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Solution of Fractional Differential Equations using Integral Operators with multi-index Mittag-Leffler matrix function in the Kernels

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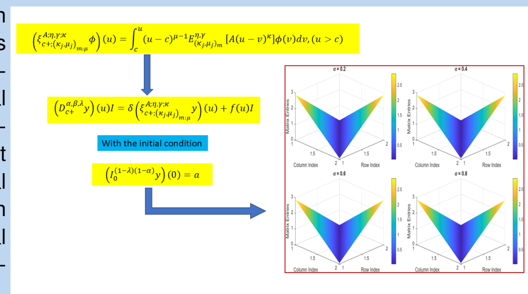
Abstract: In this article, we introduce a new integral operator associated with multi-index Mittag-Leffler matrix function. We present the boundedness of this operator in the Lebesgue space $\mathcal{L}(c, d)$. Furthermore, we analyze the composition of this new operator with standard Riemann-Liouville fractional integral and derivative operators. Subsequently, we formulate first-order Volterra integro-differential equation involving a multi-index integral operator and obtain its explicit solution using the Laplace transform method. In addition, we present numerical and graphical analysis of the solution to the Volterra integro-differential equation involving the multi-index integral operator. These results enhance the analytical framework of special matrix functions and contribute to the study of fractional integral operators and their applications.



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Introduction

Fractional calculus has emerged as an effective tool for analyzing complex phenomena across various scientific [1] and engineering fields [2, 3]. In recent year, fractional differential equations have been formulated by extending classical integer-order derivatives to arbitrary order. The fractional derivatives play crucial role in these equations, making them highly effective for modeling wide range of physical phenomena. Subsequently, integral operators are employed to describe intermediate processes and critical phenomena in physics, fluid mechanics [4], and various other fields[5, 6, 7, 8, 9].

The Riemann-Liouville fractional integral operator, along with its Laplace transform and power function formula are defined as follows [5]:

$$(\mathcal{I}^{\alpha}_{c+}\phi)(u) = \frac{1}{\Gamma(\alpha)}\int_c^u (u-v)^{\alpha-1}\phi(v)\mathrm{d}v, \quad (\Re(\alpha) > 0), \tag{1}$$

$$\mathcal{L}\{\mathcal{I}^{\alpha}_{c+}\phi; s\} = s^{-\alpha}\Phi(s), \ (\Re(\alpha) > 0, \ \Re(s) > 0), \tag{2}$$

and

$$\mathcal{I}^{\alpha}_{c+}(u-c)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\alpha)}(u-c)^{\alpha+\nu-1}, \quad (\Re(\alpha) > 0, \nu > 0). \tag{3}$$

The Riemann-Liouville derivative operator and Caputo derivative are given as below [5]:

$$\mathcal{D}^{\alpha}_{c+}\phi = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n (\mathcal{I}^{n-\alpha}_{c+}\phi)(u), \quad (\Re(\alpha) > 0; \ n = [\Re(\alpha)] + 1), \tag{4}$$

and

$$(\mathcal{D}^{\alpha}_{c+}\phi)(u) = (\mathcal{I}^{n-\alpha}_{c+}\mathcal{D}^{\alpha}_{c+}\phi)(u) = \frac{1}{\Gamma(n-\alpha)}\int_c^u (u-v)^{n-\alpha-1}\phi^n(v)\mathrm{d}v, \tag{5}$$

$$(n-1 < \alpha \leq n, \quad n \in \mathbb{N}).$$

The Hilfer fractional derivative operator is given as follows [10]:

$$\mathcal{D}^{\alpha,\lambda}_{c+} = \left(\mathcal{I}^{\lambda(1-\alpha)}_{c+}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n (\mathcal{I}^{(1-\lambda)(1-\alpha)}_{c+}\phi)\right)(u). \tag{6}$$

Garg et al. [11, p.1072, Eqs. (17) and (22)], presented the generalized composite fractional derivative operator, along with its power function formula and Laplace transform, which are defined as in the following forms:

$$\mathcal{D}^{\alpha,\beta,\lambda}_{c+} = \left(\mathcal{I}^{\lambda(n-\alpha)}_{c+}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n (\mathcal{I}^{(1-\lambda)(n-\alpha)}_{c+}\phi)\right)(u), \tag{7}$$

$$(n-1 < \alpha, \ \beta < n, \ 0 \leq \lambda \leq 1, \ n \in \mathbb{N}),$$

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$$\mathcal{D}_{c+}^{\alpha,\beta;\lambda}(u-c)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma\big(\lambda(\alpha-\beta)+\nu-\alpha\big)}(u-c)^{\lambda(\alpha-\beta)+\nu-\alpha-1}, \tag{8}$$

$$(\nu > 0,\, n-1 < \alpha, \beta \leq n,\, 0 \leq \lambda \leq 1,\, n \in \mathbb{N}),$$

and

$$\mathcal{L}\left[\mathcal{D}_{c+}^{\alpha,\beta;\lambda}\phi(u); s\right] = s^{\lambda(\beta-\alpha)+\alpha}F(s) - \sum_{r=0}^{n-1} s^{\,n-r-1-\lambda(n-\beta)} \times \left(\mathcal{D}^r\left(\mathcal{I}_{c+}^{(1-\lambda)(1-\alpha)}f\right)(u)\right)\Big|_{u\to 0}, \tag{9}$$

$$(\Re(s) > 0, n-1 < \alpha, \beta \leq n, 0 \leq \lambda \leq 1, n \in \mathbb{N}).$$

Remark 1. *On substituting $\alpha = \beta$ in Eq.*(7)*, the generalized composite fractional derivative operator reduces to the Hilfer fractional derivative operator given in Eq.*(6)*. Further, we put $\alpha = \beta$ and $\lambda = 0$, then we have Riemann-Liouville fractional derivative operator give in Eq.*(4)*.*

The classical Mittag-Leffler function $E_\kappa(z)$ is defined by the following series representation [12, Eq.(1)]:

$$E_\kappa(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+1)}, \quad (z \in \mathbb{C}; \Re(\kappa) > 0). \tag{10}$$

Wiman [13] generalized the Mittag-Leffler function $E_\kappa(z)$, which is defined as follows:

$$E_{\kappa,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+\mu)}, \quad (\delta, \lambda \in \mathbb{C}; \Re(\kappa) > 0, \Re(\mu) > 0). \tag{11}$$

Prabhakar [14] presented the three-parameter Mittag-Leffler function $\mathbb{E}_{\kappa,\mu}^{\eta}(z)$, which is given below:

$$E_{\kappa,\mu}^{\eta}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_n}{\Gamma(\kappa n+\mu)} \frac{z^n}{n!}, \tag{12}$$

$$(\delta, \lambda, \mu \in \mathbb{C}; \Re(\kappa) > 0, \Re(\mu) > 0, \Re(\eta) > 0).$$

Srivastava and Tomovski [10] further generalized the Mittag-Leffler function $\mathbb{E}_{\kappa,\mu}^{\eta,\gamma}(z)$, which is given as follows:

$$E_{\kappa,\mu}^{\eta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\Gamma(\kappa n+\mu)} \frac{z^n}{n!}, \tag{13}$$

$$(\kappa, \mu, \eta, \gamma \in \mathbb{C}; \Re(\kappa) > \max\{0, \Re(\gamma) - 1\}; \Re(\gamma) > 0).$$

The multi-index Mittag-Leffler function is given in the following form [12, Eq.(8)]:

$$E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\prod\limits_{j=1}^{m}\Gamma(\kappa_j n+\mu_j)} \frac{z^n}{n!}, \tag{14}$$

$$(\delta_j, \lambda_j, \mu, \kappa \in \mathbb{C}, \Re(\mu_j) > 0\, (j = 1, 2, \cdots, m);$$

$$\Re\left(\sum_{j=1}^{m}\kappa_j\right) > max\{0, \Re(\gamma) - 1\}).$$

We now proceed to discuss the theory of special matrix functions introduced by Jódar and Cortés [15]. They defined the properties of some special matrix functions. Çekim [16] investigates the generalization of Beta, Gamma, hypergeometric and confluent hypergeometric matrix functions along with their integral representations and recurrence relations. Abdalla and Bakhet [17] introduced extensions of the Gamma and Beta matrix functions and established several of their fundamental properties. Altin et al. [18] presented a matrix formulation of the Appell hypergeometric function and examined its properties. Numerous researchers have investigated and systematically studied extended Gamma, Beta functions [19], hypergeometric function [20] and Mittag-Leffler function [21, 22] with matrix arguments. These functions play a significant role in several areas, including the developing theory of orthogonal matrix polynomials, mathematical physics, and theoretical physics [23].

