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Abstract

In this paper, we introduce a new extension of Mittag-Leffler function. We investigate its basic properties, including recurrence relations, differential formulas, integral representations, Laplace transform and Mellin transform. Furthermore, we establish fractional integral and differential operators associated with this extended Mittag-Leffler type function. Several interesting special cases of our main results are derived.

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1 Introduction

The Mittag-Leffler function is highly useful and important due to its special properties and significant potential in solving applied problems. Therefore, it gained more attention in several areas of applied mathematics and engineering sciences, such as statistical distributions [12], stochastic processes [10], mechanical relaxation [11], modeling of processes (diffusion) [15], electrotechnics [19], complex network [2], etc. Moreover, the functions of the Mittag-Leffler type appear in many fields of science, such as physics [32], chemistry [14], biology [8], economics [30].

The real importance of the Mittag-Leffler function has been recognized when its special role in fractional calculus has been discovered. The fractional calculus extends the conventional integral and differential operators. The concept of fractional calculus emerged in History as a response to the query of whether derivatives of integer order can be expanded to fractional order. There are numerous definitions of fractional derivatives and integrals, each with distinct properties and capacities to model various behaviors.

In [3], Atangana and Baleanu introduced a new fractional derivative known as the ABC-fractional derivative, which utilizes the Mittag–Leffler function as its kernel. The ABC-fractional differential operator is more suitable for accurately describing real-world phenomena, as it is a nonlocal operator with a nonsingular kernel. The ABC-fractional derivative has gained popularity in recent years primarily due to its wide-ranging applications [4, 22, 31]. For further details on the fundamental advancements in ABC-type derivatives, readers can refer to recent works [1, 7, 9, 29].

Definition 1 *([16]). The Mittag-Leffler function $E_\alpha(z)$ is defined by*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha, z \in \mathbb{C}, \Re(\alpha) > 0. \tag{1.1}$$

Definition 2 *([33]). The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.2}$$

Definition 3 *([20]). The three-parameter Mittag-Leffler function* $E^{\gamma}_{\alpha,\beta}(z)$ *is defined by*

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0, \tag{1.3}$$

where $(\gamma)_n$ *denotes the Pochhammer symbol defined in terms of the familiar Gamma function* Γ *by (see, e.g., [26])*

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}:=\{1,2,...\}).\end{cases}$$

Definition 4 *([17]). The extended Mittag-Leffler function* $E^{\gamma;c}_{\alpha,\beta}(z;p)$ *is defined by*

$$E^{\gamma;c}_{\alpha,\beta}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.4}$$

where $\alpha,\beta,\gamma,c\in\mathbb{C}$*;* $\Re(\alpha)>0$*,* $\Re(\beta)>0$*,* $\Re(\gamma)>0$*,* $\Re(c)>0$*,* $p\in\mathbb{R}^+_0$ *and* $B_p(x,y)$ *is an extension of beta function defined in [5, 21] as follows:*

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt,\quad \Re(x)>0,\Re(y)>0,\Re(p)>0, \tag{1.5}$$

and $B_0(x,y)=B(x,y)$ *is the familiar beta function (see, [26]).*

Definition 5 *([18]). The Mittag-Leffler-type function of arbitrary order* $E^{j,k}_{\alpha,\beta}(z)$ *is defined by*

$$E^{j,k}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{nj+k}}{\Gamma(\beta+\alpha(nj+k))},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,j\geq 1,k\geq 0. \tag{1.6}$$

Definition 6 *[27]. The Fox-Wright function is defined as*

$${}_p\Psi_q\left[\begin{array}{c}(d_1,D_1),...,(d_p,D_p)\\(e_1,E_1),...,(e_q,E_q)\end{array}\middle|\,z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(d_i+D_i n)}{\prod_{j=1}^{q}\Gamma(e_j+E_j n)}\frac{z^n}{n!}, \tag{1.7}$$

where $d_i,D_i,e_j,E_j,z\in\mathbb{C}$*,* $\Re(d_i)>0,\Re(D_i)>0$*,* $i=1,...,p$*,* $\Re(e_i)>0,\Re(E_i)>0$*,* $j=1,...,q$ *and* $1+\Re\left(\sum_{j=1}^{q}E_j-\sum_{i=1}^{p}D_i\right)\geq 0$*.*

