



High-Precision Matrix Computational Technique for Fractional Differential Equations using Generalized Symmetric Polynomials

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Abstract: In this work, a generalized main class of symmetric orthogonal polynomials is proposed and utilized within a spectral framework for solving fractional differential equations (FDEs) using tau method. Emphasis is placed on the monic symmetric and shifted monic symmetric representations of several kinds Chebyshev polynomials. These polynomials are employed to develop general unified operational differentiation matrices, which convert the original differential equations into equivalent systems of algebraic equations. The symmetry and orthogonality properties inherent to these polynomial families enhance both the precision and stability of the numerical schemes. A comparative study is conducted to assess the performance of each type in terms of accuracy, convergence behavior, and computational efficiency. The results confirm that the proposed method provides a reliable and adaptable framework for effectively solving a wide range of linear and nonlinear differential equations within the context of numerical analysis and applied mathematics.

Keywords: Symmetric orthogonal polynomials, Monic and shifted monic forms, Chebyshev polynomials, Spectral Tau method.

1 Introduction

Fractional differential equations (FDEs), which involve derivatives of arbitrary (non-integer) order, have attracted increasing interest due to their effectiveness in modeling real-world phenomena characterized by memory, nonlocality, and hereditary behavior [intro1, intro2]. Unlike classical differential models, FDEs can describe complex dynamic systems such as viscoelastic materials, anomalous diffusion, biological tissues, and electrical circuits more accurately [intro3, intro4, intro5, intro6]. Their applications span various scientific and engineering fields, including bioengineering, control theory, porous media, and thermal science.

However, solving FDEs analytically is challenging, especially for nonlinear, variable-order, or multiterm systems. As a result, the development of accurate and efficient numerical methods has become essential. Among the various numerical approaches, spectral and pseudospectral methods based on orthogonal polynomials have proven highly effective for approximating smooth solutions with a high degree of accuracy and rapid convergence [intro7, intro8, intro9, intro10, intro11, intro12, intro13]. These methods transform FDEs into systems of algebraic equations by constructing operational matrices of fractional derivatives, significantly simplifying the computational complexity [hakim1, hakim2, hakim3, hakim4].

Traditionally, classical orthogonal polynomials such as Legendre, Chebyshev kinds, and Jacobi polynomials have been used as basis functions in spectral methods [intro16, intro14, intro15]. Although successful, these classical families may lack the flexibility to handle more intricate problems involving variable-order operators, mixed-type equations, or nonlocal kernels. Using these generalized polynomials, we develop new operational matrices for fractional derivatives, tailored to various definitions such as the Caputo, Riemann–Liouville, and Atangana–Baleanu

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types. These matrices enable the transformation of both linear and nonlinear FDEs into algebraic systems, which are solved efficiently using spectral collocation techniques. The methodology is applicable to a wide range of problems, including variable-order, multi-order, and non-singular kernel cases.

Recent research has shown the effectiveness of using fifth- and sixth-kind Chebyshev polynomials to solve mixed-type and multi-order FDEs [me fifth, Atta2024]. For example, Obeid et al. (2023) constructed an operational matrix using shifted fifth-kind Chebyshev polynomials to tackle mixed fractional models with promising accuracy [Obeid2023]. Sadri and Aminikhah (2022) applied a fifth-kind-based algorithm to solve time-fractional diffusion-wave equations with multi-term structure [Sadri2022]. Similarly, novel operational matrix strategies have been proposed using seventh-kind Chebyshev and shifted Jacobi polynomials for variable-order and strongly singular problems [seventh1, seventh2, seventh3].

In this paper, we generalize and consolidate these recent developments by proposing a unified operational matrix approach based on a generalized class of shifted monic orthogonal polynomials, denoted $\mathcal{P}_j^{(\varepsilon, r, \kappa, \rho)}(x)$. This general framework encompasses several known polynomial families as special cases, including the fifth-, and sixth-kind Chebyshev polynomials, as well as shifted Legendre, Jacobi polynomials and other polynomials. Using this unified basis, we derive new operational matrices for fractional derivatives, focusing on the Caputo and Riemann fractional derivative definitions, which are widely used in modern fractional calculus due to their nonlocal and nonsingular characteristics. Therefore, the symmetric polynomials are used here with the spectral tau approach as a numerical computational matrix method. The proposed methodology is applied to a broad class of linear and nonlinear fractional differential equations (FDEs), accommodating fractional-order operators and diverse boundary conditions. These experiments demonstrate superior performance in terms of convergence and approximation accuracy when compared to existing techniques that rely on traditional polynomial bases. By building upon and extending the latest advances in orthogonal polynomial-based spectral methods, this work offers a comprehensive and adaptable framework for solving a wide variety of fractional differential models.

The paper is organized as: **section 2**, contains preliminaries while **section 3**, shows the extended Sturm–Liouville of the basic class of symmetric orthogonal polynomials. In addition to, **section 4**, drives the operational matrices of derivatives using shifted monic symmetric orthogonal polynomials of both ordinary and fractional cases. And, the approximation and related theories are introduced in **section 5**. Furthermore, the method of solution based on tau spectral approximation to Linear and Nonlinear FDEs using shifted monic symmetric orthogonal polynomials is obtained in **section 6**. Finally, the test examples and applications to the method is given in **section 7**, while the conclusion were written in **section 8**.

2 Preliminaries

This section discusses essential mathematical definitions and properties, focusing on symmetric orthogonal polynomials, and introduces monic forms. It also covers fractional derivatives, emphasizing the Caputo definition to facilitate initial conditions in differential equations.

Definition 2.1 The left-sided Caputo fractional derivative of order $\xi \in (k-1, k)$ for a Lebesgue integrable function $\Psi(x)$ is defined by [caputo1, caputo2]:

$${}^C D_{x_0+}^\xi \Psi(x) = \frac{1}{\Gamma(k-\xi)} \int_{x_0}^x \frac{\Psi^{(k)}(s)}{(x-s)^{\xi-k+1}} ds, \quad \xi > 0, x > x_0, \quad (1)$$

where $k \in \mathbb{N}$ is the smallest integer greater than ξ , $\Gamma(\cdot)$ is the Gamma function, and $\Psi^{(k)}(x)$ denotes the k -th derivative of $\Psi(x)$. The integration is taken over the interval $[x_0, x]$.

Definition 2.2. For the power function x^ν , the Caputo fractional derivative of order ξ , with $\xi > 0$, is given by [caputo1, caputo2]:

$${}^C D_{x_0+}^\xi x^\nu = \begin{cases} 0 & \text{if } \nu < \lceil \xi \rceil, \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\xi)} x^{\nu-\xi} & \text{if } \nu \geq \lceil \xi \rceil, \end{cases} \quad (2)$$

where $\lceil \cdot \rceil$ denotes the ceiling function, and $\nu \in \mathbb{N}$.

Definition 2.3. For a constant function $\Psi(x) = C$, the Caputo fractional derivative of order ξ is [caputo1, caputo2]:

$${}^C D_{x_0+}^\xi C = 0, \quad \xi > 0. \quad (3)$$

Definition 2.4. Let $\{\mathcal{P}_j(x)\}_{j=0}^\infty$ be a sequence of polynomials orthogonal with respect to a weight function $w(x)$ over the symmetric interval $[-1, 1]$. Then, the orthogonality condition holds:

$$\int_{-1}^1 \mathcal{P}_j(x) \mathcal{P}_k(x) w(x) dx = \begin{cases} 0, & j \neq k, \\ 1 & j = k, \end{cases} \quad (4)$$

By the definition of orthogonality in the Hilbert space $L_w^2[-1, 1]$, two functions $\mathcal{P}_j(x)$ and $\mathcal{P}_k(x)$, where $j \neq k$, are orthogonal with respect to the inner product:

$$\langle \mathcal{P}_j, \mathcal{P}_k \rangle = \int_{-1}^1 \mathcal{P}_j(x) \mathcal{P}_k(x) w(x) dx. \quad (5)$$

Since $\mathcal{P}_j$ and $\mathcal{P}_k$ are elements of an orthogonal polynomial system, their inner product is zero whenever $j \neq k$.