A square matrix $A \in \mathbb{C}^{r\times r}$ is called positive stable if $\Re(\zeta) > 0$, $\forall \zeta \in \sigma(A)$, spectrum of A is denoted by $\sigma(A)$ and spectral abscissa of A is defined as [24]:

$$\mathfrak{A}(A) = max\{\Re(z) : z \in \sigma(A)\}, \quad \mathfrak{B}(A) = min\{\Re(z) : z \in \sigma(A)\},$$
$$\text{and} \quad \mathfrak{B}(A) = -\mathfrak{A}(-A). \tag{15}$$

The Euclidean norm of A is defined as:

$$||A|| = \sup_{\mathsf{x}\neq 0} \frac{||A\mathsf{x}||}{||\mathsf{x}||} = max\{\sqrt{\zeta} : \zeta \in \sigma(A^*A)\}, \tag{16}$$

where, $||\mathsf{x}|| = (\mathsf{x}^*\mathsf{x})^{\frac{1}{2}}$ is Euclidean norm of x, for any vector $\mathsf{x} \in \mathbb{C}^{r\times r}$, and A^* is conjugate transpose of A.

Jódar and Cortés [15] introduced the Gamma matrix function $\Gamma(A)$, which is defined as follows:

$$\Gamma(A) = \int_0^\infty e^{-u}u^{A-I}du, \tag{17}$$

where $A \in \mathbb{C}^{r\times r}$ and A is a positive stable matrix.

If $A + nI$ is a non-singular matrix for every $n \in \mathbb{N}$, then the inverse Gamma matrix function [15] is defined by

$$\Gamma^{-1}(A) = A(A+I)\cdots(A+(n-1)I)\Gamma^{-1}(A+nI), \quad n \in \mathbb{N}. \tag{18}$$

The Pochhammer matrix symbol is defined in [15] as follows:

$$(A)_n = \begin{cases} I, & n = 0, \\ A(A+I)\cdots(A+(n-1)I), & n \in \mathbb{N}. \end{cases} \tag{19}$$

From Eqs. (18) and (19), we have

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A). \tag{20}$$

A limiting representation of the Gamma matrix function was presented in [15] as follows:

$$\Gamma(A) = \lim_{n\to\infty}(n-1)!(A)_n^{-1}n^A, \quad n \in \mathbb{N}. \tag{21}$$

Let A, B be positive stable matrices in $\mathbb{C}^{r\times r}$. The Beta matrix function $\mathbb{B}(A, B)$ is defined as follows [15]:

$$\mathbb{B}(A, B) = \int_0^1 u^{A-I}(1-u)^{B-I}du. \tag{22}$$

If $A, B, A + B \in \mathbb{C}^{r\times r}$, A and B are commutative matrices and $A + nI, B + nI$ and $A + B + nI$ are invertible matrices, then defined in [15] such as:

$$\mathbb{B}(A, B) = \Gamma(A)\Gamma(B)\Gamma^{-1}(A+B). \tag{23}$$

Çekim [16] explored the generalizations of Beta matrix functions, which are given below:

$$\mathbb{B}_q(A, B) = \int_0^1 u^{A-I}(1-u)^{B-I}e^{\left(\frac{-q}{u(1-u)}\right)}du, \tag{24}$$

where A and B are positive stable matrices in $\mathbb{C}^{r\times r}$, and $\Re(q) > 0$.

Abdalla and Bakhet [17] also generalized the Beta matrix functions in the following form:

$$\mathbb{B}(A, B; C) = \int_0^1 u^{A-I}(1-u)^{B-I}e^{\left(\frac{-C}{u(1-u)}\right)}du, \tag{25}$$

where $A, B, C \in \mathbb{C}^{r\times r}$ are positive stable matrices.

Garrappa et al. [25] defined the two-parameter Mittag-Leffler matrix function in the following form:

$$E_{\kappa,\mu}(A) = \sum_{n=0}^{\infty} \frac{A^n}{\Gamma(\kappa n+\mu)}, \tag{26}$$

where $A \in \mathbb{C}^{r\times r}$ is a positive stable matrix and $\Re(\kappa), \Re(\mu) > 0$.

Goyal et al. [26] proposed a new extension of the Beta matrix function based on a two-parameter Mittag-Leffler matrix function, defined as follows:

$$\mathbb{B}_{\kappa,\mu}^{C}(A, B) = \int_0^1 u^{A-I}(1-u)^{B-I}E_{\kappa,\mu}\left(\frac{-C}{u(1-u)}\right)du, \tag{27}$$

where $\Re(\kappa), \Re(\mu) > 0$ and A, B, C are positive stable matrices in $\mathbb{C}^{r\times r}$.

Bhammar and Jana [19] introduced the four-parameter Mittag-Leffler matrix function, defined as follows:

$$E^{\eta,\gamma}_{\kappa,\mu}(A) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\Gamma(\kappa n+\mu)} \frac{A^n}{n!}, \tag{28}$$

where $A \in \mathbb{C}^{r\times r}$ is a positive stable matrix, $\Re(\mu), \Re(\kappa), \Re(\eta) > 0$ and $\gamma \in (0,1) \cup \mathbb{N}$.

Sharma et al. [27] introduced generalized Mittag-Leffler function with matrix arguments, is defined as follows:

$$E^{\eta,\gamma;\sigma}_{\kappa,\mu,\rho}(A) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\Gamma(\kappa n+\mu)(\rho)_{\sigma n}} \frac{A^n}{n!}, \tag{29}$$

$(A \in \mathbb{C}^{r\times r}, \kappa, \mu, \eta, \gamma \in \mathbb{C},\ \sigma,\rho \in \mathbb{R}, \min\{\Re(\kappa), \Re(\mu), \Re(\eta), \Re(\gamma)\} > 0,$ and $\sigma \leq \Re(\kappa) + \rho)$.

Pal et al. [21] examined matrix analogue of the four-parameter Mittag-Leffler function and demonstrated its convergence on $|z| = 1$:

$$E^{\eta,\mathsf{Q}}_{\kappa,\mathsf{S}}(z) = \Gamma^{-1}(\mathsf{Q}) \sum_{n=0}^{\infty} \Gamma(\mathsf{Q}+\eta nI)\Gamma^{-1}(\mathsf{S}+\kappa nI) \frac{z^n}{n!}, \tag{30}$$

where $\kappa, \eta \in \mathbb{R}^+$ and S, Q are positive stable matrices in $\mathbb{C}^{r\times r}$.

Sharma et al. [28] introduced an integral operator involving following extended Mittag-Leffler matrix function:

$$E^{\mathsf{P},\mathsf{Q}}_{\mathsf{R},\mathsf{S}}(\mathsf{z}) = \Gamma^{-1}(\mathsf{Q}) \sum_{\mathsf{n}=0}^{\infty} \Gamma(\mathsf{Pn}+\mathsf{Q})\Gamma^{-1}(\mathsf{Rn}+\mathsf{S}) \frac{\mathsf{z}^{\mathsf{n}}}{\mathsf{n}!}, \tag{31}$$

where $\mathsf{P}, \mathsf{Q}, \mathsf{R}$, and S are positive stable matrices in $\mathbb{C}^{r\times r}$ and $\mathsf{Rn}+\mathsf{S}$ is non-singular matrix for every $n \geq 0$.