Definition 7 *The H-function of two variables is defined as*

$$H^{0,B:A_1,B_1;A_2,B_2}_{C,D:C_1,D_1;C_2,D_2}\left[\begin{array}{c|c}z_1 & (a_j;\alpha_j^{(1)},\alpha_j^{(2)})_{1,C}:(c_j^{(1)},\gamma_j^{(1)})_{1,C_1};(c_j^{(2)},\gamma_j^{(2)})_{1,C_2}\\ z_2 & (b_j;\beta_j^{(1)},\beta_j^{(2)})_{1,D}:(d_j^{(1)},\delta_j^{(1)})_{1,D_1};(d_j^{(2)},\delta_j^{(2)})_{1,D_2}\end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(\eta_1,\eta_2)\prod_{i=1}^{2}(\theta_i(\eta_i)z_i^{\eta_i})d\eta_1 d\eta_2, \tag{1.8}$$

where $\omega=\sqrt{-1}$*, and*

$$\phi(\eta_1,\eta_2)=\frac{\prod_{j=1}^{B}\Gamma\left(1-a_j+\sum_{i=1}^{2}\alpha_j^{(i)}\eta_i\right)}{\prod_{j=1}^{D}\Gamma\left(1-b_j+\sum_{i=1}^{2}\beta_j^{(i)}\eta_i\right)\prod_{j=B+1}^{C}\Gamma\left(a_j-\sum_{i=1}^{2}\alpha_j^{(i)}\eta_i\right)},$$

$$\theta_i(\eta_i)=\frac{\prod_{j=1}^{A_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\eta_i\right)\prod_{j=1}^{B_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\eta_i\right)}{\prod_{j=A_i+1}^{D_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\eta_i\right)\prod_{j=B_i+1}^{C_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\eta_i\right)},\quad (i=1,2).$$

For conditions of convergence for the H-function of two variables (1.8), one can refer to [28].

Definition 8 *([25]). The Laplace transform of the function $f(z)$ is defined as*

$$\mathcal{L}\{f(z);s\} = \int_0^\infty e^{-sz} f(z)dz, \quad \Re(s) > 0. \tag{1.9}$$

Definition 9 *([25]). The Mellin transform of the function $f(z)$ is given as*

$$\mathcal{M}\{f(z);s\} = \int_0^\infty z^{s-1} f(z)dz = f^*(s), \quad \Re(s) > 0, \tag{1.10}$$

and the inverse Mellin transform is defined as

$$f(z) = \mathcal{M}^{-1}\{f^*(s);z\} = \frac{1}{2\pi i}\int_L f^*(s) z^{-s} ds, \tag{1.11}$$

where L is a contour of integration that begins at $-i\infty$ and ends at $i\infty$.

Definition 10 *([23]). The left-sided Riemann-Liouville fractional integral operator I^ν_{a+} and the right-sided Riemann-Liouville fractional integral operator I^ν_{b-} are defined by*

$$\left(I^\nu_{a+} f\right)(x) = \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1} f(t)dt, \quad \left(\Re(\nu) > 0, x > a\right), \tag{1.12}$$

$$\left(I^\nu_{b-} f\right)(x) = \frac{1}{\Gamma(\nu)}\int_x^b (t-x)^{\nu-1} f(t)dt, \quad \left(\Re(\nu) > 0, x < b\right). \tag{1.13}$$

Definition 11 *([23]). The left-sided Riemann-Liouville fractional derivative operator D^ν_{a+} and the right-sided Riemann-Liouville fractional derivative operator D^ν_{b-} are defined by*

$$\left(D^\nu_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I^{m-\nu}_{a+} f\right)(x), \quad \left(\Re(\nu) > 0, m = [\Re(\nu)] + 1\right), \tag{1.14}$$

$$\left(D^\nu_{b-} f\right)(x) = (-1)^m \left(\frac{d}{dx}\right)^m \left(I^{m-\nu}_{b-} f\right)(x), \quad \left(\Re(\nu) > 0, m = [\Re(\nu)] + 1\right), \tag{1.15}$$

where $\Re(\nu)$ denotes the real part of the complex number $\nu \in \mathbb{C}$ and $[\Re(\nu)]$ represents the integral part of $\Re(\nu)$.

