Assume that I is not a prime hyperideal of R. Then, $I^2 \subseteq \phi(I)$ by Theorem 3.12. Hence $\sqrt{I}=\sqrt{\phi(I)}$. Since $\phi(I) \subseteq \phi(J) \subseteq J = \sqrt{J}$, we have $\sqrt{I}=\sqrt{\phi(J)} \subseteq J$. Thus, $I \subseteq J$, a contradiction. Therefore, I is a prime hyperideal of R.

Theorem 3.14 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function, and I_1 and I_2 be ϕ −prime hyperideals of R. If $\phi(I_1)=\phi(I_2)=\phi(I_1 \cap I_2)$, then $I_1 \cap I_2$ is a $\phi-2$ −absorbing hyperideal of R.*

Proof. Let $a,b,c \in R$ such that $a \bullet b \bullet c \subseteq (I_1 \cap I_2)-\phi(I_1 \cap I_2)$, let $a \bullet b \nsubseteq I_1 \cap I_2$ and $b \bullet c \nsubseteq I_1 \cap I_2$. Then, $a,b,c \notin I_1 \cap I_2$. If $a \in I_1 \cap I_2$, then $a \in I_1$ and $a \in I_2$. Since I_1 and I_2 are hyperideals, we have $a \bullet b \subseteq I_1$ and $a \bullet b \subseteq I_2$. Then, $a \bullet b \subseteq I_1 \cap I_2$ which is a contradiction. Thus, $a \notin I_1 \cap I_2$. Similarly, $b \notin I_1 \cap I_2$ and $c \notin I_1 \cap I_2$. We have three cases:

Case (1): $a \notin I_1$ and $a \notin I_2$. Since, $c \notin I_1 \cap I_2$, we have three cases again. Assume that $c \notin I_1$ and $c \notin I_2$. Since $a \bullet b \bullet c \subseteq (I_1 \cap I_2)-\phi(I_1 \cap I_2)$. Then, $a \bullet b \bullet c \subseteq I_1-\phi(I_1)$. Since I_1 is a ϕ −prime hyperideal of R and $a \bullet c \nsubseteq I_1$, $a \bullet b \bullet c \subseteq I_1-\phi(I_1)$, then $b \in I_1$ by Lemma 3.9. Thus, $a \bullet b \subseteq I_1$. Similarly, Since I_2 is a ϕ −prime hyperideal of R and $a \bullet c \nsubseteq I_2$, $a \bullet b \bullet c \subseteq I_2-\phi(I_2)$, we have $b \in I_2$ by Lemma 3.9. Thus, $a \bullet b \subseteq I_2$. Hence, $a \bullet b \subseteq I_1 \cap I_2$, a contradiction. Thus, $c \in I_1$ or $c \in I_2$. Now, assume that $c \notin I_1$ and $c \in I_2$. Since, I_1 is a ϕ −prime hyperideal of R and $a \bullet c \nsubseteq I_1$, $a \bullet b \bullet c \subseteq I_1-\phi(I_1)$, we have $b \in I_1$. Thus, $b \bullet c \subseteq I_1$. Since, $c \in I_2$, then $b \bullet c \subseteq I_2$ and thus $b \bullet c \subseteq I_1 \cap I_2$, a contradiction. Similarly $c \notin I_2$ and $c \in I_1$ lead to a contradiction. Thus, if $a \notin I_1 \cap I_2$, then $a \in I_1$ or $a \in I_2$.

Case (2): $a \in I_1$ and $a \notin I_2$. We show that $c \in I_2$. Assume that $c \notin I_2$. Since I_2 is a ϕ −prime hyperideal of R, we have $a \bullet c \nsubseteq I_2$. Since, whenever $a \bullet b \bullet c \subseteq I_2-\phi(I_2)$, $a \bullet c \nsubseteq I_2$ and also I_2 is a ϕ −prime hyperideal of R, then $b \in I_2$ by Lemma 3.9. Thus, $a \bullet b \subseteq I_1 \cap I_2$, a contradiction. Thus, $c \in I_2$ and we get $c \notin I_1$. Therefore, $a \bullet c \subseteq I_1 \cap I_2$.