Definition 1. *[29] If $A \in \mathbb{C}^{r\times r}$, $\kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$, and $\Re(\kappa_j) > 0\,(j = 1, 2, \cdots, m)$ such that $\Re\left(\sum_{j=1}^m \kappa_j\right) > \max\{0, \Re(\gamma) - 1\}$, then the multi-index Mittag-Leffler matrix function is defined as:*

$$E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}(A) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{A^n}{n!}. \tag{32}$$

Remark 2. *On substituting $m = 1$ in Eq.(32), then we have four-parameter Mittag-Leffler matrix function given in Eq.(28).*

Remark 3. *On putting $m = 1$, and $\eta = \gamma = 1$ in Eq.(32), then we have two-parameter Mittag-Leffler matrix function defined in Eq.(26).*

Definition 2. *[29] If $A \in \mathbb{C}^{r\times r}$, $\mathbb{A}_1, \cdots, \mathbb{A}_r \in \mathbb{R}^+$ and $\mathbb{B}_1, \cdots, \mathbb{B}_s \in \mathbb{R}^+$ such that $1 + \sum_{j=1}^s \mathbb{B}_j - \sum_{j=1}^r \mathbb{A}_j \geq 0$, then the generalized Fox-Wright hypergeometric matrix function is defined as follows:*

$${}_r\Psi_s \left[\begin{array}{l} (\delta_1, \mathbb{A}_1), \cdots, (\delta_r, \mathbb{A}_r); \\ (\lambda_1, \mathbb{B}_1), \cdots, (\lambda_s, \mathbb{B}_s); \end{array} A \right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{r} \Gamma(\delta_j + \mathbb{A}_j n)}{\prod\limits_{j=1}^{s} \Gamma(\lambda_j + \mathbb{B}_j n)} \frac{A^n}{n!}. \tag{33}$$

Numerous functions arise as special cases of the Fox-Wright hypergeometric matrix function through appropriate specialization of its parameters. A few of these functions are presented here.

(i) The relationship between the multi-index Mittag-Leffler matrix function and the Fox-Wright hypergeometric matrix function can be expressed in the following manner:

$$E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}(A) = \Gamma^{-1}(\eta)\, {}_1\Psi_m \left[\begin{array}{c} (\eta, \gamma); \\ (\kappa_j, \mu_j)_1^m; \end{array} A \right]. \tag{34}$$

(ii) If we put m=1 in Eq.(34), then it reduces to the confluent hypergeometric matrix function:

$$E^{\eta,\gamma}_{\kappa,\mu}(A) = \Gamma^{-1}(\eta)\, {}_1\Psi_1 \left[\begin{array}{l} (\eta, \gamma); \\ (\kappa, \mu); \end{array} A \right]. \tag{35}$$

Now, we establish the convergence condition of the multi-index Mittag-Leffler matrix function and Fox-Wright hypergeometric matrix function.

Theorem 1. *[29] Let $\kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$ and $\Re(\mu_j) > 0\,(j = 1, 2, \cdots, m)$ and A be positive stable matrix in $\mathbb{C}^{r\times r}$ such that $\Re\left(\sum_{j=1}^m \kappa_j\right) > max\{0, \Re(\gamma) - 1\}$. Then the matrix series*

$$E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}(A) = \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{A^n}{n!}, \tag{36}$$

converges absolutely for every matrix $A \in \mathbb{C}^{r\times r}$.

Proof. Let $\|\cdot\|$ be any submultiplicative matrix norm. For the nth term of the series, we estimate

$$\begin{aligned} \left\| \frac{(\eta)_{\gamma n}}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{A^n}{n!} \right\| &\leq \frac{|(\eta)_{\gamma n}|}{\left|\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)\right| |n!|} \|A^n\|, \\ &\leq \frac{|(\eta)_{\gamma n}|}{\left|\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)\right| |n!|} \|A\|^n. \end{aligned} \tag{37}$$

Using the generalized Pochhammer symbol, we obtain

$$\left\| \frac{(\eta)_{\gamma n}}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{A^n}{n!} \right\| \leq \frac{1}{|\Gamma(\eta)|} \frac{|\Gamma(\eta+\gamma n)|}{\left|\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)\right| |n!|} \|A\|^n. \tag{38}$$

For large n, Stirling's asymptotic formula [30],

$$\Gamma(an+b) \sim \sqrt{2\pi}\,(an)^{an+b-\frac{1}{2}} e^{-an}, \qquad a > 0, \tag{39}$$

yields,

$$\frac{\Gamma(\eta+\gamma n)}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{1}{n!} = O\left(n^{-\left(\Re\left(\sum\limits_{j=1}^{m}(\kappa_j)-\gamma\right)+1\right)n} \right), \quad \text{as } n \to \infty. \tag{40}$$

Consequently, there exists a constant $C > 0$ such that

$$\left\| \frac{(\mu)_{\kappa n}}{\prod\limits_{j=1}^{m} \Gamma(\kappa_j n+\mu_j)} \frac{A^n}{n!} \right\| \leq C\, \|A\|^n\, n^{-\left(\Re\left(\sum\limits_{j=1}^{m}(\kappa_j)-\gamma\right)+1\right)n}. \tag{41}$$

Hence, under the condition $\Re\left(\sum_{j=1}^m \kappa_j\right) > max\{0, \Re(\gamma) - 1\}$, the series converges absolutely for every $A \in \mathbb{C}^{r\times r}$. □

Theorem 2. *Let $A \in \mathbb{C}^{r\times r}$, $\mathbb{A}_1, \cdots, \mathbb{A}_r \in \mathbb{R}^+$ and $\mathbb{B}_1, \cdots, \mathbb{B}_s \in \mathbb{R}^+$ such that $1 + \sum_{j=1}^s \mathbb{B}_j - \sum_{j=1}^r \mathbb{A}_j \geq 0$, then the matrix series*

$${}_r\Psi_s \left[\begin{array}{l} (\delta_1, \mathbb{A}_1), \cdots, (\delta_r, \mathbb{A}_r); \\ (\lambda_1, \mathbb{B}_1), \cdots, (\lambda_s, \mathbb{B}_s); \end{array} A \right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{r} \Gamma(\delta_j + \mathbb{A}_j n)}{\prod\limits_{j=1}^{s} \Gamma(\lambda_j + \mathbb{B}_j n)} \frac{A^n}{n!}, \tag{42}$$

converges absolutely for every matrix $A \in \mathbb{C}^{r\times r}$.