Here, we recall the left and right-sided Riemann-Liouville fractional integrations of a power function are defined in [13] by

$$\left(I^\nu_{0+} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)} x^{\lambda+\nu-1}, \quad \left(\Re(\nu) > 0, \Re(\lambda) > 0\right), \tag{1.16}$$

$$\left(I^\nu_{-} t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda+\nu-1}, \quad \left(0 < \Re(\nu) < 1 - \Re(\lambda)\right), \tag{1.17}$$

respectively. The left and right-sided Riemann-Liouville fractional differentiations of a power function are defined, respectively, by (see [13])

$$\left(D^\nu_{0+} t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\nu)} x^{\lambda-\nu-1}, \quad \left(\Re(\nu) > 0, \Re(\lambda) > 0\right) \tag{1.18}$$

and

$$\left(D^\nu_{-} t^{\lambda-1}\right)(x) = \frac{\Gamma(1+\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda-\nu-1}, \quad \left(\Re(\nu) > 0, \Re(\lambda) < \Re(\nu) - [\Re(\nu)]\right). \tag{1.19}$$

Next, motivated by the above such extensions of the Mittag-Leffler function, we define a new extension of the arbitrary order Mittag-Leffler-type function (1.6) and investigate its certain properties.

2 Basic properties

Definition 12 . *An extension of various functions described above can be given by the definition of extended Mittag-Leffler-tpe function of arbitrary order, represents as follows:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} z^{nj+k}, \tag{2.1}$$

where $\alpha, \beta, \gamma, c \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$; $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$.

We consider special cases enumerated by equation (2.1)

(i) Setting $p = 0$ and $\gamma = 1$ in (2.1) yields (1.6), as established by Pathan and Bin-Saad [18].

(ii) Setting $j = 1$ and $k = 0$ in (2.1) yields (1.4), as established by Özarslan and Yilmaz [17].

(iii) Setting $k = p = 0$ and $j = 1$ in (2.1) yields (1.3), as established by Prabhakar [20].

The following theorems list some basic properties of this function.

Theorem 2.1 *Let* $\alpha, \beta, \gamma, c, z \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$, *then*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p). \tag{2.2}$$

Proof. Applying (2.1) to the right side of (2.2), we get

$$\begin{aligned}
&\beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p) \\
&= \beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p) + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)} z^{nj+k} \\
&= \beta E_{\alpha,\beta+1}^{\gamma,c;j,k}(z;p) + \alpha \sum_{n=0}^{\infty} \frac{(nj+k) B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)} z^{nj+k} \\
&= \sum_{n=0}^{\infty} \frac{(\beta+\alpha(nj+k)) B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)} z^{nj+k} \\
&= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} z^{nj+k},
\end{aligned}$$

which, in terms of (2.1), yields the desired formula (2.2).

This result generalizes the Corollary 4 in [17], if we consider $j = 1$ and $k = 0$ in (2.2), we have the well-known result of [17].

Theorem 2.2 *Let* $\alpha, \beta, \gamma, c, z \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$, $m \in \mathbb{N}$, *then*

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{\gamma,c;j,k}(\omega z^\alpha;p)\right] = z^{\beta-m-1} E_{\alpha,\beta-m}^{\gamma,c;j,k}(\omega z^\alpha;p). \tag{2.3}$$

Proof. From (2.1), we find that

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{\gamma,c;j,k}(\omega z^\alpha;p)\right] \\
&= \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} z^{\alpha(nj+k)+\beta-1}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma, c-\gamma)\Gamma(\beta-m+\alpha(nj+k))} z^{\alpha(nj+k)+\beta-m-1}$$

$$= z^{\beta-m-1} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta-m+\alpha(nj+k))} (\omega z^{\alpha})^{nj+k}. \tag{2.4}$$

Now, using (2.1) in (2.4) we obtain the desired formula.