Definition 2.5. [Masjed-Jamei2006, me symmetric] Let $\{\Theta_j(x)\}_{j=0}^\infty$ be a sequence of symmetric functions satisfying

$$\Theta_j(x) = (-1)^j \Theta_j(-x), \quad \text{for all } j \in \mathbb{N}_0.$$

Suppose that each $\Theta_j(x)$ is a solution of the second-order differential equation

$$\mathcal{A}(x) \Theta_j''(x) + \mathcal{B}(x) \Theta_j'(x) + \left[\mu_j \mathcal{C}(x) + \mathcal{D}(x) + \frac{1 - (-1)^j}{2} \mathcal{E}(x) \right] \Theta_j(x) = 0, \quad (6)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ are fixed coefficient functions, and $\{\mu_j\}$ is a real sequence of parameters. If the coefficient functions satisfy:

- $\mathcal{A}(x), \mathcal{C}(x), \mathcal{D}(x), \mathcal{E}(x)$ are even functions,
- $\mathcal{B}(x)$ is an odd function,

then the functions $\Theta_j(x)$ form an orthogonal system on the symmetric interval $[-\beta, \beta]$ with respect to the weight function $\omega^*(x)$.

$$\int_{-\beta}^{\beta} \omega^*(x) \Theta_j(x) \Theta_k(x) dx = \begin{cases} 0, & j \neq k, \\ \int_{-\beta}^{\beta} \omega^*(x) \Theta_j^2(x) dx, & j = k, \end{cases} \quad (7)$$

where the weight function $\omega^*(x)$ is given by:

$$\omega^*(x) = \frac{\mathcal{C}(x)}{\mathcal{A}(x)} \exp \left(\int \frac{\mathcal{B}(x)}{\mathcal{A}(x)} dx \right). \quad (8)$$

Furthermore, $\omega^*(x)$ must be a positive, even function over $[-\beta, \beta]$, and the endpoint $x = \beta$ must be a root of the function

$$\mathcal{K}(x) = \exp \left(\int \frac{\mathcal{B}(x)}{\mathcal{A}(x)} dx \right). \quad (9)$$

3 Extended Sturm–Liouville of a Basic Class of Symmetric Orthogonal Polynomials.

In this section, we introduce a fundamental class of symmetric orthogonal polynomials constructed through an extended Sturm–Liouville theorem (according to Definition 2.5.). This framework accommodates even and odd coefficient functions and yields a set of orthogonal polynomials defined over symmetric intervals.

3.1 Generating Symmetric Orthogonal Polynomials.

By utilizing the generalized symmetric differential equation presented earlier, we now introduce a specific selection of coefficient functions as follows:

$$\begin{aligned} \mathcal{A}(x) &= x^2(\varepsilon x^2 + r), \quad (\text{even function}), \\ \mathcal{B}(x) &= x(\kappa x^2 + \rho), \quad (\text{odd function}), \\ \mathcal{C}(x) &= x^2, \quad \mathcal{D}(x) = 0, \quad \mathcal{E}(x) = -\rho, \quad (\text{even functions}), \end{aligned}$$

where, $\varepsilon, r, \kappa, \rho \in \mathbb{R}$ are free real parameters. The associated eigenvalue for each $j \in \mathbb{N}$ is given by:

$$\mu_j = -j(\kappa + (j-1)\varepsilon).$$

Substituting these expressions into the generalized symmetric differential equation yields the following second-order differential equation satisfied by the symmetric orthogonal polynomial $\Theta_j(x)$:

$$x^2(\varepsilon x^2 + r) \Theta_j''(x) + x(\kappa x^2 + \rho) \Theta_j'(x) - \left(j(\kappa + (j-1)\varepsilon) x^2 - \left(1 - \frac{(-1)^j \rho}{2} \right) \right) \Theta_j(x) = 0. \quad (10)$$

To construct polynomial solutions for the proposed second-order differential equation, we begin by analysing the even-degree case: Set $j = 2n$, substitute into the differential equation to obtain:

$$x^2(\varepsilon x^2 + r) \Theta_{2n}''(x) + x(\kappa x^2 + \rho) \Theta_{2n}'(x) - \left(2n(\kappa + (2n-1)\varepsilon) - \left(1 - \frac{(-1)^{2n} \rho}{2} \right) \right) \Theta_{2n}(x) = 0. \quad (11)$$

Solving this equation yields the polynomial solution $\mathcal{S}_{2n}^{(\varepsilon, r, \kappa, \rho)}(x)$ expressed as:

$$\mathcal{S}_{2n}^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^n \binom{n}{k} \prod_{i=0}^{n-(k+1)} \left(\frac{(2i-1+2n)\varepsilon + \kappa}{(2i+1)r + \rho} \right) x^{2n-2k}. \quad (12)$$

For the odd-degree case:

Set $j = 2n+1$, leading to the simplified equation:

$$x^2(\varepsilon x^2 + r) \Theta_{2n+1}''(x) + x(\kappa x^2 + \rho) \Theta_{2n+1}'(x) - \left((2n+1)(\kappa + 2n\varepsilon) - \left(1 - \frac{(-1)^{2n+1} \rho}{2} \right) \right) \Theta_{2n+1}(x) = 0, \quad (13)$$

which results in the solution:

$$\mathcal{S}_{2n+1}^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^n \binom{n}{k} \prod_{i=0}^{n-(k+1)} \left(\frac{(2i+1+2n)\varepsilon + \kappa}{(2i+3)r + \rho} \right) x^{2n+1-2k}. \quad (14)$$

Finally, by unifying both even and odd cases, the general polynomial solution of (10) can be written as:

$$\mathcal{S}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{\lfloor m/2 \rfloor}{k} \prod_{i=0}^{\lfloor m/2 \rfloor - (k+1)} \left(\frac{(2i+1+(-1)^m + 2\lfloor m/2 \rfloor)\varepsilon + \kappa}{(2i+1+(-1)^m + 2)r + \rho} \right) x^{m-2k}. \quad (15)$$

In addition, we call:

$$\mathcal{C}_l = \frac{(2\ell+1+(-1)^m + 2\lfloor m/2 \rfloor)\varepsilon + \kappa}{(2\ell+1+(-1)^m + 2)r + \rho}. \quad (16)$$

3.2 Construction of Monic Symmetric Orthogonal Polynomials.

In 2006, Masjed-Jamei [Masjed-Jamei2006], introduced a monic family of symmetric orthogonal polynomials, denoted by $\overline{\mathcal{S}}_m^{(\varepsilon, r, \kappa, \rho)}(x)$, which are characterized by having leading coefficients equal to unity. These polynomials are defined in terms of the original family $\mathcal{S}_m^{(\varepsilon, r, \kappa, \rho)}(x)$ as follows:

$$\overline{\mathcal{S}}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \left(\prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{1}{\mathcal{C}_j} \right) \mathcal{S}_m^{(\varepsilon, r, \kappa, \rho)}(x). \quad (17)$$

The monic polynomials satisfy the recurrence relation:

$$\overline{\mathcal{S}}_{m+1}^{(\varepsilon, r, \kappa, \rho)}(x) = x \cdot \overline{\mathcal{S}}_m^{(\varepsilon, r, \kappa, \rho)}(x) + \Lambda_m^{\varepsilon, r, \kappa, \rho} \cdot \overline{\mathcal{S}}_{m-1}^{(\varepsilon, r, \kappa, \rho)}(x), \quad m \geq 0, \quad (18)$$

with initial conditions:

$$\overline{\mathcal{S}}_0^{(\varepsilon, r, \kappa, \rho)}(x) = 1, \quad \overline{\mathcal{S}}_1^{(\varepsilon, r, \kappa, \rho)}(x) = x.$$

Here, the recurrence coefficient $\Lambda_m^{\varepsilon, r, \kappa, \rho}$ is given by:

$$\Lambda_m^{\varepsilon, r, \kappa, \rho} = \frac{\varepsilon r m^2 + ((\kappa - 2\varepsilon)r - (-1)^m \varepsilon \rho)m + \frac{1}{2}\rho(\kappa - 2\varepsilon)(1 - (-1)^m)}{(2\varepsilon m + \kappa - \varepsilon)(2\varepsilon m + \kappa - 3\varepsilon)}. \quad (19)$$

The explicit forms of the first few monic polynomials are as follows:

$$\begin{aligned} \overline{\mathcal{S}}_2^{(\varepsilon, r, \kappa, \rho)}(x) &= \frac{r+\rho}{\varepsilon+\kappa} + x^2, \\ \overline{\mathcal{S}}_3^{(\varepsilon, r, \kappa, \rho)}(x) &= \frac{(3r+\rho)x}{3\varepsilon+\kappa} + x^3, \\ \overline{\mathcal{S}}_4^{(\varepsilon, r, \kappa, \rho)}(x) &= \frac{(r+\rho)(3r+\rho)}{(3\varepsilon+\kappa)(5\varepsilon+\kappa)} + \frac{2(3\rho+\rho)x^2}{5\varepsilon+\kappa} + x^4, \\ \overline{\mathcal{S}}_5^{(\varepsilon, r, \kappa, \rho)}(x) &= \frac{(3r+\rho)(5r+\rho)x}{(5\varepsilon+\kappa)(7\varepsilon+\kappa)} + \frac{2(5r+\rho)x^3}{7\varepsilon+\kappa} + x^5. \end{aligned}$$

This monic formulation not only provides a normalized representation of the symmetric orthogonal polynomials but also facilitates algebraic handling and numerical computations. Furthermore, by assigning suitable values to the parameters ε , r , κ , and ρ , this generalized class can reproduce Chebyshev polynomials of different kinds and other known polynomials, making it a versatile tool in the study of orthogonal systems. The generalized monic function $\overline{\mathcal{S}}_m^{(\varepsilon, r, \kappa, \rho)}(x)$ possesses significant flexibility, enabling the generation of various classical orthogonal polynomial families through specific choices of the parameters ε , r , κ , and ρ .