Proof. Let $\|\cdot\|$ be any submultiplicative matrix norm. For the nth term of the series, we estimate

$$\left\| \frac{\prod_{j=1}^{r}\Gamma(\delta_j+\mathbb{A}_j n)}{\prod_{j=1}^{s}\Gamma(\lambda_j+\mathbb{B}_j n)} \frac{A^n}{n!} \right\| \leq \frac{\left|\prod_{j=1}^{r}\Gamma(\delta_j+\mathbb{A}_j n)\right|}{\left|\prod_{j=1}^{s}\Gamma(\lambda_j+\mathbb{B}_j n)\right|} \frac{\|A\|^n}{|n!|}$$

For large n, Stirling's asymptotic formula [30],

$$\Gamma(an+b)\sim\sqrt{2\pi}\,(an)^{an+b-\frac{1}{2}}e^{-an}, \qquad a>0, \tag{43}$$

$$\frac{\prod_{j=1}^{r}\Gamma(\delta_j+\mathbb{A}_j n)}{\prod_{j=1}^{s}\Gamma(\lambda_j+\mathbb{B}_j n)} \frac{1}{n!} = O\left(n^{-\left(\Re\left(\sum_{j=1}^{s}\mathbb{B}_j-\sum_{j=1}^{r}\mathbb{A}_j\right)+1\right)n}\right), \text{as } n\to\infty.$$

Consequently, there exists a constant $C>0$ such that

$$\left\| \frac{\prod_{j=1}^{r}\Gamma(\delta_j+\mathbb{A}_j n)}{\prod_{j=1}^{s}\Gamma(\lambda_j+\mathbb{B}_j n)} \frac{A^n}{n!} \right\| \leq C\|A\|^n n^{-\left(\Re\left(\sum_{j=1}^{s}\mathbb{B}_j-\sum_{j=1}^{r}\mathbb{A}_j\right)+1\right)n}$$

Hence, under the condition $1+\sum_{j=1}^{s}\mathbb{B}_j-\sum_{j=1}^{r}\mathbb{A}_j\geq 0$, the series converges absolutely for every $A\in\mathbb{C}^{r\times r}$. ☐

Theorem 3. *[29] If $A\in\mathbb{C}^{r\times r}$ and $\kappa,\mu,\kappa_j,\mu_j,\eta,\gamma\in\mathbb{C}$, $\Re(\kappa_j)>0\,(j=1,2,\cdots,m)$ such that $\Re\left(\sum_{j=1}^{m}\kappa_j\right)>max\{0,\Re(\gamma)-1\}$, then the Laplace transform of multi-index Mittag-Leffler matrix function is defined as:*

$$\int_0^\infty \tau^{\mu-1}e^{-s\tau}E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}(A\tau^\beta)\mathrm{d}\tau=\frac{s^{-\mu}}{\Gamma(\eta)}{}_2\Psi_m\left[\begin{array}{cc}(\eta,\gamma),(\mu,\kappa); & \dfrac{A}{}\\ (\mu_j,\kappa_j)_1^m; & s^\kappa\end{array}\right]. \tag{44}$$

Proof. Let left hand side of Eq.(44) be represented by $\mathfrak{Q}$ and with the help of Eq.(32), we have

$$\mathfrak{Q}=\int_0^\infty\tau^{\mu-1}e^{-s\tau}\sum_{n=0}^{\infty}\frac{(\eta)_{\gamma n}}{\prod_{j=1}^{m}\Gamma(\kappa_j n+\mu_j)}\frac{(A\tau^\beta)^n}{n!}d\tau.$$

Next, interchanging the order of integration and summations, we have

$$\mathfrak{Q}=\frac{s^{-\mu}}{\Gamma(\eta)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta+\gamma n)\Gamma(\beta n+\mu)}{\prod_{j=1}^{m}\Gamma(\kappa_j n+\mu_j)}\frac{A^n}{s^\beta n!}.$$

Finally, making use of definition (42), we obtain our desired result. ☐

Multi-index Integral Operator with Matrix Arguments

Numerous researchers[31, 32] constructed integral operators using special kernels and exhibit distinctive key features. Prabhakar [14] introduced fractional integral operator that involve generalized Mittag-Leffler function in the kernel. Subsequently, Srivastva et al.[12] investigated the fractional integral operator that contains multi-index Mittag-Leffler function in its kernel. Bansal et al.[33] established an integral operator involving the family of incomplete H-function in kernel. Recently, Sharma et al. [28] developed integral operator with matrix arguments associated four-parameter Mittag-Leffler matrix function in the kernel.
Motivated by the preceding studies, we introduced new integral operator involving multi-index Mittag-Leffler matrix function as kernel. Furthermore, we establish the boundedness of this operator on the space $\mathcal{L}(c,d)$. Next, we derive power formula of multi-index Mittag-Leffler matrix function under generalized composite fractional derivative operator.

Definition 3. *Let $A\in\mathbb{C}^{r\times r}$ and $u,v,\kappa,\mu,\kappa_j,\mu_j,\eta,\gamma\in\mathbb{C}$, $\Re(\kappa_j)>0\,(j=1,2,\cdots,m)$ such that $\Re\left(\sum_{j=1}^{m}\kappa_j\right)>\max\{0,\Re(\gamma)-1\}$, $(u>c)$. Then, the integral operator involving multi-index Mittag-Leffler matrix function as a kernel is defined as follows:*

$$(\xi^{A;\eta,\gamma;\kappa}_{c+;(\kappa_j,\mu_j)_m;\mu}\phi)(u)=\int_c^u(u-v)^{\mu-1}E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[A(u-v)^\kappa]\phi(v)\mathrm{d}v. \tag{45}$$

Remark 4. *On putting $m=1$ in above equation, we have new integral operator involving four-parameter Mittag-Leffler function with matrix arguments.*

Theorem 4. *Under the given condition of definition 3 and let $[c,d](d>c)$ be closed interval and ϕ be Lebesgue measurable function on $\mathcal{L}(c,d)$ of the real line $\mathbb{R}$ given by*

$$\mathcal{L}(c,d)=\left\{\phi:\ \|\phi\|_1=\int_c^d|\phi(u)|\,\mathrm{d}u<\infty\right\}. \tag{46}$$

Then integral operator $\xi^{A;\eta,\gamma;\kappa}_{c+;(\kappa_j,\mu_j)_m;\mu}$ is bounded on $\mathcal{L}(c,d)$:

$$\|\xi^{A;\eta,\gamma;\kappa}_{c+;(\kappa_j,\mu_j)_m;\mu}\phi\|_1\leq k\,\|\phi\|_1, \tag{47}$$

where the constant

$$k=(d-c)^{\Re(\mu)}\sum_{n=0}^{\infty}\frac{|(\eta)_{\gamma n}|}{\{\Re(\kappa)n+\mu\}\prod_{j=1}^{m}\Gamma(\Re(\kappa_j)n+\Re(\mu_j))}\times\frac{\|A\|^n(d-c)^{\Re(\kappa)n}}{n!}. \tag{48}$$

Proof. Let n^{th} term of series (48) is k_n then the relation

$$\left\|\frac{k_{n+1}}{k_n}\right\| = \frac{|\Gamma(\eta+\gamma+\gamma n)|(\Re(\kappa)n+\mu)\prod_{j=1}^{m}\Gamma(\Re(\kappa_j)n+\Re(\mu_j))\|A\|(d-c)^{\Re(\kappa)}}{(n+1)|\Gamma(\eta+\gamma n)|(\Re(\kappa)n+\Re(\kappa)+\mu)\prod_{j=1}^{m}\Gamma(\Re(\kappa_j)(n+1)+\Re(\mu_j))},$$

$$\to 0 \quad (n\to+\infty).$$

With the help of Eqs.(45) and (46), we have

$$\left\|\xi^{A;\eta,\gamma;\kappa}_{c+;(\kappa_j,\mu_j)_m;\mu}\phi\right\|_1=\int_c^d\left|\left|\int_c^u(u-v)^{\Re(\mu)-1}E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[A(u-v)^\kappa]\times\phi(v)\mathrm{d}v\right|\right|\mathrm{d}u.$$

Now, interchanging the order of integration by the Dirichlet formula [34, Eq.(1.32)], we get

$$\left\|\xi^{A;\eta,\gamma;\kappa}_{a+;(\kappa_j,\mu_j)_m;\mu}\phi\right\|_1\leq\int_c^d\left[\int_v^d(u-v)^{\Re(\mu)-1}\left\|E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[A(u-v)^\kappa]\right\|\mathrm{d}u\right]\times|\phi(v)|\,\mathrm{d}v.$$

Substituting $u-v=t$ in above equation, we have

$$\begin{aligned}\left\|\xi^{A;\eta,\gamma;\kappa}_{a+;(\kappa_j,\mu_j)_m;\mu}\phi\right\|_1&\leq\int_c^d\left[\int_0^{d-v}t^{\Re(\mu)-1}\left\|E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[At^\kappa]\right\|\mathrm{d}t\right]|\phi(v)|\,\mathrm{d}v,\\&\leq\int_c^d\left[\int_0^{d-c}t^{\Re(\mu)-1}\left\|E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[At^\kappa]\right\|\mathrm{d}t\right]|\phi(v)|\,\mathrm{d}v.\end{aligned}$$