This result generalizes the Theorem 11 in [17], if we consider $j=1$ and $k=0$ in (2.3), we have the well-known result of [17].

Theorem 2.3 *Let* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(\nu)>0$ *and* $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, *then*

$$\frac{1}{\Gamma(\nu)}\int_0^1 t^{\beta-1}(1-t)^{\nu-1} E_{\alpha,\beta}^{\gamma,c;j,k}(zt^{\alpha};p)dt = E_{\alpha,\beta+\nu}^{\gamma,c;j,k}(z;p), \tag{2.5}$$

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}^{\gamma,c;j,k}(\omega t^{\alpha};p)dt = z^{\beta} E_{\alpha,\beta+1}^{\gamma,c;j,k}(\omega z^{\alpha};p). \tag{2.6}$$

Proof. We have

$$\int_0^1 t^{\beta-1}(1-t)^{\nu-1} E_{\alpha,\beta}^{\gamma,c;j,k}(zt^{\alpha};p)dt$$

$$= \int_0^1 t^{\beta-1}(1-t)^{\nu-1} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} (zt^{\alpha})^{nj+k} dt$$

$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} z^{nj+k} \int_0^1 t^{\beta+\alpha(nj+k)-1}(1-t)^{\nu-1} dt$$

$$= \Gamma(\nu) \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\nu+\alpha(nj+k))} z^{nj+k}$$

$$= \Gamma(\nu) E_{\alpha,\beta+\nu}^{\gamma,c;j,k}(z;p),$$

hence (2.5) is proved.

Next, we have

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}^{\gamma,c;j,k}(\omega t^{\alpha};p)dt$$

$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k))} \int_0^z t^{\beta+\alpha(nj+k)-1} dt$$

$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n\ \omega^{nj+k}}{n!\ B(\gamma, c-\gamma)(\beta+\alpha(nj+k))\Gamma(\beta+\alpha(nj+k))} z^{\beta+\alpha(nj+k)}$$

$$= z^{\beta} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{n!\ B(\gamma, c-\gamma)\Gamma(\beta+\alpha(nj+k)+1)} (\omega z^{\alpha})^{nj+k}$$

$$= z^{\beta} E_{\alpha,\beta+1}^{\gamma,c;j,k}(\omega z^{\alpha};p),$$

which is the result (2.6).

When $p=0$ and $\gamma=1$, Theorem 2.3 reduces to the following corollary.

Corollary 2.1 *The following results hold:*

$$\frac{1}{\Gamma(\nu)}\int_0^1 t^{\beta-1}(1-t)^{\nu-1} E_{\alpha,\beta}^{j,k}(zt^{\alpha})dt = E_{\alpha,\beta+\nu}^{j,k}(z) \tag{2.7}$$

and

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}^{j,k}(\omega t^{\alpha})dt = z^{\beta} E_{\alpha,\beta+1}^{j,k}(\omega z^{\alpha}). \tag{2.8}$$

3 Integral representations

Theorem 3.1 *Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$, *then*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \frac{z^k}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1} (1-t)^{c-\gamma-1} e^{-\frac{p}{t(1-t)}} E_{\alpha j,\beta+\alpha k}^{c} \left(tz^j\right) dt. \tag{3.1}$$

Proof. Using (1.5) in (3.1), we obtain

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \sum_{n=0}^{\infty} \frac{z^k}{B(\gamma, c-\gamma)} \left\{ \int_0^1 t^{\gamma+n-1} (1-t)^{c-\gamma-1} e^{-\frac{p}{t(1-t)}} dt \right\} \frac{(c)_n \, (z^j)^n}{n! \, \Gamma(\alpha n j + \beta + \alpha k)},$$

which can be written as

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \frac{z^k}{B(\gamma, c-\gamma)} \int_0^1 t^{\gamma-1} (1-t)^{c-\gamma-1} e^{-\frac{p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{(c)_n \, (tz^j)^n}{\Gamma(\alpha n j + \beta + \alpha k) \, n!} dt.$$

Applying (1.3) in above equation, we obtain the required result.

Setting $t = sin^2\theta$ and $t = \frac{u}{1+u}$ in Theorem 3.1 yields Corollaries 3.1 and 3.2, respectively.