Case (3): Assume that $a \in I_2$ and $a \notin I_1$, we show that $c \in I_1$. Assume that $c \notin I_1$. Since I_2 is a ϕ −prime hyperideal of R, then $a \bullet c \nsubseteq I_1$. Since, whenever $a \bullet b \bullet c \subseteq I_1-\phi(I_1)$, $a \bullet c \nsubseteq I_1$ and I_1 is a ϕ −prime hyperideal of R, then $b \in I_1$ by Lemma 3.9. Thus, $a \bullet b \subseteq I_1 \cap I_2$, a contradiction. Since, $c \in I_1$ and $c \notin I_1 \cap I_2$, we have $c \notin I_2$ and hence $a \bullet c \subseteq I_1 \cap I_2$. Thus, $I_1 \cap I_2$ is a $\phi-2$ −absorbing hyperideal of R.

Example 3.15 *Let $(\mathbb{Z}_6, \oplus, \odot)$ be a ring such that the binary operations $\oplus$, $\odot$ defines as follows:*

$\overline{a} \oplus \overline{b}$ and $\overline{a} \odot \overline{b}$ are remainder of $\frac{a+b}{6}$ and $\frac{a.b}{6}$ where $+$ and . are ordinary addition and multiplication for all $\overline{a},\overline{b} \in \mathbb{Z}_6$. For $\overline{a},\overline{b} \in \mathbb{Z}_6$, we define the hyperoperation $\overline{a} \bullet \overline{b} = \{\overline{0}, \overline{ab}, \overline{2ab}, \overline{3ab}, \overline{4ab}, \overline{5ab}\}$. One can easily see that $(\mathbb{Z}_6, \oplus, \bullet)$ is a commutative hyperring. Now, let $I_1=\{0\}$ and $I_2=\{\overline{0},\overline{2},\overline{4}\}$. Then, $I_1 \cap I_2 = \{\overline{0}\}$. Clearly, $\{\overline{0}\}$ is a ϕ_n-2 −absorbing hyperideal of $(\mathbb{Z}_6, \oplus, \bullet)$ $\forall\, n \geq 2$, but it is not a ϕ_n-2 −prime hyperideal of $(\mathbb{Z}_6, \oplus, \bullet)$.

Theorem 3.16 *Let J and P be proper hyperideals of a multiplicative hyperring $(R,+,\bullet)$ such that $J \subseteq P$, and let $n \geq 2$ be positive integer. If P is a ϕ_n-2 −absorbing primary hyperideal of R, then P/J is a ϕ_n-2 −absorbing primary hyperideal of R/J.*

Proof. Assume that P is a ϕ_n-2 −absorbing primary hyperideal of R. Let $a,b,c \in R$ such that $(a+J)\star(b+J)\star(c+J) \in P/J-(P/J)^n$. Since $J \subseteq P$, we have $a \bullet b \bullet c \subseteq P-P^n$. Thus, $a \bullet b \subseteq P$ or $a \bullet c \subseteq \sqrt{P}$ or $b \bullet c \subseteq \sqrt{P}$. Now, $J \subseteq P$ implies $\sqrt{P/J}=\sqrt{P}/J$. Thus, $(a+J)\star(b+J) \subseteq P/J$ or $(a+J)\star(c+J) \subseteq \sqrt{P}/J$ or $(b+J)\star(c+J) \subseteq \sqrt{P}/J$. Hence, P/J is a ϕ_n-2 −*absorbing primary hyperideal* of R/J.

Theorem 3.17 *Let $J \subseteq P$ be proper hyperideals of a multiplicative hyperring $(R,+,\bullet)$. If P is a ϕ_w-2 −absorbing primary hyperideal of R, then P/J is a ϕ_w-2 −absorbing primary hyperideal of R/J.*

Proof. Proceed similar as Theorem 3.16.