Let $\int_0^{d-c}t^{\Re(\mu)-1}\left\|E^{\eta,\gamma}_{(\kappa_j,\mu_j)_m}[At^\kappa]\right\|\mathrm{d}t=G$ then we have

$$\left\|\xi^{A;\eta,\gamma;\kappa}_{a+;(\kappa_j,\mu_j)_m;\mu}\phi\right\|_1\leq\int_c^d G|\phi(v)|\,\mathrm{d}v. \tag{49}$$

Using Eq.(32), we obtain

$$G\leq\int_0^{d-c}t^{\Re(\mu)-1}\sum_{n=0}^{\infty}\frac{|(\eta)_{\gamma n}|}{\prod_{j=1}^{m}\Gamma(\Re(\kappa_j)n+\Re(\mu_j))}\frac{\|A\|^n t^{\Re(\kappa)\,n}}{n!}\mathrm{d}t.$$

Further, interchanging the order of integration and summation under the stated condition, we have

$$G \leq \sum_{n=0}^{\infty} \frac{|(\eta)_{\gamma n}|}{\prod_{j=1}^{m} \Gamma(\Re(\kappa_j)n + \Re(\mu_j))} \frac{\|A\|^n}{n!} \int_0^{d-c} t^{\Re(\kappa)n+\Re(\mu)-1} \mathrm{d}t,$$

$$\leq (d-c)^{\Re(\mu)} \sum_{n=0}^{\infty} \frac{|(\eta)_{\gamma n}|}{\{\Re(\kappa)n+\mu\} \prod_{j=1}^{m} \Gamma(\Re(\kappa_j)n + \Re(\mu_j))} \tag{50}$$

$$\times \frac{\|A\|^n (d-c)^{\Re(\kappa)n}}{n!},$$

$$= k \tag{51}$$

Finally, using Eqs.(49), (50) and (46), we arrive at the required result. □

Theorem 5. *Let* $A \in \mathbb{C}^{r\times r}$ *and* $u, v, w, \kappa, \mu, \kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$, $\Re(\kappa_j) > 0\,(j = 1, 2, \cdots, m)$ *such that* $\Re\left(\sum_{j=1}^{m} \kappa_j\right) > max\{0, \Re(\gamma) - 1\}$. *Also let* $u > c, c \in \mathbb{R}^+, 0 < \alpha, \beta < 1, 0 \leq \lambda \leq 1$*, then the image formula of multi-index Mittag-Leffler matrix function under generalized composite fractional derivative operator is defined as follows:*

$$\mathcal{D}_{c+}^{\alpha,\beta,\lambda}((v-c)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(v-c)^{\kappa}])(u)$$
$$= \frac{(u-c)^{\omega+\lambda(\alpha-\beta)-\alpha-1}}{\Gamma(\eta)} \tag{52}$$
$$\times {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega+\lambda(\alpha-\beta)-\alpha,\kappa); \end{array} A(u-c)^{\kappa}\right].$$

Proof. To prove Eq.(52), with the help of Eqs.(32) and (8)

$$\mathcal{D}_{c+}^{\alpha,\beta,\lambda}((v-c)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(v-c)^{\kappa}])(u)$$
$$= \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\prod_{j=1}^{m} \Gamma(\kappa_j n + \mu_j)} \frac{A^n}{n!} \mathcal{D}_{c+}^{\alpha,\beta,\lambda}\left((v-c)^{\kappa n+\omega-1}\right)(u),$$
$$= \sum_{n=0}^{\infty} \frac{(\eta)_{\gamma n}}{\prod_{j=1}^{m} \Gamma(\kappa_j n + \mu_j)} \frac{A^n}{n!} \frac{\Gamma(\kappa n+\omega)}{\Gamma(\kappa n+\omega+\lambda(\alpha-\beta)-\alpha)}$$
$$\times (u-c)^{\kappa n+\omega+\lambda(\alpha-\beta)-\alpha-1}.$$

Making use of Eq.(42), we obtain the required result. □

Corollary 1. *Let* $A \in \mathbb{C}^{r\times r}$ *and* $\kappa, \mu, \kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$, $\Re(\kappa_j) > 0\,(j = 1, 2, \cdots, m)$ *such that* $\Re\left(\sum_{j=1}^{m} \kappa_j\right) > max\{0, \Re(\gamma) - 1\}$. *Also let* $u > c, c \in \mathbb{R}^+, 0 < \alpha < 1, 0 \leq \lambda \leq 1$*, then the image formula of multi-index Mittag-Leffler matrix function under Hilfer and Riemann-Liouville fractional derivative and integral operator are defined as in the following forms:*

$$\mathcal{D}_{c+}^{\alpha;\lambda}((v-c)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(v-c)^{\kappa}])(u)$$
$$= \frac{(u-c)^{\omega-\alpha-1}}{\Gamma(\eta)} {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega-\alpha,\kappa); \end{array} A(u-c)^{\kappa}\right], \tag{53}$$

$$\mathcal{D}_{c+}^{\alpha}((v-c)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(v-c)^{\kappa}])(u)$$
$$= \frac{(u-c)^{\omega-\alpha-1}}{\Gamma(\eta)} {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega-\alpha,\kappa); \end{array} A(u-c)^{\kappa}\right], \tag{54}$$

and

$$\mathcal{I}_{c+}^{\alpha}((v-c)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(v-c)^{\kappa}])(u)$$
$$= \frac{(u-c)^{\omega+\alpha-1}}{\Gamma(\eta)} {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega+\alpha,\kappa); \end{array} A(u-c)^{\kappa}\right]. \tag{55}$$

Proof. Substituting the specific value of parameters in Theorem 5, the generalized composite fractional derivative reduces to Eqs.(53),(54) and (55). □

Composition Results for Multi-index Integral Operators

In this part, we analyze compositions of the generalized integral operator given in Eq.(45) with standard fractional operators, including Hilfer fractional operator, Riemann-Liouville fractional integral and derivative operators.

Theorem 6. *Under the given condition of definition 3, the following composition relation of multi-index integral operator and Riemann-Liouville fractional integral and derivative operator holds true:*

$$\mathcal{I}_{c+}^{\alpha}\, \xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa} \phi = \xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\, \mathcal{I}_{c+}^{\alpha} \phi, \tag{56}$$

and

$$\mathcal{D}_{c+}^{\alpha}\, \xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa} \phi = \xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\, \mathcal{D}_{c+}^{\alpha} \phi. \tag{57}$$

Proof. Let $\mathfrak{L}$ be the left hand side of Eq.(56), with help of Eqs. (1) and (32), we have

$$\mathfrak{L} = \frac{1}{\Gamma(\alpha)} \int_c^u (u-x)^{\alpha-1} \int_c^x (x-v)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(x-v)^{\kappa}]\phi(v)\mathrm{d}v\mathrm{d}x.$$

Applying the Dirichlet formula [34, Eq.(1.32)], we obtain

$$\mathfrak{L} = \frac{1}{\Gamma(\alpha)} \int_c^u \left[\int_v^u (u-x)^{\alpha-1}(x-v)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(x-v)^{\kappa}]\mathrm{d}x\right] \phi(v)\mathrm{d}v.$$

Putting the $x - v = y$ in above equation, we get

$$\mathfrak{L} = \frac{1}{\Gamma(\alpha)} \int_c^u \left[\int_0^{u-v} (u-v-y)^{\alpha-1} y^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[Ay^{\kappa}]dy\right] \phi(v)\mathrm{d}v.$$