Corollary 3.1 *The following result holds true:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \frac{2z^k}{B(\gamma, c-\gamma)} \int_0^{\frac{\pi}{2}} sin^{2\gamma-1}\theta \; cos^{2c-2\gamma-1}\theta \; e^{-\frac{p}{sin^2\theta cos^2\theta}} E_{\alpha j,\beta+\alpha k}^{c} \left(z^j sin^2\theta\right) d\theta. \tag{3.2}$$

Corollary 3.2 *The following result holds true:*

$$E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) = \frac{z^k}{B(\gamma, c-\gamma)} \int_0^{\infty} \frac{u^{\gamma-1}}{(1+u)^c} \; e^{-\frac{p(1+u)^2}{u}} E_{\alpha j,\beta+\alpha k}^{c} \left(\frac{uz^j}{1+u}\right) du. \tag{3.3}$$

These results generalize Corollaries 2 and 3 in [17], if we consider $j = 1$ and $k = 0$ in (3.2) and (3.3), we obtain the well-known results of [17].

4 Integral transforms

Here, we evaluate the Laplace and Mellin transforms of the extended Mittag-Leffler function (2.1).

Theorem 4.1 *(Laplace transform): Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$, $\Re(\sigma) > 0$, $\Re(q) > 0$, $\Re(s) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$, *then*

$$\int_0^{\infty} z^{q-1} e^{-sz} E_{\alpha,\beta}^{\gamma,c;j,k}(xz^{\sigma};p)\, dz = \frac{s^{-(q+\sigma k)} x^k}{\Gamma(\gamma)\Gamma(c-\gamma)}$$
$$\times H_{1,1:0,2;2,2}^{0,1:1,0;1,2} \left[\begin{array}{c|ccc} p & (1-\gamma;1,1): & -; & (1-q-\sigma k, \sigma j), (1-c,1) \\ -\left(\frac{x}{s^{\sigma}}\right)^j & (1-c;1,2): & (0,1),(c-\gamma,1); & (0,1),(1-\beta-\alpha k, \alpha j) \end{array} \right]. \tag{4.1}$$

Proof. We find from (2.1) that

$$\int_0^{\infty} z^{p-1} e^{-sz} E_{\alpha,\beta,\gamma,\delta}^{j,k}(xz^{\sigma})\, dz$$
$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)\,(c)_n \; x^{nj+k}}{n! \; B(\gamma, c-\gamma)\,\Gamma(\beta + \alpha(nj+k))} \int_0^{\infty} z^{q+\sigma nj+\sigma k-1} e^{-sz} dz$$
$$= \frac{s^{-(q+\sigma k)} x^k}{\Gamma(\gamma)\Gamma(c-\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)\,\Gamma(q+\sigma k+\sigma nj)\,\Gamma(c-n)\, x^{nj}}{n! \; \Gamma(\beta+\alpha(nj+k))\, s^{\sigma nj}}. \tag{4.2}$$

Evaluating $B_p(\gamma+n, c-\gamma)$ in (4.2) using the series form of (1.5), we obtain

$$\int_0^\infty z^{p-1} e^{-sz} E^{j,k}_{\alpha,\beta,\gamma,\delta}(xz^\sigma)\,dz$$
$$= \frac{s^{-(q+\sigma k)}x^k}{\Gamma(\gamma)\Gamma(c-\gamma)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(q+\sigma k+\sigma nj)\,\Gamma(c-n)\,\Gamma(\gamma+n-m)\,\Gamma(c-\gamma-m)}{\Gamma(\beta+\alpha(nj+k))\,\Gamma(c+n-2m)\,m!\,n!}(-p)^m\left(\frac{x}{s^\sigma}\right)^{nj}.$$

In view of (1.8), we get the desired result.

When $j=1$ and $k=1$, Theorem 4.1 reduces to the following corollary.