Definition 3.18 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function, and $I \subseteq J$ be proper hyperideals of R. The proper hyperideal J/I of R/I is called a ϕ_I-2 −absorbing primary hyperideal of R/I if whenever $a,b,c \in R/I$ with $a \bullet b \bullet c \subseteq J/I-(\phi(J)+I)/I$ implies $a \bullet b \subseteq J/I$ or $a \bullet c \subseteq \sqrt{J/I}$ or $b \bullet c \subseteq \sqrt{J/I}$.*

Theorem 3.19 *Let $(R,+,\bullet)$ be a strongly distributive multiplicative hyperring, $\phi: L(R) \longrightarrow L(R) \cup \{\emptyset\}$ be a function, J be a proper hyperideal of R. Suppose that I is a proper hyperideal of R with $I \subseteq \phi(J)$. Then, the following assertions are equivalent.*

(1) J is a $\phi-2$ −*absorbing primary hyperideal* of R.

(2) J/I is a ϕ_I-2 −*absorbing primary hyperideal* of R/I.

(3) J/I^n is a $\phi_{I^n}-2$ −*absorbing primary hyperideal* of R/I^n for every positive integer $n \geq 1$.

Proof. $(1) \Longrightarrow (2)$ Assume that J is a $\phi-2$ −absorbing primary hyperideal of R, and let $a,b,c \in R$ such that $(a+I)\star(b+I)\star(c+I)=\{x+I: x \in a \bullet b \bullet c\} \subseteq J/I-(\phi(J)+I)/I$. Now, since R is a strongly distributive multiplicative hyperring and I is a hyperideal of R, then R/I is a ring by (Davvaz &

Leoreanu-Fotea, 2007, Corollary 4.3.6). Thus, $(a+I)\star(b+I)\star(c+I)=a\bullet b\bullet c+I\subseteq J/I-(\phi(J)+I)/I$. Thus, $a\bullet b\bullet c\subseteq J-\phi(J)$. Thus, $a\bullet b\subseteq J$ or $a\bullet c\subseteq\sqrt{J}$ or $b\bullet c\subseteq\sqrt{J}$. Since, $I\subseteq\phi(J)\subseteq J$, we have $\sqrt{J/I}=\sqrt{J}/I$. Thus, $(a+I)\star(b+I)\subseteq J/I$ or $(a+I)\star(c+I)\subseteq\sqrt{J}/I$ or $(b+I)\star(c+I)\subseteq\sqrt{J}/I$. Hence, J/I is a ϕ_I-2-absorbing primary hyperideal of R/I.

(2) $\Longrightarrow$ (3) Assume that (2) hold and let $n\geq 1$ be positive integer. Since $I\subseteq\phi(J)$, we have $I^n\subseteq I\subseteq\phi(J)$. Suppose that $a,b,c\in R$ with $(a+I^n)\star(b+I^n)\star(c+I^n)=\{y+I^n: y\in a\bullet b\bullet c\}\subseteq J/I^n-\phi(J+I^n)/I^n$. Thus, $a\bullet b\bullet c\nsubseteq\phi(J)$. Since $I\subseteq\phi(J)$ and $a\bullet b\bullet c\nsubseteq\phi(J)$, we have $a\bullet b\bullet c\nsubseteq I$. Thus, $(a+I)\star(b+I)\star(c+I)\subseteq J/I-\phi(J+I)/I$. Since $\sqrt{J}/I=\sqrt{J/I^n}=\sqrt{J}/I^n$ and J/I is a ϕ_I-2-absorbing primary hyperideal of R, we conclude that $a\bullet b\subseteq J$ or $a\bullet c\subseteq\sqrt{J}$ or $b\bullet c\subseteq\sqrt{J}$. Thus, $a\bullet b+I^n\subseteq J/I^n$ or $a\bullet c+I^n\subseteq\sqrt{J}/I^n$ or $b\bullet c+I^n\subseteq\sqrt{J}/I^n$. Note that I^n is a hyperideal of R and R is a strongly distributive multiplicative hyperring, so R/I^n is a ring.

(3) $\Longrightarrow$ (1) Assume that (3) hold and let $n=1$. Assume that $a,b,c\in R$ with $a\bullet b\bullet c\subseteq J-\phi(J)$. Since, $I\subseteq\phi(J)$, then $a\bullet b\bullet c\nsubseteq I$. Since, $I\subseteq\phi(J)\subset J$, we have $(a+I)\star(b+I)\star(c+I)=a\bullet b\bullet c+I\subseteq J/I-\phi(J)/I$. Since, $\sqrt{J/I}=\sqrt{J}/I$ and J/I is a ϕ_I-2-absorbing primary hyperideal of R, we conclude that $a\bullet b\subseteq J$ or $a\bullet c\subseteq\sqrt{J}$ or $b\bullet c\subseteq\sqrt{J}$. Hence, J is a $\phi-2-$absorbing primary hyperideal of R.