Using Eq.(55), we have

$$\mathfrak{L} = \int_c^u \frac{(u-v)^{\alpha+\omega-1}}{\Gamma(\eta)} {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega+\alpha,\kappa); \end{array} A(u-v)^{\kappa}\right] \phi(v)\mathrm{d}v.$$

Let $\mathfrak{D}$ be right hand side of Eq.(56), with help of Eqs.(32) and (1), we obtain

$$\mathfrak{D} = \int_c^x (x-v)^{\omega-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(x-v)^{\kappa}] \frac{1}{\Gamma(\alpha)} \int_c^v (v-u)^{\alpha-1}\phi(u)\mathrm{d}x\mathrm{d}u.$$

Applying the Dirichlet formula [34, Eq.(1.32)], we obtain

$$\mathfrak{D} = \frac{1}{\Gamma(\alpha)} \int_c^x \left[\int_u^x (x-v)^{\omega-1}(v-u)^{\alpha-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[A(x-v)^{\kappa}]\mathrm{d}x\right] \phi(u)\mathrm{d}u.$$

Substituting the $v - u = y$ in above equation, we have

$$\mathfrak{D} = \frac{1}{\Gamma(\alpha)} \int_c^u \left[\int_0^{x-u} (x-u-y)^{\omega-1} y^{\alpha-1} E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[Ay^{\kappa}]dy\right] \phi(v)\mathrm{d}v.$$

Using Eq.(55), we get

$$\mathfrak{D} = \int_c^u \frac{(x-u)^{\alpha+\omega-1}}{\Gamma(\eta)} {}_2\Psi_{m+1}\left[\begin{array}{c} (\eta,\gamma),(\omega,\kappa); \\ (\mu_j,\kappa_j),(\omega+\alpha,\kappa); \end{array} A(x-u)^{\kappa}\right] \phi(v)\mathrm{d}v,$$
$$= (\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\, \mathcal{I}_{c+}^{\alpha} \phi)(u).$$

Similarly, Eq.(57) can be established using an analogous argument. □

Theorem 7. *Under the given condition of definition 3, the following relation of multi-index integral operator and Hilfer fractional derivative operator holds true:*

$$\mathcal{D}_{c+}^{\alpha;\lambda}\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa} \phi\right) = \mathcal{I}_{c+}^{(1-\lambda)(1-\alpha)} \xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa} \phi. \tag{58}$$

Proof. Firstly, taking left hand side of Eq.(58) and using Eq.(6), we have

$$\mathcal{D}_{c+}^{\alpha;\lambda}\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right)=\mathcal{I}^{\lambda(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}u}\left(\mathcal{I}^{(1-\lambda)(1-\alpha)}\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right).$$

Using Eqs.(45) and (1), we have

$$\begin{aligned}
&\mathcal{D}_{c+}^{\alpha;\lambda}\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right)\\
&=\mathcal{I}^{\lambda(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u\frac{(u-v)^{\mu+(1-\lambda)(1-\alpha)-1}}{\Gamma(\eta)}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha),\kappa);\end{array}A(u-v)^{\kappa}\right]\phi(v)\mathrm{d}v,\\
&=\mathcal{I}^{\lambda(1-\alpha)}\int_0^u\frac{(u-v)^{\mu+(1-\lambda)(1-\alpha)-2}}{\Gamma(\eta)}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha)-1,\kappa);\end{array}A(u-v)^{\kappa}\right]\phi(v)\mathrm{d}v,\\
&=\frac{1}{\Gamma(\lambda(1-\alpha))}\int_c^x(x-s)^{\lambda(1-\alpha)-1}ds\int_c^s\frac{(s-v)^{\mu+(1-\lambda)(1-\alpha)-2}}{\Gamma(\eta)}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha)-1,\kappa);\end{array}A(s-v)^{\kappa}\right]\phi(v)\mathrm{d}v.
\end{aligned}\tag{59}$$

Next, taking right hand side of Eq.(58) and using Eqs.(45) and (1), we have

$$\begin{aligned}
&\mathcal{I}_{c+}^{(1-\lambda)(1-\alpha)}\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\\
&=\frac{1}{\Gamma(\eta)\Gamma(\lambda(1-\alpha))}\int_c^x\phi(v)dv\\
&\times\int_0^{x-v}(x-\tau-v)^{\lambda(1-\alpha)-1}\tau^{\mu+(1-\lambda)(1-\alpha)-2}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha)-1,\kappa);\end{array}A\tau^{\kappa}\right]d\tau.\\
&=\frac{1}{\Gamma(\eta)}\frac{c}{x}\mathcal{I}_{0+}^{\lambda(1-\alpha)}\tau^{\mu+(1-\lambda)(1-\alpha)-2}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha)-1,\kappa);\end{array}A\tau^{\kappa}\right]\phi(v)\mathrm{d}v,\\
&=\frac{1}{\Gamma(\eta)\Gamma(\lambda(1-\alpha))}\int_c^x(x-v)^{\omega+(1-\lambda)(1-\alpha)-1}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\omega,\kappa);\\(\mu_j,\kappa_j),(\omega+(1-\lambda)(1-\alpha),\kappa);\end{array}A\tau^{\kappa}\right]\phi(v)\mathrm{d}v.
\end{aligned}\tag{60}$$

Finally, from Eqs.(59) and (60), we obtain the required result. □

Analytical Solution to Volterra-Integro Differential Equation

Several authors have obtained solutions to the Volterra integro-differential equation using various methods. Kilbas et al. [31] derived the solution to Cauchy-type problem involving fractional integro-differential equation whose kernel contains the generalized Mittag-Leffler function. Srivastava and Tomovski [10] introduced an integral operator with generalized Mittag-Leffler functions and derived solutions of fractional differential equations by applying the Laplace transform. Furthermore, Srivastava et al. [32] investigated an integral operator involving Fox's H-function and obtained solutions of fractional differential equations associated this operator. Bansal et al. [33] derived solution of first-order Volterra-type integro-differential equation by applying Laplace transform. Motivated by these works, we obtain the solution of the Volterra integro-differential equation involving multi-index integral operator by applying the Laplace transform.

Laplace Transform of multi-index integral operator

The well known Laplace transform of the function $\phi(u)$ is defined by Sneddon [35]:

$$L\{\phi(u)\}=\int_0^{\infty}e^{-su}\phi(u)\mathrm{d}u.\tag{61}$$

Theorem 8. *Under the given condition of definition 3, then Laplace transform of integral matrix operator is represented as follows:*

$$\begin{aligned}L\left\{\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right)(u);s\right\}&=\frac{s^{-\mu}}{\Gamma(\eta)}\\
&\times{}_2\Psi_m\left[\begin{array}{c}(\eta,\gamma),(\mu,\kappa);\\(\mu_j,\kappa_j);\end{array}As^{-\kappa}\right]\Phi(s).\end{aligned}\tag{62}$$

Proof. To prove Eq.(62), taking the Laplace transform of integral matrix operator and apply convolution theorem, we have

$$\begin{aligned}L\left\{\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right)(u);s\right\}&=L\left\{v^{\mu-1}E_{(\kappa_j,\mu_j)_m}^{\eta,\gamma}[Av^{\kappa}];s\right\}\\
&\times L\left\{\phi(v);s\right\}.\end{aligned}$$

Making use of Theorem 3, we have

$$\begin{aligned}L\left\{\left(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi\right)(u);s\right\}&=\frac{s^{-\mu}}{\Gamma(\eta)}\\
&\times{}_2\Psi_m\left[\begin{array}{c}(\eta,\gamma),(\mu,\kappa);\\(\mu_j,\kappa_j);\end{array}As^{-\kappa}\right]\Phi(s).\end{aligned}$$

