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Trivial Ring Extension of Suitable-Like Conditions and some properties

التمديدات الحلقية البديهية للحلقات المتكافئة وشبيهاتها وبعض الخصائص

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Abstract

We investigate the transfer of the notion of suitable rings along with related concepts, such as potent and semipotent rings, in the general context of the trivial ring extension, then we put these results in use to enrich the literature with new illustrative and counter examples subject to these ring-theoretic notions. Also we discuss some basic properties of the mentioned notions.

Keywords: Trivial ring extension, idealization, clean ring, potent ring, semipotent ring, von Neumann regular ring, suitable rings.

ملخص

نبحث في هذا المنشور عملية انتقال مفهوم الحلقات المكافئة و بعض المفاهيم المرتبطة كمفهومي الحلقات القوية و شبه القوية في الحالة العامة من التمديدات الحلقية البديهية، ومن ثم نستخدم النتائج الجديدة في تزويد المطبوعات بامثلة توضيحية تخضع للمفاهيم المدروسة. كما نناقش بعض الخصائص الحلقية لهذه المفاهيم.

الكلمات المفتاحية : التمديدات الحلقية البديهية، العمل بالمثالية، الحلقة النظيفة، الحلقة القوية، الحلقة شبه القوية، حلقة فون يومان المنتظمة، الحلقات المكافئة.

1 Introduction

Let R be a ring and I be an ideal of R. We say that idempotents lift modulo I if for each x in R such that $x - x^2 \in I$, there is an idempotent e in R such that $e - x \in I$. R is called a suitable ring if idempotents lift modulo every left ideal of R. In order to provide examples of suitable rings, Nicholson in 1977 introduced the concept of the clean rings (Nicholson, 1977). He defined a ring to be clean if all of it's elements are clean. Where the clean element is the element that can be written as a sum of a unit and an idempotent. Common examples of clean rings are local rings, Boolean rings, Division rings. The class of clean rings lies strictly inside the class of suitable rings. For more details about these rings we refer the reader to (Nicholson, 1977, 1975; Nicholson & Zhou, 2005a, 2005b) .

Let $J(R)$ denotes the Jacobson radical of R. R is called semipotent ring if any ideal I that is not contained in $J(R)$ contains a non zero idempotent, equivalently, R is semipotent if and only if for any $a \in R \setminus J(R)$, there is a non zero $x \in R$ such that $xax = x$. A semipotent ring is called potent if idempotents lift modulo $J(R)$. It is easy to see from the definitions that potent rings are semipotent and suitable rings are potent. Hence we have the following diagram of implications

$$\text{Clean} \Rightarrow \text{Suitable} \Rightarrow \text{Potent} \Rightarrow \text{Semipotent}$$

Where the first implication is reversible when the idempotents are central elements (particularly, if the ring is commutative) (Nicholson, 1977, Proposition 1.8) and the others are irreversible in general even if the ring is commutative. For counter examples see (Nicholson, 1977, P. 272) and (Nicholson & Zhou, 2005b, Example 25).

Recall that a ring R is called von Neumann regular ring if for each $x \in R$, there is $a \in R$ such that $xax = x$. By (Nicholson, 1977, P 271.), von Neumann regular rings are suitable while the converse is not true in general even if the ring is commutative. We will provide a counter example for the last mentioned fact.

In this article we will focus on the commutative case only. From the above discussion, if R is a commutative ring, then:

R is von Neumann regualar $\Rightarrow$ R is suitable $\Rightarrow$ R is potent $\Rightarrow$ R is semipotent

Where the aforementioned implications are irreversible in general. Examples are rare in the literature. We will use the trivial ring extensions to construct new such examples. Also to provide new classes of rings subject to the previous ring theoretic notions.

For a ring R and an R-module M, the trivial ring extension of R by M is the ring $R \ltimes M$ where the underlying group is $R \times M$ and the multiplication is defined by $(a,m)(b,f) = (ab, af + bm)$. The ring $R \ltimes M$ is also called the idealization of M over R and introduced in 1962 by Nagata (Nagata, 1962) in order to facilitate interaction between rings and their modules and also to provide families of examples of commutative rings containing zero-divisors (reduced elements). For more details on commutative trivial extensions (or idealizations), we refer the reader to (Abuhlail, Jarrar, & Kabbaj, 2011; Adarbeh & Kabbaj, 2017; Anderson & Winders, 2009; C. Bakkari & Mahdou, 2010; Damiano & Shapiro, 1984; Fossum, 1973; Goto, 1982; S. Goto, 2013; Glaz, 1989; Huckaba, 1988; Gulliksen, 1974; Kourki, 2009; Levin, 1985; ?, ?; Olberding, 2014; Palmér, 1973; Popescu, 1985; Reiten, 1972; Roos, 1981; Salce, 2009).