Monic Chebyshev polynomials of the first kind can be obtained by choosing $\varepsilon = -1$, $r = 1$, $\kappa = -1$, and $\rho = 0$, giving:

$$\overline{T}_m(x) = 2^{m-1} \overline{\mathcal{S}}_{-1,1,-1,0}^m(x).$$

Monic Chebyshev polynomials of the second kind are derived when $\varepsilon = -1$, $r = 1$, $\kappa = -3$, and $\rho = 0$:

$$\overline{U}_m(x) = 2^m \overline{\mathcal{S}}_{-1,1,-3,0}^m(x).$$

Setting $\varepsilon = -1$, $r = 1$, $\kappa = -2$, and $\rho = 0$, the function simplifies to the even-degree monic Legendre polynomials:

$$\overline{P}_m(x) = \frac{(2m)!}{(m!)^2 2^{2m}} \overline{\mathcal{S}}_{-1,1,-2,0}^{2m}(x).$$

The ultraspherical (Gegenbauer) polynomials $C_m^a(x)$ arise by assigning $\varepsilon = -1$, $r = 1$, $\kappa = -(2a + 1)$, and $\rho = 0$, resulting in:

$$C_m^a(x) = \frac{2^m (a)_m}{m!} \overline{\mathcal{S}}_{-1,1,-(2a+1),0}^{2m}(x),$$

where $(a)_m$ is the Pochhammer symbol [po] and, $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$.

In [Masjed-Jamei2006], within the research of Masjed-Jamei, after the proposal of symmetric polynomials with their four parameters, Chebyshev polynomials of the fifth and sixth kinds were introduced by selecting parameter values that had not existed before. More recently, the same idea was utilized, using specific numbers to develop the seventh kind.

For the monic Chebyshev polynomials of the fifth kind, the parameters are set as $\varepsilon = -1$, $r = 1$, $\kappa = -3$, and $\rho = 2$:

$$\overline{\mathcal{S}}^m(x) = \overline{\mathcal{S}}_{-1,1,-3,2}^m(x).$$

The monic Chebyshev polynomials of the sixth kind are obtained by choosing $\varepsilon = -1$, $r = 1$, $\kappa = -5$, and $\rho = 2$:

$$\overline{\mathcal{Y}}_m(x) = \overline{\mathcal{S}}_{-1,1,-5,2}^m(x).$$

Lastly, the monic Chebyshev polynomials of the seventh kind are generated by selecting the parameter values $\varepsilon = -1$, $r = 1$, $\kappa = -5$, and $\rho = 4$. Under this configuration, the polynomials are given by:

$$\overline{\mathcal{X}}_m(x) = \overline{\mathcal{S}}_{-1,1,-5,4}^m(x).$$

This unified formulation provides a powerful and elegant framework for studying a broad spectrum of classical orthogonal polynomials and offers a basis for further analytical and computational advancements.

Favard's theorem [111] establishes that the three-term recurrence relation implies the following orthogonality property:

$$\begin{aligned} \left\langle \overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(x), \overline{\mathcal{F}}_n^{(\varepsilon, r, \kappa, \rho)}(x) \right\rangle &= \int_{-\beta}^{\beta} \Theta^{(\varepsilon, r, \kappa, \rho)}(x) \overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(x) \overline{\mathcal{F}}_n^{(\varepsilon, r, \kappa, \rho)}(x) dx \\ &= \left((-1)^m \prod_{\nu=0}^m \Lambda_{\nu}^{(\varepsilon, r, \kappa, \rho)} \int_{-\beta}^{\beta} \Theta^{(\varepsilon, r, \kappa, \rho)}(x) dx \right) \delta_{m,n}, \end{aligned} \quad (20)$$

where: $\Lambda_{\nu}^{(\varepsilon, r, \kappa, \rho)}$ is defined as in equation (19), $\delta_{m,n}$ is the Kronecker delta function, The weight function $\Theta^{(\varepsilon, r, \kappa, \rho)}(x)$ is given by:

$$\Theta^{(\varepsilon, r, \kappa, \rho)}(x) = \exp \left(\int \frac{\rho + x^2(\kappa - 2\varepsilon)}{x(r + \varepsilon x^2)} dx \right). \quad (21)$$

Additionally, the expression $(\varepsilon x^2 + r)\Theta^{(\varepsilon, r, \kappa, \rho)}(x)$ must vanish at the boundaries of the orthogonality interval.

3.3 Shifted monic class of symmetric orthogonal polynomials.

In this section, we introduce a shifted version of the monic symmetric orthogonal polynomials (MCSOP) for $\beta > 0$, defined over the interval $S = [0, \beta]$, via an appropriate variable transformation, the polynomials denoted by $\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)$, for $m \in \mathbb{N}_0$.

$$\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(2x - \beta), \quad m \in \mathbb{N}_0, \quad x \in S, \quad (22)$$

where, $\overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}$ denotes the standard monic symmetric orthogonal polynomial (MCSOP) of degree m .

The polynomial system $\{\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)\}_{m=0}^{\infty}$ forms a complete orthogonal set with respect to the weight function:

$$\Theta^{(\varepsilon, r, \kappa, \rho)}(x) = \exp \left(\int \frac{\rho + (2x - \beta)^2(\kappa - 2\varepsilon)}{(2x - \beta)(r + \varepsilon(2x - \beta)^2)} dx \right), \quad (23)$$

which, satisfies the inner product defined by:

$$\langle \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}, \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)} \rangle = \mathcal{N}_m \delta_{m,n}, \quad (24)$$

where, $\delta_{m,n}$ is the Kronecker delta and the normalization constant $\mathcal{N}_m$ is given by:

$$\mathcal{N}_m = (-1)^m \left(\prod_{\nu=0}^m \Lambda_{\nu}^{(\varepsilon, r, \kappa, \rho)} \right) \int_S \Theta^{(\varepsilon, r, \kappa, \rho)}(x) dx. \quad (25)$$

Lemma 3.1. Let $x \in S$ and $m \in \mathbb{N}_0$. Then, the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)$ admit the following series expansion:

$$\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} x^s, \quad (26)$$

where,

$$\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} = \binom{\lfloor \frac{m}{2} \rfloor}{k} \binom{m-2k}{s} \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{1}{\mathcal{C}_j^{(\varepsilon, r, \kappa, \rho)}} \prod_{l=0}^{\lfloor \frac{m}{2} \rfloor - (k+1)} \mathcal{C}_l^{(\varepsilon, r, \kappa, \rho)} (2)^s (-\beta)^{m-2k-s}. \quad (27)$$

Proof. We begin by considering the monic form of the symmetric polynomials $\overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(x)$, which can be expressed using the series expansion from equation (17):

$$\overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor}{k} \prod_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \frac{1}{\mathcal{C}_j^{(\varepsilon, r, \kappa, \rho)}} \prod_{l=0}^{\lfloor \frac{m}{2} \rfloor - (k+1)} \mathcal{C}_l^{(\varepsilon, r, \kappa, \rho)} x^{m-2k}. \quad (28)$$

The shifted polynomial is obtained through the transformation $x = 2x - \beta$, giving:

$$\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(2x - \beta). \quad (29)$$

Substituting the expansion of $\overline{\mathcal{F}}_m^{(\varepsilon, r, \kappa, \rho)}(x)$ and applying the binomial theorem to $(2x - \beta)^{m-2k}$, we obtain:

$$(2x - \beta)^{m-2k} = \sum_{s=0}^{m-2k} \binom{m-2k}{s} (2x)^s (-\beta)^{m-2k-s}.$$

Expanding and collecting terms, we derive the double summation representation:

$$\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} x^s,$$

with coefficients $\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)}$ as previously defined. This completes the proof. ■

4 Operational Matrix of Derivatives Using Shifted Monic Symmetric Orthogonal Polynomials.

In this section, we construct the operational matrices corresponding to the differentiation operators using the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)$. These matrices are then applied to approximate a given function over the interval $S = [0, \beta]$.