The proof of the next result is easily verified, and thus we omit the proof.

Lemma 3.20 *Let $(R,+,\bullet)$ be a strongly distributive multiplicative hyperring and $\phi: L(R)\longrightarrow L(R)\cup\{\emptyset\}$ be a function. Set $R/\{\emptyset\}=R$, and let J be a proper hyperideal of R. Then, J is a prime (primary, $2-$absorbing primary, respectively) hyperideal of R if and only if $J/\phi(J)$ is a prime (primary, $2-$absorbing primary, respectively) hyperideal of $R/\phi(J)$.*

Theorem 3.21 *Let $(R,+,\bullet)$ be a multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup\{\emptyset\}$ be a function, and I be a proper hyperideal of R. Then, the following assertions are equivalent.*

(1) I is a $\phi-$*primary hyperideal* of R.

(2) For each $a\in R-\sqrt{I}$, $(I:_R a)=I\cup(\phi(I):_R a)$.

(3) For each $a\in R-\sqrt{I}$, either $(I:_R a)=I$ or $(I:_R a)=(\phi(I):_R a)$.

Proof. (1) $\Longrightarrow$ (2) Assume that I is a $\phi-$primary hyperideal of R. Clearly, $I\cup(\phi(I):_R a)\subseteq(I:_R a)$. On the other hand, for every $x\in(I:_R a)$, if $x\bullet a\subseteq\phi(I)$, then $x\in(\phi(I):_R a)$. Otherwise, from $x\bullet a\subseteq I-\phi(I)$ and $a\not\in\sqrt{I}$, we get $x\in I$. Hence, $(I:_R a)\subseteq I\cup(\phi(I):_R a)$.

(2) $\Longrightarrow$ (3) It is clear since $(I:_R a)$ is a hyperideal of R.

(3) $\Longrightarrow$ (1) Assume that $a,b\in R$ with $a\bullet b\subseteq I-\phi(I)$. Obviously, $(I:_R a)\neq(\phi(I):_R a)$. If $a\not\in\sqrt{I}$, then by (3), we have $(I:_R a)=I$. This implies that $b\in I$, that is I is $\phi-$primary hyperideal of R.

Theorem 3.22 *Let $(R,+,\bullet)$ be a strongly distributive multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup\{\emptyset\}$ be a function, I be a $\phi-2-$absorbing primary hyperideal of R and P be a hyperideal of R. If $a\bullet b\bullet P\subseteq I-\phi(I)$ and $a\bullet b\nsubseteq I$ for any $a,b\in R$, then $a\bullet P\subseteq\sqrt{I}$ or $b\bullet P\subseteq\sqrt{I}$.*

Proof. Assume that $a\bullet P\nsubseteq\sqrt{I}$ and $b\bullet P\nsubseteq\sqrt{I}$ for some $a,b\in R$. Since, $a\bullet P=\cup\ a\bullet j_\alpha\nsubseteq\sqrt{I}$ and $b\bullet P=\cup\ b\bullet j_\alpha\nsubseteq\sqrt{I}$ for all $j_\alpha\in P$, then there exists j_α such that $a\bullet j_\alpha\nsubseteq\sqrt{I}$ and $b\bullet j_\alpha\nsubseteq\sqrt{I}$. We may assume that $a\bullet j_1\nsubseteq\sqrt{I}$ and $b\bullet j_2\nsubseteq\sqrt{I}$ for some $j_1,j_2\in P$. Also, for all j_α we have $a\bullet b\bullet j_\alpha\subseteq I-\phi(I)$. Since $a\bullet b\bullet j_1\subseteq I-\phi(I)$, $a\bullet b\nsubseteq I$ and $a\bullet j_1\nsubseteq\sqrt{I}$, we have $b\bullet j_1\subseteq\sqrt{I}$. Similarly, since $a\bullet b\bullet j_2\subseteq I-\phi(I)$, $a\bullet b\nsubseteq I$ and $b\bullet j_2\nsubseteq\sqrt{I}$, we have $a\bullet j_2\subseteq\sqrt{I}$. Now, since I is a $\phi-2-$absorbing primary hyperideal of R, whenever $a\bullet b(j_1+j_2)\subseteq I-\phi(I)$ and $a\bullet b\nsubseteq I$, we have $a\bullet(j_1+j_2)\subseteq\sqrt{I}$ or $b\bullet(j_1+j_2)\subseteq\sqrt{I}$. Assume that $a\bullet(j_1+j_2)=a\bullet j_1+a\bullet j_2\subseteq\sqrt{I}$. Since $a\bullet j_2\subseteq\sqrt{I}$, we have $a\bullet j_1\subseteq\sqrt{I}$ a contradiction. Similarly, let $b\bullet(j_1+j_2)=b\,j_1+b\bullet j_2\subseteq\sqrt{I}$. Since $b\bullet j_1\subseteq\sqrt{I}$, we have $b\bullet j_2\subseteq\sqrt{I}$, a contradiction. Thus, $a\bullet P\subseteq\sqrt{I}$ or $b\bullet P\subseteq\sqrt{I}$.