The first section is devoted to some basic properties of the suitable-like conditions. Namely, we provide a characterization of the (semi)suitable rings using the factor ring through an ideal contained in the Jacobson (nil) radical of the ring. Also we prove that the (semi)potent rings stable under taking the quotient modulo an ideal contained in the nilradical of the ring.

In the second section, we put the first section results in use to investigate the transfer of the potent rings along with related concepts, such as (semi)suitable and semipotent rings, in the most general case of the trivial ring extensions. Lastly, we will use the results to establish a new classes of examples subject to the mentioned ring theoretic notions.

Throughout, R denotes a commutative ring; $J(R)$ denotes the Jacobson radical of R; $Nil(R)$ denotes the nilradical of R; $Q(R)$ denotes the total ring of fractions of R; $Spec(R)$ denotes the set of all prime ideals of R; $Max(R)$ denotes the set of all maximal ideals of R.

2 Basic Properties

The following proposition provides a characterization of suitable rings.

Proposition 2.1. *Let R be a ring . Then R is a suitable ring if and only if $\frac{R}{I}$ is a suitable ring and idempotents lift modulo I, for every ideal I subset of $J(R)$.*

Proof. Since in the commutative context, suitable rings are clean. The result could be directly concluded from (Immormino, 2013, Theorem 1.6) □

A ring R is called semisuitable ring if idempotents lift modulo $J(R)$. Trivially, R is potent rings if and only if R is semipotent and semisuitable. The following is an example of a semisuitable ring which is not semipotent and hence not potent.

Example 2.2. Let $\mathbb{Z}$ be the ring of integers under the usual addition and multiplication. Then $\mathbb{Z}$ is a semisuitable ring which is not semipotent. Indeed, since $J(\mathbb{Z}) = 0$, $x - x^2 \in J(\mathbb{Z})$ if and only if $x - x^2 = 0$, if and only if $x = 0$ or $x = 1$. But 1 and 0 are the only possible idempotents of $\mathbb{Z}$. Hence idempotents of $\mathbb{Z}$ lift modulo $J(\mathbb{Z})$ and consequently, $\mathbb{Z}$ is semisuitable. For $\mathbb{Z}$ is not semipotent, notice that $2\mathbb{Z}$ is an ideal that is not contained in $J(\mathbb{Z})$ and doesn't contain any non zero idempotent.

The following fact characterizes the semisuitable rings.

Theorem 2.3. *Let R be a ring. Then R is a semisuitable ring if and only if $\frac{R}{I}$ is a semisuitable ring and idempotents lift modulo I, for every ideal I subset of $J(R)$.*

Proof. First notice that since $I \subseteq J(R)$, all the maximal ideals of R contains I and hence, $J(\frac{R}{I}) = \frac{J(R)}{I}$. Assume that R is semisuitable, we have to show that $\frac{R}{I}$ is semisuitable. Indeed, if $(x+I)^2-(x+I) \in J(\frac{R}{I}) = \frac{J(R)}{I}$, then $x^2-x \in J(R)$. But R being semisuitable implies that $x-e \in J(R)$ for some idempotent e in R. Hence, $(x+I)-(e+I) \in J(\frac{R}{I})$ and $(e+I)$ is an idempotent in $\frac{R}{I}$. Therefore $\frac{R}{I}$ is semisuitable. Now, idempotents lift modulo I can be easily concluded from (Nicholson & Zhou, 2005b, Lemma 5). Conversely, assume that $\frac{R}{I}$ is semisuitable and idempotents lift modulo I. If $x-x^2 \in J(R)$, then $(x+I)-(x+I)^2 \in \frac{J(R)}{I} = J(\frac{R}{I})$. $\frac{R}{I}$ being semisuitable implies that $(x+I)-(e+I) \in J(\frac{R}{I})$ for some $(e+I)$ idempotent in $\frac{R}{I}$. Since idempotents lift modulo I, $(e+I) = (e_0+I)$ for some e_0 idempotent in R. Hence $x-e_0 \in J(R)$ for some e_0 idempotent of R. Thus R is semisuitable. $\square$

Next, we see that the quotient of a semipotent ring through an ideal contained in the nilradical is also a semipotent ring.