Proposition 4.1. Let $\{\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)\}_{m=0}^n$ be a set of shifted monic symmetric orthogonal polynomials defined on the interval $S = [0, \beta]$. Suppose these polynomials can be expressed in terms of the monomial basis $\{x^s\}_{s=0}^n$ as:

$$\mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{X}(x), \quad (30)$$

where,

$$\mathcal{G}_n(x) = [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)}(x)]^T, \\ \mathbf{X}(x) = [1, x, \dots, x^n]^T,$$

and $\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \in \mathbb{R}^{(n+1) \times (n+1)}$ is the Coefficient lower triangle matrix, in general takes the form:

$$\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} = \begin{pmatrix} b_{0,0}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & 0 & \dots & 0 \\ b_{1,0}^{(\varepsilon, r, \kappa, \rho)} & b_{1,1}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & \dots & 0 \\ b_{2,0}^{(\varepsilon, r, \kappa, \rho)} & b_{2,1}^{(\varepsilon, r, \kappa, \rho)} & b_{2,2}^{(\varepsilon, r, \kappa, \rho)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n,0}^{(\varepsilon, r, \kappa, \rho)} & b_{n,1}^{(\varepsilon, r, \kappa, \rho)} & b_{n,2}^{(\varepsilon, r, \kappa, \rho)} & b_{n,3}^{(\varepsilon, r, \kappa, \rho)} & \dots & b_{n,n}^{(\varepsilon, r, \kappa, \rho)} \end{pmatrix}, \quad (31)$$

where, $b_{i,j}^{(\varepsilon, r, \kappa, \rho)}$ are the internal elements of $\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)}$ and they directly given from relation (27).

Proposition 4.2. Furthermore, the operational matrix of ordinary derivative $\mathcal{D}^{(1)} \in \mathbb{R}^{(n+1) \times (n+1)}$ (first order derivative), defined by:

$$\frac{d}{dx} \mathcal{G}_n(x) = \mathcal{D}^{(1)} \cdot \mathcal{G}_n(x), \quad (32)$$

where, $\mathcal{D}^{(1)}$ is the lower triangular operational matrix of derivatives, an $(n+1) \times (n+1)$ matrix, in general, defined as:

$$\mathcal{D}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ d_{1,0}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & 0 & \dots & 0 \\ 0 & d_{2,1}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & \dots & 0 \\ d_{3,0}^{(\varepsilon, r, \kappa, \rho)} & 0 & d_{3,2}^{(\varepsilon, r, \kappa, \rho)} & 0 & \dots & 0 \\ 0 & d_{4,1}^{(\varepsilon, r, \kappa, \rho)} & 0 & d_{4,3}^{(\varepsilon, r, \kappa, \rho)} & \dots & 0 \\ d_{5,0}^{(\varepsilon, r, \kappa, \rho)} & 0 & d_{5,2}^{(\varepsilon, r, \kappa, \rho)} & 0 & d_{5,4}^{(\varepsilon, r, \kappa, \rho)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n,0}^{(\varepsilon, r, \kappa, \rho)} & d_{n,1}^{(\varepsilon, r, \kappa, \rho)} & d_{n,2}^{(\varepsilon, r, \kappa, \rho)} & d_{n,3}^{(\varepsilon, r, \kappa, \rho)} & \dots & d_{n,n}^{(\varepsilon, r, \kappa, \rho)} \end{bmatrix},$$

where, $d_{i,j}^{(\varepsilon, r, \kappa, \rho)}$ are the internal elements of $\mathcal{D}^{(1)}$, and, $\mathcal{D}^{(1)} = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{P} \cdot \left(\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1}$, in addition, $\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)}$ is previously defined.

proof. From the relation (30) $\mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{X}(x)$, we differentiate both sides with respect to x :

$$\frac{d}{dx} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{d}{dx} \mathbf{X}(x).$$

The derivative of the monomial vector is

$$\frac{d}{dx} \mathbf{X}(x) = \mathbf{P} \cdot \mathbf{X}(x),$$

where, $\mathbf{P}$ is the monomial basis differentiation matrix.

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 3 & 0 & \dots & 0 \\ 0 & 0 & 0 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n & 0 \end{bmatrix}.$$

Substituting this into the previous expression yields:

$$\frac{d}{dx} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{P} \cdot \mathbf{X}(x).$$

Since, $\mathbf{X}(x) = \left(\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{G}_n(x)$, obtains:

$$\frac{d}{dx} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{P} \cdot \left(\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{G}_n(x).$$

Hence, the operational matrix of derivative in the polynomial basis is

$$\mathcal{D}^{(1)} = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{P} \cdot \left(\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1}.$$

■

Remark: Let $\mathcal{G}_n(x) = [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)}(x)]^T$ denote the vector of the shifted monic symmetric orthogonal polynomials over the interval $S = [0, \beta]$. Then, the m -th derivative of this vector with respect to x can be expressed using an operational matrix $\mathcal{D}^{(m)}$ as follows:

$$\frac{d^m \mathcal{G}_n(x)}{dx^m} = \mathcal{D}^{(m)} \mathcal{G}_n(x), \quad (33)$$

where,

$$\mathcal{D}^{(m)} = (\mathcal{D}^{(1)})^m, \quad m = 1, 2, 3, \dots, \quad (34)$$

and the first-order derivative matrix $\mathcal{D}^{(1)}$ is constructed via the similarity transformation:

$$\mathcal{D}^{(1)} = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathbf{P} \cdot (\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)})^{-1}, \quad (35)$$

then,

$$\mathcal{D}^{(m)} = (\mathcal{D}^{(1)})^m = \overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)} \cdot (\mathbf{P})^m \cdot (\overline{\mathcal{B}}_n^{(\varepsilon, r, \kappa, \rho)})^{-1}, \quad (36)$$

where, $\mathbf{P}$ and $\overline{\mathcal{B}}_n^{(r, s, p, q)}$ are previously defined.

For example, if n, β taken as 4, 1, then,

$$\begin{aligned} \mathcal{G}_4(x) &= [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_4^{(\varepsilon, r, \kappa, \rho)}(x)]^T, \\ \mathbf{X}(x) &= [1, x, x^2, x^3, x^4]^T, \\ \overline{\mathcal{B}}_4^{(\varepsilon, r, \kappa, \rho)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 + \frac{r+\rho}{\varepsilon+\kappa} & -4 & 4 & 0 & 0 \\ -1 - \frac{3\varepsilon+\kappa}{3r+\rho} & 6 + \frac{2(3r+\rho)}{3\varepsilon+\kappa} & -12 & 8 & 0 \\ 1 + \frac{2(3r+\rho)}{5\varepsilon+\kappa} + \frac{(r+\rho)(3r+\rho)}{(3\varepsilon+\kappa)(5\varepsilon+\kappa)} & -8 - \frac{8(3r+\rho)}{5\varepsilon+\kappa} & 24 + \frac{8(3r+\rho)}{5\varepsilon+\kappa} & -32 & 16 \end{pmatrix}, \\ \mathbf{P} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}. \end{aligned}$$

So,

$$\mathcal{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ -\frac{4(mn+4np+3pq)}{(m+p)(m+3p)} & 0 & 6 & 0 & 0 \\ 0 & -\frac{16p(n+3q)}{m^2+8mp+15p^2} & 0 & 8 & 0 \\ \frac{16(2mnp(n-q)-m^2nq+p^2(11n^2+23nq+15q^2))}{(m+p)(m+3p)(m+5p)(m+7p)} & 0 & -\frac{8(mn+5p(2n+3q))}{(m+5p)(m+7p)} & 0 & 10 & 0 \end{pmatrix},$$

and,

$$\mathcal{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \\ 0 & 24 & 0 & 0 & 0 \\ -\frac{32\varepsilon(\rho+3r)}{\kappa^2+8\kappa\varepsilon+15\varepsilon^2} - \frac{32(\kappa\rho+4\rho\varepsilon+3\varepsilon r)}{(\kappa+\varepsilon)(\kappa+3\varepsilon)} & 0 & 48 & 0 & 0 \end{pmatrix}.$$