Theorem 3.23 *Let $(R,+,\bullet)$ be a strongly distributive multiplicative hyperring, $\phi: L(R)\longrightarrow L(R)\cup\{\emptyset\}$ be a function, and I be a proper hyperieal of R. Then, I is a $\phi-2-$absorbing primary hyperideal of R if and only if $I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$, then $I_1\bullet I_2\subseteq I$ or $I_2\bullet I_3\subseteq\sqrt{I}$ or $I_1\bullet I_3\subseteq\sqrt{I}$ for any hyperideals I_1,I_2,I_3 of R.*

Proof. Let I be a $\phi-2-$absorbing primary hyperideal of R, $I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$ and $I_1\bullet I_2\nsubseteq I$. Claim that $I_2\bullet I_3\subseteq\sqrt{I}$ or $I_1\bullet I_3\subseteq\sqrt{I}$. Assume that $I_1\bullet I_3\nsubseteq\sqrt{I}$ and $I_2\bullet I_3\nsubseteq\sqrt{I}$. Thus, there exist $j_1\in I_1$ and $j_2\in I_2$ such that $j_1\bullet I_3\nsubseteq\sqrt{I}$ and $j_2\bullet I_3\nsubseteq\sqrt{I}$. By Theorem 3.22, we get $j_1\bullet j_2\subseteq I$. Since $I_1\bullet I_2\nsubseteq I$, we have $a\bullet b\nsubseteq I$ for some $a\in I_1$ and $b\in I_2$. Since $a\bullet b\bullet I_3\subseteq I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$ and $a\bullet b\nsubseteq I$, then by Theorem 3.22, $a\bullet I_3\subseteq\sqrt{I}$ or $b\bullet I_3\subseteq\sqrt{I}$.

Case (1): Assume that $a\bullet I_3\subseteq\sqrt{I}$ and $b\bullet I_3\nsubseteq\sqrt{I}$. Since $j_1\bullet b\bullet I_3\subseteq I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$, $b\bullet I_3\nsubseteq\sqrt{I}$ and $j_1\bullet I_3\nsubseteq\sqrt{I}$, we have $j_1\bullet b\subseteq I$ by Theorem 3.22. Since $(a+j_1)\bullet b\bullet I_3\subseteq I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$ and $b\bullet I_3\nsubseteq\sqrt{I}$, we have $(a+j_1)\bullet I_3\subseteq\sqrt{I}$ or $(a+j_1)\bullet b\subseteq I$ by Theorem 3.22. Assume that $(a+j_1)\bullet I_3\subseteq\sqrt{I}$. Then for every $j_3\in I_3$, since R is strongly distributive, we conclude that $(a+j_1)\bullet I_3=\cup\ (a+j_1)\bullet j_3=a\bullet j_3+j_1\bullet j_3=a\bullet I_3+j_1\bullet I_3\subseteq\sqrt{I}$. Since $\sqrt{I}$ is a hyperideal and $a\bullet I_3\subseteq\sqrt{I}$, we get $j_1\bullet I_3\subseteq\sqrt{I}$, a contradiction. Now, suppose that $(a+j_1)\bullet b=a\bullet b+j_1\bullet b\subseteq I$. Since I is a hyperideal and $j_1\bullet b\subseteq I$, we have $a\bullet b\subseteq I$, a contradiction.