Proposition 2.4. *Let R be a commutative ring. If R is semipotent, then $\frac{R}{I}$ is semipotent, for every ideal I subset of $Nil(R)$.*

Proof. Assume that R is semipotent and $I \subseteq Nil(R)$. We have to show that $\frac{R}{I}$ is semipotent. Indeed, let $\frac{K}{I}$ be an ideal of $\frac{R}{I}$ that is not contained in $J(\frac{R}{I})$. As $I \subseteq Nil(R) \subseteq J(R)$ (Hungerford, 1974, Theorem 2.12, Page 430)(Also one may use the fact that $Nil(R)$ is the intersection of all the prime ideals of R while $J(R)$ is the intersection of all the maximal ideals of R. Now maximals are primes finishes the conclusion), $J(\frac{R}{I}) = \frac{J(R)}{I}$. Hence K is not contained in $J(R)$. But R being semipotent implies K contains a non zero idempotent e. Since $(e+I)^2 = e^2+I = e+I$, $e+I$ is an idempotent element of $\frac{R}{I}$. It remains to show that $e+I$ is a non zero element of $\frac{R}{I}$. On the contrary, assume that $e+I = 0+I$ or equivalently $e \in I$. On the other hand $I \subseteq Nil(R)$ which implies that $e^n = 0$ for some minimal positive integer $n \geq 2$ ($n = 1$ implies $e = 0$, absurd). Using $e^2 = e$, we obtain

$e^2e^{n-2} = ee^{n-2} = e^{n-1} = 0$ which contradicts the fact that n is minimal. Hence $e+I \neq 0+I$. Therefore $\frac{R}{I}$ is semipotent. $\square$

Corollary 2.5. *Let R be a commutative ring. Then R is a (semi)suitable ring if and only if $\frac{R}{I}$ is a (semi)suitable ring, for every ideal I subset of $Nil(R)$.*

Proof. Combine both of Proposition 2.1 and Theorem 2.3 with the fact that idempotents lift modulo every ideal in the $Nil(R)$ (Koh, 1974). $\square$

Corollary 2.6. *Let R be a commutative ring. If R is a (semi)potent ring, then $\frac{R}{I}$ is a (semi)potent ring, for every ideal I subset of $Nil(R)$.*

Proof. The result follows directly from Theorem 2.3 and Proposition 2.4. $\square$

The following proposition deals with the behaviour of the semisuitability via the direct product of rings. It deserves to recall that if R_1 and R_2 are rings, then $J(R_1 \times R_2) = J(R_1) \times J(R_2)$. (Hungerford, 1974, Theorem 2.17, Page 432)

Proposition 2.7. *A direct product of rings $\prod_{i \in I} R_i$ is semisuitable if and only if each R_i is semisuitable.*

Proof. For simplicity we may choose $I = \{1,2\}$ and the general case has an identical proof. Assume that $R_1 \times R_2$ is semisuitable and $x_1^2 - x_1 \in J(R_1)$. Then $(x_1,0)^2 - (x_1,0) \in J(R_1) \times J(R_2) = J(R_1 \times R_2)$. Since $R_1 \times R_2$ is semisuitable, there is an idempotent (e_1,e_2) in $R_1 \times R_2$ such that $(x_1,0) - (e_1,e_2) \in J(R_1) \times J(R_2)$. Consequently $x_1 - e_1 \in J(R_1)$ with e_1 idempotent in R_1 and hence R_1 is semisuitable. Similar arguments show that R_2 is semisuitable. Conversely, assume that both R_1 and R_2 are semisuitable and let $(x_1,x_2)^2 - (x_1,x_2) = (x_1^2 - x_1, x_2^2 - x_2) \in J(R_1) \times J(R_2)$. Then $x_i^2 - x_i \in J(R_i)$ for $i = 1,2$. But R_i being semisuitable implies that there is an idempotent $e_i \in R_i$ such that $x_i - e_i \in J(R_i)$ for $i = 1,2$. Hence $(x_1,x_2) - (e_1,e_2) \in J(R_1) \times J(R_2) = J(R_1 \times R_2)$. Observing that for $i = 1,2$, e_i is idempotent in R_i implies that (e_1, e_2) is an idempotent in $R_1 \times R_2$ finishes the proof of $R_1 \times R_2$ is semisuitable. $\square$

Remark 2.8. Notice that the suitable, potent, and semipotent notions are not local notions. Indeed, if we consider a non semipotent ring R (e.g., $\mathbb{Z}$), then since the local rings are suitable, $R_{\mathfrak{m}}$ is suitable for any maximal $\mathfrak{m}$ (hence, potent and semipotent). While R from the assumption is not semipotent and obviously, neither potent nor suitable. A similar argument can be used to prove that also the semisuitability is not a local notion. For example of non semisuitable ring we refer the reader to (Nicholson & Zhou, 2005b, Example 25).