□

Theorem 9. *Let* $A\in\mathbb{C}^{r\times r}$ *and* $u,v,\kappa,\mu,\kappa_j,\mu_j,\eta,\gamma\in\mathbb{C}$, $\Re(\kappa_j)>0\,(j=1,2,\cdots,m)$ *such that* $\Re\left(\sum_{j=1}^m\kappa_j\right)>max\{0,\Re(\gamma)-1\}$. *Also let* $u>c,(c\in\mathbb{R}^+=[0,\infty)),0<\alpha,\beta<1,0\leq\lambda\leq1$, *then the fractional differential equations is*

$$(\mathcal{D}_{c+}^{\alpha,\beta,\lambda}y)(u)I=\delta(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi)(u)+f(u)I,\tag{63}$$

with initial condition

$$\left(\mathcal{I}_0^{(1-\lambda)(1-\alpha)}y\right)(0)=a,\tag{64}$$

has a solution

$$\begin{aligned}y(u)I&=\frac{aIu^{\lambda(1-\alpha)+\alpha-1}}{\Gamma(1-(\lambda-1)(\alpha-1))}+\frac{\delta u^{\mu+\alpha-\lambda(\alpha-\beta)}}{\Gamma(\eta)}\\
&\times{}_2\Psi_{m+1}\left[\begin{array}{c}(\eta,\gamma),(\mu,\kappa);\\(\mu_j,\kappa_j),(\mu+\alpha-\lambda(\alpha-\beta)+1,\kappa);\end{array}Au^{-\kappa}\right]\\
&+\frac{1}{\Gamma(\lambda(\beta-\alpha)+\alpha)}\int_0^u(u-v)^{\lambda(\beta-\alpha)+\alpha-1}f(v)I\mathrm{d}v.\end{aligned}\tag{65}$$

Proof. Applying the Laplace transform in Eq.(63) and using Eqs.(9) and Theorem 8, we have

$$\begin{aligned}&s^{-\lambda(\alpha-\beta)+\alpha}Y(s)I-s^{-\lambda(1-\beta)}\left(\mathcal{I}^{(1-\lambda)(1-\alpha)y}\right)(0)I\\
&=\frac{\delta s^{-\mu-1}}{\Gamma(\eta)}{}_2\Psi_m\left[\begin{array}{c}(\eta,\gamma),(\mu,\kappa);\\(\mu_j,\kappa_j);\end{array}As^{-\kappa}\right]+\Phi(s)I.\end{aligned}$$

After straightforward simplification, we have

$$\begin{aligned}Y(s)I&=aIs^{\lambda(\alpha-1)-\alpha}+\frac{\delta s^{-\mu-\alpha+\lambda(\alpha-\beta)-1}}{\Gamma(\eta)}\\
&\times{}_2\Psi_m\left[\begin{array}{c}(\eta,\gamma),(\mu,\kappa);\\(\mu_j,\kappa_j);\end{array}As^{-\kappa}\right]+s^{\lambda(\alpha-\beta)-\alpha}\Phi(s)I.\end{aligned}$$

Taking inverse Laplace transform of above equation, we obtain

$$\begin{aligned}y(u)I&=\frac{aIu^{\lambda(1-\alpha)+\alpha-1}}{\Gamma(1-(\lambda-1)(\alpha-1))}+\frac{\delta}{\Gamma(\eta)}\sum_{n=0}^{\infty}\frac{(\eta+\gamma n)(\mu+\kappa n)}{\prod_{j=1}^{m}\Gamma(\kappa_j n+\mu_j)}\frac{A^n}{n!}\\
&\times L^{-1}\left\{s^{-\kappa n-\mu-\alpha+\lambda(\alpha-\beta)-1}\right\}\\
&+\frac{1}{\Gamma(\lambda(\beta-\alpha)+\alpha)}\int_0^u(u-v)^{\lambda(\beta-\alpha)+\alpha-1}f(v)I\mathrm{d}v.\end{aligned}$$

We obtain the required solution, after straightforward simplification. □

Corollary 2. *Let $A \in \mathbb{C}^{r\times r}$ and $u, v, \kappa, \mu, \kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$, $\Re(\kappa_j) > 0\,(j = 1, 2, \cdots, m)$ such that $\Re\left(\sum_{j=1}^{m}\kappa_j\right) > max\{0, \Re(\gamma) - 1\}$. Also let $u > c, (c \in \mathbb{R}^+ = [0,\infty)), 0 < \alpha < 1, 0 \le \lambda \le 1$, then the fractional differential equations is*

$$(\mathcal{D}_{c+}^{\alpha,\lambda} y)(u)I = \delta(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi)(u) + f(u)I, \tag{66}$$

with initial condition

$$\left(\mathcal{I}_0^{(1-\lambda)(1-\alpha)} y\right)(0) = a, \tag{67}$$

has a solution

$$\begin{aligned} y(u)I &= \frac{aIu^{\lambda(1-\alpha)+\alpha-1}}{\Gamma(1-(\lambda-1)(\alpha-1))} + \frac{\delta u^{\mu+\alpha}}{\Gamma(\eta)} \\ &\times {}_2\Psi_{m+1}\left[\begin{matrix} (\eta,\gamma),(\mu,\kappa); \\ (\mu_j,\kappa_j),(\mu+\alpha+1,\kappa); \end{matrix}\; Au^{-\kappa}\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v) I \mathrm{d}v. \end{aligned} \tag{68}$$

Proof. On taking $\alpha = \beta$ in Theorem 9, the generalized composite fractional derivative operator reduces to the Hilfer fractional derivative operator, thereby yielding the required result. □

Corollary 3. *Let $A \in \mathbb{C}^{r\times r}$ and $u, v, \kappa, \mu, \kappa_j, \mu_j, \eta, \gamma \in \mathbb{C}$, $\Re(\kappa_j) > 0\,(j = 1, 2, \cdots, m)$ such that $\Re\left(\sum_{j=1}^{m}\kappa_j\right) > max\{0, \Re(\gamma) - 1\}$. Also let $u > c, (c \in \mathbb{R}^+ = [0,\infty)), 0 < \alpha < 1$, then the fractional differential equations is*

$$(\mathcal{D}_{c+}^{\alpha} y)(u)I = \delta(\xi_{c+;(\kappa_j,\mu_j)_m;\mu}^{A;\eta,\gamma;\kappa}\phi)(u) + f(u)I, \tag{69}$$

with initial condition

$$\left(\mathcal{I}_0^{(1-\alpha)} y\right)(0) = a, \tag{70}$$

has a solution

$$\begin{aligned} y(u)I &= \frac{aIu^{\alpha-1}}{\Gamma(1+(\alpha-1))} + \frac{\delta u^{\mu+\alpha}}{\Gamma(\eta)} \\ &\times {}_2\Psi_{m+1}\left[\begin{matrix} (\eta,\gamma),(\mu,\kappa); \\ (\mu_j,\kappa_j),(\mu+\alpha+1,\kappa); \end{matrix}\; Au^{-\kappa}\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v) I \mathrm{d}v. \end{aligned} \tag{71}$$

Proof. On taking $\alpha = \beta$ and $\lambda = 0$ in Theorem 9, the generalized composite fractional derivative operator reduces to the Riemann-Liouville fractional derivative operator, thereby yielding the required result. □

Results and Discussion

In this section, we present numerical and graphical illustrations to investigate the effects of the fractional-order parameters α, β and λ on the solution matrix under various fractional derivative operators. Using MATLAB, numerical computations and graphical representations are obtained to illustrate the behavior of proposed solution.

For $m = 1$, the fractional-order parameters α, β and γ are varies, while all remaining parameters are fixed as $\delta = \eta = \gamma = \mu = \kappa = \kappa_1 = \mu_1 = a = u = 1$ with $f(v) = 1$ and matrix $A = \begin{pmatrix} 0.5 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}$, the solution matrices, together with their corresponding trace and determinant values are computed from Theorem 9 (Eq.(65)), Corollary 2 (Eq.(68)) and Corollary 3 (Eq. (71)), and are presented in Tables 1, 2 and 3 respectively.