Case (2): Assume that $a\bullet I_3\nsubseteq\sqrt{I}$ and $b\bullet I_3\subseteq\sqrt{I}$. Then, $a\bullet j_2\subseteq I$ by Theorem 3.22. Since $a\bullet(b+j_2)\bullet I_3\subseteq I_1\bullet I_2\bullet I_3\subseteq I-\phi(I)$, $a\bullet I_3\nsubseteq\sqrt{I}$, we have $a\bullet(b+j_2)\subseteq I$ or $(b+j_2)\bullet I_3\subseteq\sqrt{I}$ by Theorem 3.22. Assume that $(b+j_2)\bullet I_3\subseteq\sqrt{I}$. Since R strongly distributive, we have $(b+j_2)\bullet I_3=\cup\ (b+j_2)\bullet j_3=b\bullet j_3+j_2\bullet j_3=b\bullet I_3+j_2\bullet I_3\subseteq\sqrt{I}$ for every $j_3\in I_3$. Since $\sqrt{I}$ is a hyperideal and $b\bullet I_3\subseteq\sqrt{I}$, we have $j_2\bullet I_3\subseteq\sqrt{I}$, a contradiction. Now, assume that $a\bullet(b+j_2)=a\bullet b+a\bullet j_2\subseteq I$. Similarly, since I is a hyperideal and $a\bullet j_2\subseteq I$, we have $a\bullet b\subseteq I$, a contradiction.

Case (3): Assume that $a\bullet I_3\subseteq\sqrt{I}$ and $b\bullet I_3\subseteq\sqrt{I}$. Since $b\bullet I_3\subseteq\sqrt{I}$ and $j_2\bullet I_3\nsubseteq\sqrt{I}$, we have $(b+j_2)\bullet I_3\nsubseteq\sqrt{I}$. By Theorem 3.22, we conclude that $j_1\bullet(b+j_2)=j_1\bullet b+j_1\bullet j_2\subseteq I$. Since $j_1\bullet j_2\subseteq I$ and $j_1\bullet b+j_1\bullet j_2\subseteq I$, we get $b\bullet j_1\subseteq I$. Since $a\bullet I_3\subseteq\sqrt{I}$ and $j_1\bullet I_3\nsubseteq\sqrt{I}$, we conclude that $(a+j_1)\bullet I_3\nsubseteq\sqrt{I}$. Hence $(a+j_1)\bullet j_2=a\bullet j_2+j_1\bullet j_2\subseteq I$ by Theorem 3.22. Since $j_1\bullet j_2\subseteq I$ and $a\bullet j_2+j_1\bullet j_2\subseteq I$, we have $a\bullet j_2\subseteq I$. Thus, $(a+j_1)\bullet(b+j_2)=a\bullet b+a\bullet j_2+b\bullet j_1+j_1\bullet j_2\subseteq I$ which leads to $a\bullet b\subseteq I$ that is a contradiction.

Let $(R_1, +_1, \bullet_1)$ and $(R_2, +_2, \bullet_2)$ be two multiplicative hyperrings. Recall from (Ardekani & Davvaz, 2014) that $(R = R_1 \times R_2, +, \bullet)$ is a multiplicative hyperrings with the operation $+$ and the hyperoperatin $\bullet$ are defined respectively as $(x,y) + (z,t) = (x +_1 z, y +_2 t)$ and $(x,y) \bullet (z,t) = \{(a,b) \in R: a \in x \bullet_1 z, b \in y \bullet_2 t\}$ for all $(x,y),(z,t) \in R$. Note that each hyperideal of R is the cartesian product of hyperideals of R_1 and R_2, respectively.