Proposition 4.3. Assume that ${}^C D_{0+}^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in \mathbb{R}$. Then, the Caputo fractional derivative of the shifted monic Symmetric polynomial vector $\mathcal{G}_n(x)$ in terms of $\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)$ can be expressed as:

$${}^C D_{0+}^\alpha \mathcal{G}_n(x) = \mathcal{O}_{(\alpha)}^{(\varepsilon, r, \kappa, \rho)} \mathcal{G}_n(x), \quad (37)$$

where, $\mathcal{O}_{(\alpha)}^{(\varepsilon, r, \kappa, \rho)}$ is referred to the lower triangular operational matrix of the Caputo fractional derivative of order α associated with the shifted monic basis, and in general it takes the following form:

$$\mathcal{O}_{(\alpha)}^{(\varepsilon, r, \kappa, \rho)} = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \mathcal{O}_{\alpha, [\alpha], 0}^{(\varepsilon, r, \kappa, \rho)} & \dots & \mathcal{O}_{\alpha, [\alpha], [\alpha]}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O}_{\alpha, n, 0}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{O}_{\alpha, n, 1}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{O}_{\alpha, n, 2}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{O}_{\alpha, n, 3}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{O}_{\alpha, n, 4}^{(\varepsilon, r, \kappa, \rho)} & \dots & \dots & \mathcal{O}_{\alpha, n, n}^{(\varepsilon, r, \kappa, \rho)} \end{pmatrix}. \quad (38)$$

Where, $\mathcal{O}_{\alpha,i,j}^{(\varepsilon,r,\kappa,\rho)}$ are the elements of $\mathcal{O}_{(\alpha)}^{(\varepsilon,r,\kappa,\rho)}$,
and,

$$\mathcal{O}_{(\alpha)}^{(\varepsilon,r,\kappa,\rho)} = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \cdot \mathcal{A}_{(\alpha)}(x) \cdot \left(\overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \right)^{-1},$$

and

$$\mathcal{A}_{(\alpha)}(x) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \frac{\Gamma(\lceil \alpha \rceil + 1)}{\Gamma(\lceil \alpha \rceil - \alpha + 1)} x^{-\alpha} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{-\alpha} \end{bmatrix}. \quad (39)$$

Proof. Let $\mathcal{G}_n(x)$ in terms of $\mathcal{B}_m^{(\varepsilon,r,\kappa,\rho)}(x)$ denote the vector of shifted monic symmetric orthogonal polynomials. According to the representation established in Section (4), this vector can be expressed in terms of the monomial basis $\mathbf{X}(x) = [1, x, x^2, \dots, x^n]^T$ via the transformation:

$$\mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \mathbf{X}(x),$$

where, $\overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)}$ is a non-singular coefficients matrix specific to the basis parameters. Applying the Caputo fractional derivative of order α , we obtain:

$${}^C D_{0+}^{\alpha} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} {}^C D_{0+}^{\alpha} \mathbf{X}(x). \quad (40)$$

By using the known result for the Caputo derivative of the monomials x^k , namely,

$${}^C D_{0+}^{\alpha} x^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}, & \text{if } k \geq \lceil \alpha \rceil, \\ 0, & \text{otherwise,} \end{cases}$$

we can express the fractional derivative of $\mathbf{X}(x)$ in terms of a lower triangular matrix:

$${}^C D_{0+}^{\alpha} \mathbf{X}(x) = \mathcal{A}_{(\alpha)}(x) \mathbf{X}(x),$$

where, $\mathcal{A}_{(\alpha)}(x)$ is defined by

$$\mathcal{A}_{(\alpha)}(x) = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \frac{\Gamma(\lceil \alpha \rceil + 1)}{\Gamma(\lceil \alpha \rceil - \alpha + 1)} x^{-\alpha} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{-\alpha} \end{bmatrix}. \quad (41)$$

Substituting into equation (40), we get:

$${}^C D_{0+}^{\alpha} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \mathcal{A}_{(\alpha)}(x) \mathbf{X}(x).$$

Since, $\mathbf{X}(x) = \left(\overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \right)^{-1} \mathcal{G}_n(x)$, it follows that:

$${}^C D_{0+}^{\alpha} \mathcal{G}_n(x) = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \mathcal{A}_{(\alpha)}(x) \left(\overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \right)^{-1} \mathcal{G}_n(x).$$

Hence, the Caputo derivative of the shifted monic basis vector is given by:

$${}^C D_{0+}^{\alpha} \mathcal{G}_n(x) = \mathcal{O}_{(\alpha)}^{(\varepsilon,r,\kappa,\rho)} \mathcal{G}_n(x),$$

where, the operational matrix $\mathcal{O}_{(\alpha)}^{(\varepsilon,r,\kappa,\rho)}$ is defined as a diagonalization:

$$\mathcal{O}_{(\alpha)}^{(\varepsilon,r,\kappa,\rho)} = \overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \mathcal{A}_{(\alpha)}(x) \left(\overline{\mathcal{B}}_n^{(\varepsilon,r,\kappa,\rho)} \right)^{-1}.$$

For example, if n, β, c , and α taken as 4, 1, 0 and $\frac{1}{2}$, then,

$$\mathcal{A}_{(0.5)}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{\pi}\sqrt{x}} & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{3\sqrt{\pi}\sqrt{x}} & 0 & 0 \\ 0 & 0 & 0 & \frac{16}{5\sqrt{\pi}\sqrt{x}} & 0 \\ 0 & 0 & 0 & 0 & \frac{128}{35\sqrt{\pi}\sqrt{x}} \end{pmatrix}.$$

5 Approximation, Error Analysis.

In this section, we present a comprehensive investigation into the convergence behavior and error estimates of the orthogonal expansion using the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)$.

Lemma 5.1 Let $\mathcal{U}(x)$ be a square-integrable function over the interval $[0, 1]$, $\mathcal{U}(x) \in L^2_{\Theta}(0, 1)$, where $\Theta(x)$ is the corresponding weight function associated with the orthogonality of the basis. Then, the function $\mathcal{U}(x)$ can be expanded in terms of the linearly independent shifted monic symmetric polynomials $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ as follows:

$$\mathcal{U}(x) = \sum_{j=0}^{\infty} \mathcal{X}_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (42)$$

where, the expansion coefficients $\mathcal{X}_j$ are given by the inner product:

$$\mathcal{X}_j = \frac{1}{\mathcal{Q}_j} \int_0^1 \mathcal{U}(x) \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \Theta(x) dx, \quad j = 0, 1, 2, \dots \quad (43)$$

where,

$$\mathcal{Q}_j = \int_0^1 (\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x))^2 \Theta(x) dx.$$

In practical applications, the infinite series is truncated to the first $m + 1$ terms to obtain an approximate solution $\mathcal{U}_m(x)$, given by:

$$\mathcal{U}_m(x) = \sum_{j=0}^m \mathcal{X}_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) = \mathcal{X}^T \psi(x), \quad (44)$$

where,

$$\mathcal{X} = [\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m]^T, \quad \psi(x) = [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)]^T.$$

Lemma 5.2 The expression for the left-sided Caputo fractional derivative of order φ applied to the general shifted monic symmetric orthogonal polynomial $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ is given by:

$${}^C D_{0+}^{\varphi} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \approx \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{E}_{(\alpha, m, n)} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (45)$$

where,

$$\mathcal{E}_{(\alpha, m, n)} = \sum_{s=0}^{m-2k} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \frac{\Gamma(s+1) \eta_j^{s, \alpha}}{\Gamma(s-\alpha+1)}, \quad (46)$$

and $\eta_j^{s, \alpha}$ is defined as:

$$\eta_j^{s, \alpha} = x^{-\alpha} \cdot \left(\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) dx. \quad (47)$$

Proof. From the analytic form of $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ and applying the Caputo derivative gives:

$$\begin{aligned} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) &= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} x^s, \\ {}^C D_{0+}^{\varphi} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) &= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} {}^C D_{0+}^{\varphi} x^s = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \frac{\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \Gamma(s+1)}{\Gamma(s-\alpha+1)} x^{s-\alpha}. \end{aligned}$$