Table 1: Numerical computations for varying fractional orders α under the generalized composite fractional derivative operator

Values of (α,β,λ)	$y(u)I$	$\mathrm{Tr}(y(u)I)$	$\mathrm{Det}(y(u)I)$
$\alpha = 0.2, \beta = 0.7, \lambda = 0.9$	$\begin{pmatrix} 2.8823 & 0.0655 \\ 0.0327 & 2.8496 \end{pmatrix}$	5.731902	8.211265
$\alpha = 0.4, \beta = 0.7, \lambda = 0.9$	$\begin{pmatrix} 2.8779 & 0.0639 \\ 0.0319 & 2.8459 \end{pmatrix}$	5.723763	8.188070
$\alpha = 0.6, \beta = 0.7, \lambda = 0.9$	$\begin{pmatrix} 2.8726 & 0.0623 \\ 0.0312 & 2.8415 \end{pmatrix}$	5.714107	8.160568
$\alpha = 0.8, \beta = 0.7, \lambda = 0.9$	$\begin{pmatrix} 2.8667 & 0.0608 \\ 0.0304 & 2.8363 \end{pmatrix}$	5.702952	8.128834

Table 2: Numerical computations for varying fractional orders $\alpha = \beta$ under the Hilfer fractional derivative operator.

Values of (α,β,λ)	$y(u)I$	$\mathrm{Tr}(y(u)I)$	$\mathrm{Det}(y(u)I)$
$\alpha = \beta = 0.2, \lambda = 0.9$	$\begin{pmatrix} 3.1927 & 0.1101 \\ 0.0551 & 3.1377 \end{pmatrix}$	6.330425	10.011750
$\alpha = \beta = 0.4, \lambda = 0.9$	$\begin{pmatrix} 3.0933 & 0.0880 \\ 0.0440 & 3.0493 \end{pmatrix}$	6.142606	9.428541
$\alpha = \beta = 0.6, \lambda = 0.9$	$\begin{pmatrix} 2.9520 & 0.0696 \\ 0.0348 & 2.9172 \end{pmatrix}$	5.869192	8.609133
$\alpha = \beta = 0.8, \lambda = 0.9$	$\begin{pmatrix} 2.7820 & 0.0543 \\ 0.0272 & 2.7548 \end{pmatrix}$	5.536793	7.662359

Table 3: Numerical computations for varying fractional orders $\alpha = \beta$ under the Riemann–Liouville fractional derivative operator.

Values of (α,β,λ)	$y(u)I$	$\mathrm{Tr}(y(u)I)$	$\mathrm{Det}(y(u)I)$
$\alpha = \beta = 0.2, \lambda = 0$	$\begin{pmatrix} 2.4609 & 0.1101 \\ 0.0551 & 2.4059 \end{pmatrix}$	4.866768	5.914537
$\alpha = \beta = 0.4, \lambda = 0$	$\begin{pmatrix} 2.5811 & 0.0880 \\ 0.0440 & 2.5371 \end{pmatrix}$	5.118220	6.544683
$\alpha = \beta = 0.6, \lambda = 0$	$\begin{pmatrix} 2.6476 & 0.0696 \\ 0.0348 & 2.6128 \end{pmatrix}$	5.260472	6.915420
$\alpha = \beta = 0.8, \lambda = 0$	$\begin{pmatrix} 2.6527 & 0.0543 \\ 0.0272 & 2.6256 \end{pmatrix}$	5.278280	6.963398

Figure 1 depicts the influence of the fractional-order parameter α on the solution matrix under the generalized composite fractional derivative operator (Table 1). As α increases from 0.2 to 0.8, only a moderate variation in the magnitude of the matrix entries is observed, whereas the overall shape of the surface remains nearly unchanged. This indicates that the fractional-order parameter primarily affects the amplitude of the solution rather than its qualitative behavior. Such stability suggests that the generalized composite fractional derivative operator preserves the essential characteristics of the matrix solution over a wide range of fractional orders. **Figure 2** illustrates the influence of the fractional-order parameters $\alpha = \beta$ on the solution matrix under the Hilfer fractional derivative operator (Table 2). The matrix values exhibit marginal decrease as $\alpha = \beta$ increases from 0.2 to 0.8, suggesting relatively weak sensitivity of the solution to the fractional parameters. These findings imply that the Hilfer fractional derivative operator provides stable and consistent dynamics while preserving the fundamental structure of the solution matrix. **Figure 3** shows the influence of the fractional-order parameters $\alpha = \beta$ on the solution matrix under the Riemann-Liouville derivative operator (Table 3). As the fractional

parameters increase from 0.2 to 0.8, only slight variations in the magnitude of the solution matrix are observed. This behavior suggests that the Riemann-Liouville derivative operator exhibits limited sensitivity to variations in the fractional order. Thus, the Riemann-Liouville derivative operator provides stable solution behavior and preserves the inherent structural characteristics of the system.

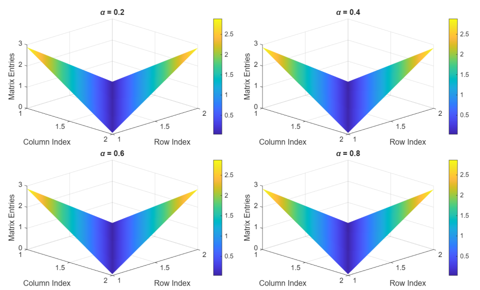

Figure 1: Influences of α on the solution matrix under generalized composite fractional derivative operator

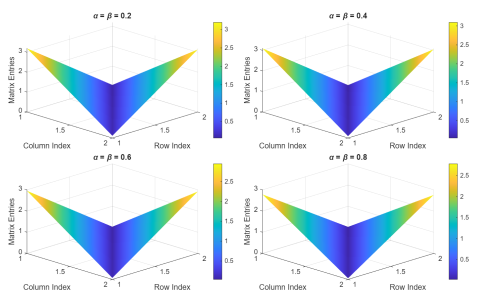

Figure 2: Influences of $\alpha = \beta$ on the solution matrix under Hilfer fractional derivative operator

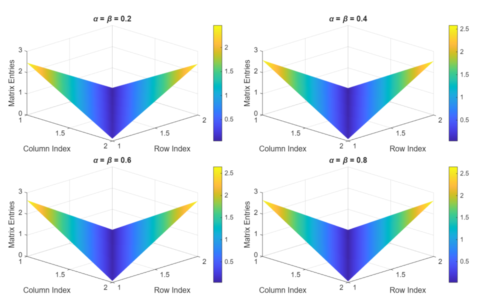

Figure 3: Influences of $\alpha = \beta$ on the solution matrix under Riemann-Liouville derivative operator

Conclusion

In this paper, we introduced a novel integral operator associated with the multi-index Mittag-Leffler matrix function and established its boundedness in the Lebesgue space $\mathcal{L}(c, d)$. We further investigated its compositional properties with the classical Riemann-Liouville fractional integral and derivative operators. Moreover, we derived the first-order Volterra integro-differential equation involving a multi-index integral operator, and its solution is obtained using the Laplace transform. Furthermore, we analyse the Numerical computations and graphical illustrations of the solution matrix. The obtained results enrich the theory of fractional calculus for matrix-valued functions and provide a useful framework for further studies on fractional integro-differential equations, operator theory, and related applications in mathematical physics and engineering sciences.

Ethics approval and consent to participate

Not applicable

Consent for publication

Not applicable

Availability of data and materials

No such data is used in this study.

Author's contribution

Karishma Sharma: conceptualization; investigation; writing -original draft; methodology; validation; visualization; writing-review and editing. **Manish Kumar Bansal**: conceptualization; investigation; writing- original draft; writing-review and editing; visualization; validation; formal analysis; methodology; supervision; project administration.

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Conflicts of interest

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