Corollary 4.1 *The following result holds true:*

$$\int_0^\infty z^{q-1} e^{-sz} E^{\gamma,c}_{\alpha,\beta}(xz^\sigma;p)\,dz = \frac{s^{-q}}{\Gamma(\gamma)\Gamma(c-\gamma)}$$
$$\times H^{0,1:1,0;1,2}_{1,1:0,2;2,2}\left[\begin{array}{c|ccc} p & (1-\gamma;1,1): & -; & (1-q,\sigma),(1-c,1) \\ -\frac{x}{s^\sigma} & (1-c;1,2): & (0,1),(c-\gamma,1); & (0,1),(1-\beta,\alpha)\end{array}\right]. \tag{4.3}$$

Theorem 4.2 *(Mellin transform): Let $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(s)>0$ and $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, then*

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k\Gamma(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)}\,{}_2\Psi_2\left[\begin{array}{c} (c,1),(\gamma+s,1) \\ (\beta+\alpha k,\alpha j),(c+2s,1)\end{array}\middle| z^j\right]. \tag{4.4}$$

Proof. Applying the Mellin transform to equation (2.1), we get

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \int_0^\infty p^{s-1}E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)\,dp. \tag{4.5}$$

Now,using (3.1) in (4.5), we have

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k}{B(\gamma,c-\gamma)}\int_0^\infty p^{s-1}\left[\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}e^{\frac{-p}{t(1-t)}}\right]E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)dtdp. \tag{4.6}$$

Changing the order of integrations in (4.6), we obtain

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)\left[\int_0^\infty p^{s-1}e^{\frac{-p}{t(1-t)}}dp\right]dt. \tag{4.7}$$

Substituting $u=\frac{p}{t(1-t)}$ in (4.7), we have

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{c-\gamma+s-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)\left[\int_0^\infty u^{s-1}e^{-u}du\right]dt$$
$$= \frac{z^k\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{c-\gamma+s-1}E^c_{\alpha j,\beta+\alpha k}\left(tz^j\right)dt. \tag{4.8}$$

Further, using (1.3) in (4.8), we find

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k\Gamma(s)}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+n+s-1}(1-t)^{c-\gamma+s-1}\sum_{n=0}^{\infty}\frac{(c)_n\,z^{nj}}{\Gamma(\alpha nj+\beta+\alpha k)\,n!}dt,$$

which gives,

$$\mathcal{M}\left[E^{\gamma,c;j,k}_{\alpha,\beta}(z;p);s\right] = \frac{z^k\Gamma(s)\Gamma(c-\gamma+s)}{\Gamma(\gamma)\Gamma(c-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(c+n)}{\Gamma(\beta+\alpha k+\alpha nj)}\frac{\Gamma(\gamma+s+n)}{\Gamma(c+2s+n)}\frac{(z^j)^n}{n!}.$$

Finally, using (1.7) in above equation, we obtain the required result.

When $s=1$, Theorem 4.2 reduces to the following corollary.

Corollary 4.2 *The following result holds true:*

$$\int_0^\infty E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)\,dp = \frac{z^k\Gamma(c-\gamma+1)}{\Gamma(\gamma)\Gamma(c-\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+1,1)\\(\beta+\alpha k,\alpha j),(c+2,1)\end{array}\middle|\, z^j\right]. \tag{4.9}$$

This result generalizes the Corollary 6 in [17], if $j = 1$ and $k = 0$, equation (4.9) coincides with the result in [17].

Corollary 4.3 *(Inverse Mellin transform): Let* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$, $\Re(s) > 0$ *and* $j \geq 1$, $k \geq 0$, $\mu > 0$, $p \in \mathbb{R}_0^+$, *then*

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;j,k}(z;p) &= \frac{z^k}{2\pi i\ \Gamma(\gamma)\Gamma(c-\gamma)}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)\Gamma(c-\gamma+s)\\ &\quad\times{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+s,1)\\(\beta+\alpha k,\alpha j),(c+2s,1)\end{array}\middle|\, z^j\right]p^{-s}ds. \end{aligned} \tag{4.10}$$

Proof. Applying the inverse Mellin transform to both sides of equation (4.4), we get the result.

This result generalizes the Corollary 7 in [17], if $j = 1$ and $k = 0$, equation (4.10) coincides with the result in [17].