Now using the inverse expansion in terms of $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$, we write:

$$x^{s-\alpha} \approx \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} x^{-\alpha} \cdot \left(\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x),$$

with

$$\eta_j^{s, \alpha} = x^{-\alpha} \cdot \left(\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \right)^{-1}. \quad (48)$$

Substituting this into the previous expression yields:

$${}^C D_{0+}^{\alpha} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \approx \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \frac{\mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \Gamma(s+1) \eta_j^{s, \alpha}}{\Gamma(s-\varphi+1)} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (49)$$

which proves the result. ■

Corollary 5.1 The left-sided Caputo's fractional derivative of order α for the vector of generalized shifted symmetric orthogonal polynomials $\Psi(x) = [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)]^T$, is given approximately by:

$${}^C D_{0+}^\alpha \Psi(x) \approx \mathcal{O}^{(\alpha)} \Psi(x). \quad (50)$$

Here, $\mathcal{O}^{(\alpha)}$ is the operational matrix of the left-sided Caputo fractional derivative of order α , such that $0 \leq \alpha \leq 1$, defined as:

$$\mathcal{O}_{(\alpha)}^{(\varepsilon, r, \kappa, \rho)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \mathcal{E}_{\alpha, 1, 0}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, 1, 1}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & 0 & \cdots & 0 \\ \mathcal{E}_{\alpha, 2, 0}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, 2, 1}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, 2, 2}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & \cdots & 0 \\ \mathcal{E}_{\alpha, 3, 0}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, 3, 1}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, 3, 2}^{(\varepsilon, r, \kappa, \rho)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathcal{E}_{\alpha, n, 0}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, n, 1}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, n, 2}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, n, 3}^{(\varepsilon, r, \kappa, \rho)} & \mathcal{E}_{\alpha, n, 4}^{(\varepsilon, r, \kappa, \rho)} & \cdots & \mathcal{E}_{\alpha, n, n}^{(\varepsilon, r, \kappa, \rho)} \end{pmatrix}, \quad (51)$$

where, the coefficients $\mathcal{E}_{(\alpha, m, n)}$ are defined by:

$$\mathcal{E}_{(\alpha, m, n)} = \frac{\mathcal{B}_{k, s}^{(\varepsilon, r, \kappa, \rho)} \Gamma(s+1)}{\Gamma(s-\alpha+1)} \cdot \left(\mathcal{B}^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x). \quad (52)$$

Proof. From Lemma 5.1 the analytic representation of $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ in the monomial form and applying the Caputo fractional derivative of order α gives:

$${}^C D_{0+}^\alpha \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \frac{\mathcal{B}_{k, s}^{(\varepsilon, r, \kappa, \rho)} \Gamma(s+1)}{\Gamma(s-\alpha+1)} x^{s-\alpha}.$$

Now using the projection:

$$x^{s-\alpha} = x^{-\alpha} \cdot \left(\mathcal{B}_{k, s}^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x),$$

we get:

$${}^C D_{0+}^\alpha \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \approx \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} \frac{\mathcal{B}_{k, s}^{(\varepsilon, r, \kappa, \rho)} \Gamma(s+1) \eta_j^{s, \alpha}}{\Gamma(s-\alpha+1)} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (53)$$

which leads to the compact form:

$${}^C D_{0+}^\alpha \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \approx \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2k} x^{-\alpha} \cdot \mathcal{E}_{(\alpha, m, n)} \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (54)$$

and consequently, the vector form:

$${}^C D_{0+}^\alpha \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) = \mathcal{O}^{(\alpha)} \cdot \Psi(x). \quad \blacksquare \quad (55)$$

Theorem 1 (see[Obaid2023]): Let $\mathcal{H}$ be a Hilbert space, and let $\bar{\mathcal{H}} \subset \mathcal{H}$ be a finite-dimensional subspace such that $\dim(\bar{\mathcal{H}}) = m$. Assume that $\{h_1, h_2, \dots, h_m\}$ forms a basis for $\bar{\mathcal{H}}$. Then, for any element $h \in \mathcal{H}$, let $\bar{h} \in \bar{\mathcal{H}}$ be its best approximation in the subspace $\bar{\mathcal{H}}$. The squared L^2 -norm of the approximation error is given by:

$$\|h - \bar{h}\|_{L^2}^2 = \frac{\text{Gram}(h, h_1, \dots, h_m)}{\text{Gram}(h_1, h_2, \dots, h_m)},$$

where, $\text{Gram}(h, h_1, \dots, h_m)$ denotes the Gram determinant, defined as:

$$\text{Gram}(h, h_1, \dots, h_m) = \begin{vmatrix} \langle h, h \rangle & \langle h, h_1 \rangle & \cdots & \langle h, h_m \rangle \\ \langle h_1, h \rangle & \langle h_1, h_1 \rangle & \cdots & \langle h_1, h_m \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle h_m, h \rangle & \langle h_m, h_1 \rangle & \cdots & \langle h_m, h_m \rangle \end{vmatrix}.$$

Lemma 5.3 Let $f(x) \in L^2[0, 1]$ be approximated using the basis $\{\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)\}_{j=0}^m$,

where $\mathcal{Y}(x) = [\mathcal{Y}_0^{(r, s, p, q)}(x), \dots, \mathcal{Y}_m^{(r, s, p, q)}(x)]^T$, such that:

$$f(x) \approx f_m(x) = \sum_{j=0}^m \mu_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) = \boldsymbol{\mu}^T \mathcal{Y}(x),$$

with coefficient vector $\boldsymbol{\mu} = [\mu_0, \mu_1, \dots, \mu_m]^T$. Then, the mean square error tends to zero as $m \rightarrow \infty$, i.e.,

$$\lim_{m \rightarrow \infty} \|f(x) - f_m(x)\|_{L^2} = 0.$$

Define the error vector due to the approximation of the Caputo fractional derivative of order $\alpha \in (0, 1)$ by:

$$\mathbf{E}^{(\alpha)} = {}^C D_{0+}^\alpha \mathcal{Y}(x) - \mathcal{O}^{(\alpha)} \mathcal{Y}(x),$$

where, $\mathbf{E}^{(\alpha)} = [\varepsilon_0^{(\alpha)}, \varepsilon_1^{(\alpha)}, \dots, \varepsilon_m^{(\alpha)}]^T$, and each error component satisfies the inequality:

$$\|\varepsilon_j^{(\alpha)}\|_{L^2} \leq \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}_{k, s}^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \left(\frac{\text{Gram}(x^{k-\alpha}, \mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})}{\text{Gram}(\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})} \right).$$

Proof. Consider the *span* of $\{\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)}(x)\} \subset L^2[0, 1]$. Suppose the function $x^{k-\alpha} \in L^2[0, 1]$ is approximated within this subspace by:

$$x^{k-\alpha} \approx \sum_{n=0}^m \lambda_n^{(\alpha)} \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)}(x).$$

Applying the theorem above, the approximation error satisfies:

$$\left\| x^{k-\alpha} - \sum_{n=0}^m \lambda_n^{(\alpha)} \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)}(x) \right\|_{L^2}^2 = \frac{\text{Gram}(x^{k-\alpha}, \mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})}{\text{Gram}(\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})}.$$

Now, to estimate the norm of the error in approximating the Caputo derivative of $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}$, we write:

$$\begin{aligned} \|\varepsilon_j^{(\alpha)}\|_{L^2} &= \left\| {}^C D_{0+}^\alpha \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)} - \sum_{n=0}^m \mathcal{E}_{(\alpha, n, m)} \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)} \right\|_{L^2}, \\ &= \left\| \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{n=0}^m \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{\eta_j^{s,\alpha} \Gamma(k+1)}{\Gamma(k-\alpha+1)} \mathcal{Y}_j(x) \right\|_{L^2}, \\ &\leq \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \left\| x^{k-\alpha} - \sum_{n=0}^m \eta_j^{s,\alpha} \mathcal{Y}_j(x) \right\|_{L^2}. \end{aligned}$$

Finally, substituting the bound from the Gram determinant result gives:

$$\|\varepsilon_j^{(\alpha)}\|_{L^2} \leq \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathcal{B}_{k,s}^{(\varepsilon, r, \kappa, \rho)} \cdot \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \left(\frac{\text{Gram}(x^{k-\alpha}, \mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})}{\text{Gram}(\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}, \dots, \mathcal{Y}_m^{(\varepsilon, r, \kappa, \rho)})} \right).$$

■

6 Linear and Nonlinear FDEs Based on Shifted Monic Symmetric Orthogonal Polynomials

This section presents the application of the operational matrix of the fractional derivative constructed using the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ for solving linear and nonlinear fractional differential equations.