Remark 3.24 *Let $(R_1, +_1, \bullet_1)$ and $(R_2, +_2, \bullet_2)$ be two multiplicative hyperrings, $R = R_1 \times R_2$, $\psi_1: L(R_1) \longrightarrow L(R_2) \cup \{\emptyset\}$ and $\psi_2: L(R_2) \longrightarrow L(R_2) \cup \{\emptyset\}$ be functions, and $\phi = \psi_1 \times \psi_2$. Let $I = I_1 \times I_2$ be a hyperideal of R, where I_1 and I_2 are hyperideals of R_1 and R_2, respectively. Suppose that $\psi_i(I_i) = \emptyset$ for some $i, 1 \le i \le 2$. Then $I - \phi(I) = I$. Hence, $\phi(I) = \emptyset$ if and only if $\psi_i(I_i) = \emptyset$ for some i, $1 \le i \le 2$. If $\phi(I) = \emptyset$, then we set $R/\phi(I) = R$.*

Theorem 3.25 *Let $(R_1, +_1, \bullet_1)$ and $(R_2, +_2, \bullet_2)$ be two multiplicative hyperrings, $R = R_1 \times R_2$, $\psi_1: L(R_1) \longrightarrow L(R_2) \cup \{\emptyset\}$ and $\psi_2: L(R_2) \longrightarrow L(R_2) \cup \{\emptyset\}$ be functions such that $\psi_2(R_2) \neq R_2$, and let $\phi = \psi_1 \times \psi_2$. Then, the following assertions are equivalent.*

(1) $I_1 \times R_2$ is a $\phi - 2$ *−absorbing primary hyperideal* of R.

(2) $I_1 \times R_2$ is a 2 *−absorbing primary hyperideal* of R.

(3) I_1 is a 2 *−absorbing primary hyperideal* of R_1.

Proof. Assume that $\psi_1(I_1) = \emptyset$ or $\psi_2(R_2) = \emptyset$. Then, $\phi(I_1 \times R_2) = \emptyset$ by Remark 3.24. Hence, $(1) \Leftrightarrow (2) \Leftrightarrow (3)$. Thus, assume that $\phi(I_1 \times R_2) \neq \emptyset$ and hence, $\psi_1(I_1) \neq \emptyset, \psi_2(R_2) \neq \emptyset$.

$(1) \Longrightarrow (2)$ It is clear that I_1 is a $\psi_1 - 2$ −absorbing primary hyperideal of R_1. If I_1 is a 2 −absorbing primary hyperideal of R_1, then we are done. Thus, assume that I_1 is not a 2 −absorbing hyperideal of R_1. Thus, there exist $a,b,c \in R_1$ with $a \bullet_1 b \bullet_1 c \subseteq \psi_1(I_1)$, $a \bullet_1 b \nsubseteq I_1$, $a \bullet_1 c \nsubseteq \sqrt{I_1}$ and $b \bullet_1 c \nsubseteq \sqrt{I_1}$. Since $\psi_2(R_2) \neq R_2$, we have $(a, 1_{R_2}) \bullet (b, 1_{R_2}) \bullet (c, 1_{R_2}) \subseteq I_1 \times R_2 - \psi_1(I_1) \times \psi_2(R_2)$. Then, $a \bullet_1 b \bullet_1 c \subseteq I_1$. Thus, $a \bullet_1 b \subseteq I_1$ or $a \bullet_1 c \subseteq \sqrt{I_1}$ or $b \bullet_1 c \subseteq \sqrt{I_1}$, a contradiction. Thus, I_1 is a 2 −absorbing primary hyperideal of R_1. Thus, $I_1 \times R_2$ is a 2 −absorbing primary hyperideal of R.

$(2) \Longrightarrow (3)$ It is clear.

$(3) \Longrightarrow (1)$ It is clear.

Theorem 3.26 *Let $(R_1, +_1, \bullet_1)$ and $(R_2, +_2, \bullet_2)$ be two multiplicative hyperrings, $R = R_1 \times R_2$, $\psi_1: L(R_1) \longrightarrow L(R_2) \cup \{\emptyset\}$ and $\psi_2: L(R_2) \longrightarrow L(R_2) \cup \{\emptyset\}$ be functions. Then the ϕ −primes of R have exactly one of the following three types:*

(1) $I_1 \times I_2$, where I_1 is a proper hyperideal of R_1 with $\psi_1(I_1) = I_1$ and I_2 is a proper hyperideal of R_2 with $\psi_2(I_2) = I_2$;

(2) $I_1 \times R_2$, where I_1 is a ψ_1 −prime hyperideal of R_1 which must be prime hyperideal if $\psi_2(R_2) \neq R_2$;

(3) $R_1 \times I_2$, where I_2 is a ψ_2 −prime hyperideal of R_2 which must be prime hyperideal if $\psi_1(R_1) \neq R_1$.