5 Fractional calculus operators

In this section, we derive certain interesting properties of $E_{\alpha,\beta}^{\gamma,c;j,k}(z;p)$ related to the operators of Riemann-Liouville fractional integral and derivative.

Theorem 5.1 *Let* $\nu,\alpha,\beta,\gamma,c,\omega \in \mathbb{C}$ *with* $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$, $x > a$. *Let* I_{a+}^ν *be the left-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega\,(t-a)^\alpha;p\right)\right]\right)(x) = (x-a)^{\beta+\nu-1}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega(x-a)^\alpha;p\right). \tag{5.1}$$

Proof. Using definitions (2.1) and (1.12), then after some simplification, we obtain

$$\begin{aligned} &\left(I_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}(\omega\,(t-a)^\alpha;p)\right]\right)(x)\\ &= \frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^\infty\frac{B_p(\gamma+n,c-\gamma)\ \omega^{nj+k}\ (c)_n}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_{a+}^\nu\left[(t-a)^{\beta+\alpha(nj+k)-1}\right]\right)(x)\\ &= \frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^\infty\frac{B_p\left(\gamma+n,c-\gamma\right)\omega^{nj+k}(c)_n}{n!\ \Gamma(\beta+\alpha(nj+k))}(x-a)^{\beta+\nu+\alpha(nj+k)-1}\\ &= (x-a)^{\beta+\nu-1}\sum_{n=0}^\infty\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n}{n!\ B\left(\gamma,c-\gamma\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma(\beta+\alpha(nj+k))}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)}(\omega(x-a)^\alpha)^{nj+k}\\ &= (x-a)^{\beta+\nu-1}\sum_{n=0}^\infty\frac{B_p(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)}(\omega(x-a)^\alpha)^{nj+k}\\ &= (x-a)^{\beta+\nu-1}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega(x-a)^\alpha;p\right). \end{aligned}$$

This result generalizes the Theorem 2.2 in [21].

Corollary 5.1 *Take* $j = 1$ *and* $k = 0$ *in (5.1), we get the result of Rahman et al. [21]*

$$\left(I_{a+}^\nu\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma;c}\left(\omega\,(t-a)^\alpha;p\right)\right]\right)(x) = (x-a)^{\beta+\nu-1}E_{\alpha,\beta+\nu}^{\gamma;c}\left(\omega(x-a)^\alpha;p\right). \tag{5.2}$$

Theorem 5.2 *Let* $\nu,\alpha,\beta,\gamma,c,\omega \in \mathbb{C}$ *with* $\Re(\nu) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$ *and* $j \geq 1$, $k \geq 0$, $p \in \mathbb{R}_0^+$. *Let* I_-^ν *be the right-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I_-^\nu\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x) = x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right). \tag{5.3}$$

Proof. Using definitions (2.1) and (1.17), then after some simplification, we obtain

$$\begin{aligned}
&\left(I_-^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)\\
&=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)\ \omega^{nj+k}(c)_n}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(I_-^{\nu}t^{-\nu-\beta-\alpha(nj+k)}\right)(x)\\
&=\frac{1}{B(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)\omega^{nj+k}(c)_n}{n!\ \Gamma\left(\beta+\alpha(nj+k)\right)}x^{-\beta-\alpha(nj+k)}\\
&=x^{-\beta}\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+n,c-\gamma\right)(c)_n}{n!\ B\left(\gamma,c-\gamma\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma(\beta+\alpha(nj+k))}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)}\left(\omega x^{-\alpha}\right)^{nj+k}\\
&=x^{-\beta}\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)}{n!\ B(\gamma,c-\gamma)}\frac{(c)_n}{\Gamma\left(\beta+\nu+\alpha(nj+k)\right)}\left(\omega x^{-\alpha}\right)^{nj+k}\\
&=x^{-\beta}E_{\alpha,\beta+\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right).
\end{aligned}$$

This completes the desired proof.
Setting $p=0$ and $\gamma=1$ in the result in Theorem 5.2, we get an interesting formula involving the Mittag-Leffler function of arbitrary order (1.6), which is asserted by the following corollary:

Corollary 5.2 *Let $\nu,\alpha,\beta,\omega\in\mathbb{C}$ such that $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $j\geq 1$, $k\geq 0$. Then the following right fractional integral formula holds true:*

$$\left(I_-^{\nu}\left[t^{-\nu-\beta}E_{\alpha,\beta}^{j,k}\left(\omega t^{-\alpha}\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta+\nu}^{j,k}(\omega x^{-\alpha}). \tag{5.4}$$

Theorem 5.3 *Let $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$ and $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$, $x>a$. Let D_{a+}^{ν} be the left-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega(t-a)^{\alpha};p\right)\right]\right)(x)=(x-a)^{\beta-\nu-1}E_{\alpha,\beta-\nu}^{\gamma,c;j,k}\left(\omega(x-a)^{\alpha};p\right). \tag{5.5}$$

Proof. For the proof of assertion (5.5), we use (1.14) and (5.1) in the left hand side of (5.5)

$$\begin{aligned}
&\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega(t-a)^{\alpha};p\right)\right]\right)(x)\\
&=\left(\frac{d}{dx}\right)^{m}\left[(x-a)^{\beta+m-\nu-1}E_{\alpha,\beta+m-\nu}^{\gamma,c;j,k}(\omega(x-a)^{\alpha},p)\right]\\
&=(x-a)^{\beta-\nu-1}E_{\alpha,\beta+m-\nu}^{\gamma,c;j,k}\left(\omega\left(x-a\right)^{\alpha};p\right),
\end{aligned}$$

This completes the proof of (5.5).

This result generalizes the Theorem 2.2 in [21].

Corollary 5.3 *Take $j=1$ and $k=0$ in (5.5), we get the result of Rahman et al. [21]*

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma;c}(\omega(t-a)^{\alpha};p)\right]\right)(x)=(x-a)^{\beta-\nu-1}E_{\alpha,\beta-\nu}^{\gamma,c}\left(\omega(x-a)^{\alpha};p\right). \tag{5.6}$$

Theorem 5.4 *Let $\nu,\alpha,\beta,\gamma,c,\omega\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(c)>0$, $\Re(\nu)>0$, $\Re(\beta)>[\Re(\nu)]+1$ and $j\geq 1$, $k\geq 0$, $p\in\mathbb{R}_0^+$. Let D_-^{ν} be the right-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_-^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)=x^{-\beta}E_{\alpha,\beta-\nu}^{\gamma,c;j,k}\left(\omega x^{-\alpha};p\right). \tag{5.7}$$

Proof. In order to the prove (5.7), we use (1.15) and (5.3) in the left hand side of (5.7)

$$\left(D_-^{\nu}\left[t^{\nu-\beta}E_{\alpha,\beta}^{\gamma,c;j,k}\left(\omega t^{-\alpha};p\right)\right]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^m \left[x^{\beta+m-\nu-1} E^{\gamma,c;j,k}_{\alpha,\beta+m-\nu}\left(\omega x^{-\alpha};p\right)\right]$$
$$= x^{-\beta} E^{\gamma,c;j,k}_{\alpha,\beta-\nu}\left(\omega x^{-\alpha};p\right),$$

which completes the desired proof.

Corollary 5.4 *If we take $p = 0$ and $\gamma = 1$ in equation (5.7), we obtain*

$$\left(D^{\nu}_{-}\left[t^{\nu-\beta} E^{j,k}_{\alpha,\beta}\left(\omega t^{-\alpha}\right)\right]\right)(x) = x^{-\beta} E^{j,k}_{\alpha,\beta-\nu}\left(\omega x^{-\alpha}\right). \tag{5.8}$$

6 Conclusion

In this paper, we derived a new extension of the Mittag-Leffler function and investigated several of its properties. The special cases of the main result for $j = 1$ and $k = 0$ are detailed in [17]. Therefore, the results introduced in this present article are new and an extension of the related outcomes in the current literature (see, e.g., [18, 21]). In addition, it is important to note that the function $E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)$ is highly compatible with fractional calculus. The newly defined Mittag-Leffler function $E^{\gamma,c;j,k}_{\alpha,\beta}(z;p)$ discussed in this article will have applications across various fields of applied sciences.

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