6.1 Linear Fractional Differential Equations

A fractional differential equation is classified as linear if it does not involve products or nonlinear functions of the unknown function $u(t)$ and its fractional derivatives. A general linear fractional differential equation can be written as [az00, az0]:

$$\sum_{j=0}^{n_1} {}^C D_{0+}^{\alpha_j} \mathcal{U}(x) = f(x), \quad (56)$$

with the corresponding initial conditions:

$${}^C D_{0+}^{\alpha_j} \mathcal{U}(x) \Big|_{x=x_j} = \xi_j, \quad j = 0, 1, \dots, n_1 - 1. \quad (57)$$

We approximate the unknown function $\mathcal{U}(x)$ using the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ as:

$$\mathcal{U}(x) \approx \mathcal{A}^T \mathcal{G}_n(x) = \sum_{j=0}^n \mathcal{A}_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x), \quad (58)$$

where,

$$\mathcal{A} = [\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n]^T,$$

is the vector of unknown coefficients and,

$$\mathcal{G}_n(x) = [\mathcal{Y}_0^{(\varepsilon, r, \kappa, \rho)}(x), \mathcal{Y}_1^{(\varepsilon, r, \kappa, \rho)}(x), \dots, \mathcal{Y}_n^{(\varepsilon, r, \kappa, \rho)}(x)]^T,$$

is the basis vector. Using the operational matrix of Caputo fractional derivative of order α , we express the fractional derivative as:

$${}^C D_{0+}^\alpha \mathcal{U}(x) \approx \mathcal{A}^T D^{(\alpha)} \mathcal{G}_n(x). \quad (59)$$

Substituting the approximation into equation (56), the residual becomes:

$$R_n(x) = \sum_{j=0}^{n_1} \mathcal{A}^T \cdot \mathcal{O}_{(\alpha_j)}^{(\varepsilon, r, \kappa, \rho)} \cdot \mathcal{G}_n(x) - \mathcal{F}^T, \quad (60)$$

$$R_n(x) = \sum_{h=0}^{n_1} \mathcal{A}^T \cdot \mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \mathcal{A}_{(\alpha)}(x) \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \mathcal{U}(x) - \mathcal{F}^T. \quad (61)$$

such that,

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{pmatrix},$$

$$\mathcal{F}_j = \frac{1}{\mathcal{M}_j} \int_0^1 \mathcal{F}(x) \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) dx.$$

Where,

$$\mathcal{M}_j = \int_0^1 (\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x))^2 \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) dx.$$

Enforcing the orthogonality condition of the residual with respect to the basis functions and using tau method:

$$\langle R_n(x), \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \rangle = \int_0^1 R_n(x) \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) dx = 0, \quad j = 0, 1, \dots, n - n_1 - 1, \quad (62)$$

where, $\Theta^{(\varepsilon, r, \kappa, \rho)}(x)$ is the corresponding weight function. The system is closed with the initial conditions formulated as:

$${}^C D_{0+}^{\alpha_j} u(x_j) = \mathcal{A}^T D^{(\alpha_j)} \mathcal{G}_n(x_j) = \xi_j, \quad j = 0, 1, \dots, n_1 - 1. \quad (63)$$

The resulting system of equations can then be solved to find $\mathcal{A}$, and the approximate solution $\mathcal{U}(x)$ is reconstructed.

6.2 Nonlinear Fractional Differential Equations

Nonlinear fractional differential equations may include products or nonlinear functions of $\mathcal{U}(x)$ and its derivatives. A general form is given by [az1, az2]:

$$\sum_{k=0}^{n_1} \sum_{j=0}^{n_2} b_{k,j}(x) \cdot \mathcal{U}^k(x) \cdot {}^C D_{0+}^{\beta_j} \mathcal{U}(x) + \sum_{k=1}^{n_3} \sum_{j=0}^{n_4} c_{k,j}(x) \cdot {}^C D_{0+}^j \mathcal{U}(x) \cdot {}^C D_{0+}^{\gamma_j} \mathcal{U}(x) = f(x), \quad (64)$$

with initial conditions:

$$\mathcal{U}(x_j) = \xi_j, \quad j = 0, 1, \dots, n_l - 1, \quad (65)$$

where $n_l = \max(n_1, n_2, n_3, n_4)$. Using the same approximation in (58):

$$\mathcal{U}(x) \approx \mathcal{A}^T \mathcal{G}_n(x), \quad (66)$$

we can write all nonlinear terms using matrix operations. The nonlinear residual becomes:

$$\begin{aligned} R_n(x) &= \sum_{k=0}^{n_1} \sum_{j=0}^{n_2} B^{k,j}(x) \cdot (\mathcal{A}^T \mathcal{G}_n(x))^k \cdot \mathcal{A}^T \mathcal{O}_{(\beta_j)} \mathcal{G}_n(x) \\ &+ \sum_{k=1}^{n_3} \sum_{j=0}^{n_4} C^{k,j}(x) \cdot \mathcal{A}^T \mathcal{D}^{(j)} \mathcal{G}_n(x) \cdot \mathcal{A}^T \mathcal{O}_{(\gamma_j)} \mathcal{G}_n(x) - \mathcal{F}, \end{aligned} \quad (67)$$

$$\begin{aligned} R_n(x) &= \sum_{k=0}^{n_1} \sum_{j=0}^{n_2} B^{k,j}(x) \cdot (\mathcal{A}^T \mathcal{G}_n(x))^k \cdot \mathcal{A}^T \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \mathcal{A}_{(\alpha)}(x) \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \right) \mathcal{G}_n(x) \\ &+ \sum_{k=1}^{n_3} \sum_{j=0}^{n_4} C^{k,j}(x) \cdot \mathcal{A}^T \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \mathcal{D}^{(j)}(x) \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \right) \mathcal{G}_n(x) \\ &\cdot \mathcal{A}^T \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \mathcal{A}_{(\gamma_j)}(x) \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \right) \mathcal{G}_n(x) - \mathcal{F}. \end{aligned} \quad (68)$$

where, $B^{k,j}(x)$ and $C^{k,j}(x)$ are diagonal matrices with diagonal entries $b_{k,j}(x)$ and $c_{k,j}(x)$, respectively, and $\mathcal{F}$ is the projection of $\mathcal{F}(x)$ onto the polynomial basis:

$$\mathcal{F}_j = \frac{1}{\mathcal{M}_j} \int_0^1 \mathcal{F}(x) \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) dx, \quad (69)$$

where,

$$\mathcal{M}_j = \int_0^1 (\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x))^2 \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) dx.$$

To determine the coefficients $\mathcal{A}$, we enforce the tau orthogonality condition:

$$\langle R_n(x), \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) \rangle = \int_0^1 R_n(x) \cdot \Theta^{(\varepsilon, r, \kappa, \rho)}(x) \cdot \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x) dx = 0, \quad j = 0, 1, \dots, n - n_1 - 1, \quad (70)$$

combined with the initial conditions:

$$\mathcal{U}(x_j) = \mathcal{A}^T \mathcal{G}_n(x_j) = \xi_j, \quad j = 0, 1, \dots, n_l - 1. \quad (71)$$

The resulting nonlinear algebraic system can be solved to obtain the coefficient vector $\mathcal{A}$, which yields the approximate solution $\mathcal{U}(x)$.

The algorithm of the method

- Construct the shifted polynomials $\{\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)\}_{j=0}^N$ on the interval $[0, 1]$, at the specific parameters.
- Evaluate the orthogonality relation with the weight function.
- Approximate the solution, at specific n .

$$\mathcal{U}(x) \approx \mathcal{A}^T \mathcal{G}_n(x) = \sum_{j=0}^n \mathcal{A}_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x),$$

where $\mathcal{A}_j$ are the unknown coefficients.

- Construct the operational matrix of the fractional derivative, using the property:

$${}^C D_{0+}^{\alpha} \mathcal{U}(x) = \mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \mathcal{A}_{(\alpha)}(x) \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} \mathcal{U}(x),$$

derive the fractional operational matrix $\mathcal{A}_{(\alpha)}(x)$ corresponding to the Caputo derivative.