3 Transfer Results and Examples

This section investigates the transfer of the von Neumann regular, suitable, potent, and semipotent rings in trivial ring extension in the most general case.

Recall that, $Spec(R \ltimes M) = \{p \ltimes M \mid p \in spec(R)\}$ and $Max(R \ltimes M) = \{m \ltimes M \mid m \in Max(R)\}$ (Huckaba, 1988, Theorem 25.1(3)). So we conclude that $J(R \ltimes M) = J(R) \ltimes M$ and $Nil(R \ltimes M) = Nil(R) \ltimes M$.
Also notice that R is a homomorphic image of $R \ltimes M$ in view of the short exact sequence, $0 \to 0 \ltimes M \to R \ltimes M \to R \to 0$. Where the third arrow is the projection map.

The following result establishes conditions under which the general case of the trivial extension of any ring inherits von Neumann-regularity, (semi)
suitability, and (semi)potency.

Theorem 3.1. *Let R be a ring and M an R-module. Then:*

1. *$R \ltimes M$ is von Neumann regular if and only if R is von Neumann regular and $M = 0$.*

2. *$R \ltimes M$ is (semi)suitable if and only if R is (semi)suitable.*

3. *$R \ltimes M$ is semipotent if and only if R is semipotent.*

4. *$R \ltimes M$ is potent if and only if R is potent.*

Proof. (1) Assume that R is von Neumann regular and m is an arbitrary element in M. Since R is a commutative von Neumann regular ring, there is $(x,y) \in R \ltimes M$ such that $(0,m)^2(x,y) = (0,m)$. Hence $(0,0) = (0,m)$. m being an arbitrary element in M implies that $M = 0$. Now, $R \ltimes M$ is von Neumann regular and $M = 0$ easily implies that R is von Neumann regular. The converse is obvious.
(2) First notice that $(0 \ltimes M) \subseteq Nil(R \ltimes M)$ (as $(0 \ltimes M)^2 = 0$). By Corollary 2.5, $R \ltimes M$ is (semi)suitable if and only if $\frac{R \ltimes M}{0 \ltimes M} \cong R$ is (semi)suitable.
(3) Assume that $R \ltimes M$ is semipotent. By Corollary 2.6, $\frac{R \ltimes M}{0 \ltimes M} \cong R$ is (semi)potent. Conversely, assume that R is semipotent and $(a,m) \in R \ltimes M \setminus J(R \ltimes M)$. We need to find a non zero element $(X,Y) \in R \ltimes M$ such that $(a,m)(X,Y)^2 = (X,Y)$. Now, $(a,m) \in R \ltimes M \setminus J(R \ltimes M)$ implies that $a \in R \setminus J(R)$. Since R is semipotent, there is a non zero $x \in R$ such that $x^2a = x$. Let $X = x$ and $Y = -x^2m$. Then:

$$\begin{aligned}
(a,m)(X,Y)^2 &= (a,m)(x,-x^2m)^2 \\
&= (a,m)(x^2,-2x^3m) \\
&= (x^2a,-2x^3am+x^2m) \\
&= (x^2a,-2xx^2am+x^2m) \\
&= (x,-2x^2m+x^2m) \\
&= (x,-x^2m) \\
&= (X,Y).
\end{aligned}$$

Therefore, $R \ltimes M$ is semipotent.
(4) Follows directly from (2) and (3).

□

The following example enrich the literature with examples which insures that the class of von Neumann regular rings lies strictly inside the class of suitable rings.

Example 3.2. Let R be a von Neumann regular ring (e.g, a field) and M a non zero R module. Then by Theorem 3.1, $R \ltimes M$ is a suitable ring which is not von Neumann regular.

In the following example, we use Example 2.2 and Theorem 3.1 to generate a class of semisuitable rings that are not suitable.

Example 3.3. Let $\mathbb{Z}$ be the ring of integers and M be any $\mathbb{Z}$-module (e.g., M is any abelian group). Then $\mathbb{Z} \ltimes M$ is another example of a semisuitable ring which is not suitable.

Recall from Example 2.2 that $\mathbb{Z}$ is semisuitable which is not suitable. Hence, by Theorem 3.1(2), $\mathbb{Z} \ltimes M$ is semisuitable and not suitable.

One may also use Theorem 3.1 again to enrich the literature with a semipotent rings that are not potent.

Example 3.4. Let R be a semipotent ring which is not potent (e.g., the ring in (Nicholson & Zhou, 2005b, Example 25)) and M is a non zero R-module. Then, by Theorem 3.1(3), $R \ltimes M$ generates a new class of semipotent rings that are not potent.

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