Proof. First of all, note that a hyperideal of R having one of these three types is a ϕ −prime hyperideal of R. Case(1) is clear since $I_1 \times I_2 - \phi(I_1 \times I_2) = \emptyset$. If I_1 is a prime hyperideal of R_1, then $I_1 \times R_2$ is a prime hyperideal of R and hence $I_1 \times R_2$ is a ϕ −prime hyperideal of R. So, assume that I_1 is a ψ_1 −prime hyperideal of R_1 and $\psi_2(R_2) \neq R_2$. Suppose that $(a_1,a_2) \bullet (b_1,b_2) \subseteq I_1 \times R_2 - \psi_1(I_1) \times \psi_2(R_2) = I_1 \times R_2 - \psi_1(I_1) \times R_2 = (I_1 - \psi_1(I_1)) \times R_2$. Then, $a_1 \bullet_1 b_1 \subseteq I_1 - \psi_1(I_1) \Longrightarrow a_1 \in I_1$ *or* $b_1 \in I_1$, so $(a_1,a_2) \subseteq I_1 \times R_2$ or $(b_1,b_2) \subseteq I_1 \times R_2$. Thus, $I_1 \times R_2$ is a ϕ −prime hyperideal of R. The proof of case (3) is similar. Next, assume that $I_1 \times I_2$ is a ϕ −prime hyperideal of R. Let $a \bullet_1 b \subseteq I_1 - \psi_1(I_1)$. Then, $(a, 0_{R_1}) \bullet (b, 0_{R_2}) = (a \bullet_1 b, 0_{R_2}) \subseteq I_1 \times I_2 - \phi(I_1 \times I_2)$. Thus, $(a, 0_{R_1}) \subseteq I_1 \times I_2$ or $(b, 0_{R_2}) \subseteq I_1 \times I_2$, i.e., $a \in I_1$ or $b \in I_1$. Thus, I_1 is a ψ_1 −prime hyperideal of R_1. Likewise, I_2 is a ψ_2 −prime hyperideal of R_2. Assume that $I_1 \times I_2 \neq \psi_1(I_1) \times \psi_2(I_2)$. Say $I_1 \neq \psi_1(I_1)$. Let $x \in I_1 - \psi_1(I_1)$. Let $y \in I_2$. Then, $(x, 1_{R_2}) \bullet (1_{R_1}, y) \subseteq I_1 \times I_2$. Thus, $(x, 1_{R_2}) \subseteq I_1 \times I_2$ or $(1_{R_1}, y) \subseteq I_1 \times I_2$. Hence $I_2 = R_2$ or $I_1 = R_1$. Assume that $I_2 = R_2$. So, $I_1 \times R_2$ is a ϕ −prime hyperideal of R, where I_1 is a ψ_1 −prime hyperideal of R_1. It remains to show that if $\psi_2(R_2) \neq R_2$, then I_1 is actually prime hyperideal of R_1. Let $a \bullet_1 b \subseteq I_1$. Now $1_{R_2} \not\subseteq \psi_2(R_2)$. Then, $(a, 1_{R_2}) \bullet (b, 1_{R_2}) \subseteq I_1 \times I_2 - \phi(I_1 \times I_2)$, so $(a, 1_{R_2}) \subseteq I_1 \times I_2$ or $(b, 1_{R_2}) \subseteq I_1 \times I_2$, that is, $a \in I_1$ or $b \in I_2$.

Conclusion

Here, we represented a new form of multiplicative hyperring theory. We discussed and proved new theorems in this area. We investigated the relation between the $\phi - 2$ −absorbing primary hyperideals and the 2 −absorbing primary hyperideals. Also, we dedicated the study to hyperideals of product of multiplicative hyperrings. We can extend the notion of $\phi - 2$ −absorbing primary hyperideals to the notion of $\phi - 2$ −absorbing quasi primary hyperideals in the next work.

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this article.

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