- Substitute the approximate solution into the FDE

$$\mathcal{D}^{\alpha} \mathcal{U}(x) + \mathcal{A}(x, \mathcal{U}(x)) = f(x).$$

- Apply the spectral tau method (70)-(71).
- Enforce the residual equation:

$$R(x) = \mathcal{D}^{\alpha} \mathcal{U}(x) + \mathcal{A}(x, \mathcal{U}(x)) - f(x) = 0.$$

- Form the algebraic system.

$$\mathbf{B} \mathbf{A} = \mathbf{F},$$

where $\mathbf{B}$ is the operational coefficient matrix and $\mathbf{F}$ is the evaluation of $f(x)$.

- Impose boundary or initial conditions.
- Solve the algebraic system.
- Reconstruct the approximate solution.

$$\mathcal{U}(x) = \mathcal{A}^T \mathcal{G}_n(x) = \sum_{j=0}^N \mathcal{A}_j \mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x).$$

7 Test Example

In this section, we present a numerical test to validate the effectiveness of the proposed spectral method based on the shifted monic symmetric orthogonal polynomials $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$. One of the main advantages of this polynomial family is its flexibility: by selecting appropriate values of the parameters ε, r, κ and ρ , it reduces to several classical orthogonal polynomials, each widely used in numerical analysis.

In the next subsection, we apply the same procedure using other polynomial families as special cases of $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ to demonstrate the generality and flexibility of the proposed approach.

7.1 Extensions to Other Special Cases of Shifted monic polynomial.

In this subsection, we illustrate the full implementation of the numerical method using the shifted Chebyshev polynomials.

Example 1. Consider the following linear fractional differential equation [AbdElhameed2023]:

$$x {}^C D_{0+}^{3/2} \mathcal{U}(x) + {}^C D_{0+}^{1/2} \mathcal{U}(x) + x^{3/2} \mathcal{U}(x) = f(x), \quad 0 < x \leq 1, \quad (72)$$

subject to the initial conditions:

$$\mathcal{U}(0) = 0, \quad \mathcal{U}'(0) = 1. \quad (73)$$

The exact solution of equation (72) with its initial condition (73) is given by:

$$\mathcal{U}(x) = x^3 - x^2.$$

Where,

$$f(x) = x^{1/2} \left(6x \left(\frac{x^2}{6} + \frac{\Gamma(\frac{7}{2}) + \Gamma(\frac{5}{2})}{\Gamma(\frac{7}{2}) \Gamma(\frac{5}{2})} \right) - 2 \left(\frac{x^2}{2} + \frac{\Gamma(\frac{5}{2}) + \Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2}) \Gamma(\frac{3}{2})} \right) \right).$$

Case 1. This case corresponds to setting the parameters of the general class as $\varepsilon = -1$, $r = 1$, $\kappa = -3$, $\rho = 2$. Denote the shifted Chebyshev polynomials of the fifth kind by $\mathcal{Y}_j^{(-1, 1, -3, 2)}(x)$. These polynomials form a complete orthogonal system on the interval $[0, 1]$ with respect to the weight function

$$\Theta(x) = \frac{(2x-1)^2}{\sqrt{x-x^2}}.$$

We now approximate the solution using a truncated expansion:

$$\mathcal{U}(x) \approx \sum_{j=0}^n \mathcal{A}_j \mathcal{Y}_j^{(-1, 1, -3, 2)}(x) = \mathcal{A}^T \mathcal{G}_n(x), \quad (74)$$

We use the operational matrix $\mathcal{O}^{(\alpha)}$ for the Caputo derivative of order α , and the transformation matrix $\mathcal{B}$ that relates the orthogonal basis to the monomial basis at $n = 5$. The residual of the differential equation is given by:

$$R_5(\tau) = \mathcal{A}^T \cdot \left[x \cdot \overline{\mathcal{B}}_{(5)}^{(-1,1,-3,2)} \mathcal{A}_{(\frac{3}{2})}(x) \left(\overline{\mathcal{B}}_{(5)}^{(-1,1,-3,2)} \right)^{-1} + \overline{\mathcal{B}}_{(5)}^{(-1,1,-3,2)} \mathcal{A}_{(\frac{1}{2})}(x) \left(\overline{\mathcal{B}}_{(5)}^{(-1,1,-3,2)} \right)^{-1} + x^{\frac{3}{2}} \right] \mathcal{G}_5^T(x) - \mathcal{F}. \quad (75)$$

Here we have,

$$\begin{aligned} \overline{\mathcal{B}}_{(5)}^{(-1,1,-3,2)} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -4 & 4 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{13}{3} & -12 & 8 & 0 & 0 \\ \frac{1}{16} & -3 & 19 & -32 & 16 & 0 \\ -\frac{3}{80} & \frac{99}{40} & -\frac{116}{5} & \frac{344}{5} & -80 & 32 \end{pmatrix}, \\ \mathcal{A}_{(\frac{3}{2})}(x) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{\sqrt{\pi} x^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{\sqrt{\pi} x^{3/2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{64}{5\sqrt{\pi} x^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{128}{7\sqrt{\pi} x^{3/2}} \end{pmatrix}, \\ \mathcal{A}_{(\frac{1}{2})}(x) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{\pi} \sqrt{x}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{3\sqrt{\pi} \sqrt{x}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{16}{5\sqrt{\pi} \sqrt{x}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{128}{35\sqrt{\pi} \sqrt{x}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{256}{63\sqrt{\pi} \sqrt{x}} \end{pmatrix}, \\ \mathcal{G}_5(x) &= \begin{pmatrix} 1 \\ -1+2x \\ -\frac{3}{4} \left(1 - \frac{4}{3} (1-4x+4x^2) \right) \\ -\frac{5}{6} \left(-1+2x - \frac{6}{5} (-1+6x-12x^2+8x^3) \right) \\ \frac{5}{16} \left(1-4(1-4x+4x^2) + \frac{16}{5} (1-8x+24x^2-32x^3+16x^4) \right) \\ \frac{7}{16} \left(-1+2x - \frac{16}{5} (-1+6x-12x^2+8x^3) + \frac{16}{7} (-1+10x-40x^2+80x^3-80x^4+32x^5) \right) \end{pmatrix}. \end{aligned}$$

To determine the coefficients $\mathcal{A}$, we apply the tau method by enforcing the orthogonality of the residual:

$$\langle R_5(x), \mathcal{Y}_j^{(-1,1,-3,2)}(x) \rangle = \frac{1}{\mathcal{B}_j} \int_0^1 R_n(x) \cdot \mathcal{Y}_j^{(-1,1,-3,2)}(x) \cdot \Theta(x) dx = 0, \quad j = 0, 1, \dots, (n-2),$$

where,

$$\mathcal{B}_j = \int_0^1 (\mathcal{Y}_j^{(-1,1,-3,2)}(x))^2 \cdot \Theta(x) dx.$$

The two remaining equations are obtained by applying the initial conditions.

$$\mathcal{U}(0) = \sum_{j=0}^n \mathcal{A}_j \cdot \mathcal{Y}_j^{(-1,1,-3,2)}(0) = 0, \quad (76)$$

$$\mathcal{U}'(0) = \sum_{j=0}^n \mathcal{A}_j \cdot \left(\frac{d}{dx} \mathcal{Y}_j^{(-1,1,-3,2)}(x) \right) \Big|_{x=0} = 0. \quad (77)$$

Solving the system composed of (75), (76), and (77) yields the unknown coefficients $\mathcal{A}$, and the approximate solution is then reconstructed using (74).

Table 1: Comparison between Exact and Approximate Solutions with Absolute Error For Example 1. case 1

x	Exact solution	Approximate Solution	Absolute Error
0.0	0.00000	0.00000	5.23886×10^{-16}
0.1	-0.00900	-0.00900	5.37764×10^{-16}
0.2	-0.03200	-0.03200	5.89806×10^{-16}
0.3	-0.06300	-0.06300	6.24500×10^{-16}
0.4	-0.09600	-0.09600	5.55112×10^{-16}
0.5	-0.12500	-0.12500	4.44089×10^{-16}
0.6	-0.14400	-0.14400	3.05311×10^{-16}
0.7	-0.14700	-0.14700	1.38778×10^{-16}
0.8	-0.12800	-0.12800	0.00000
0.9	-0.08100	-0.08100	0.00000
1.0	0.00000	0.00000	1.04083×10^{-17}

Table (1) shows that the approximate solution matches the exact solution with an extremely small absolute error, confirming the accuracy and reliability of the proposed numerical method.

Figure (1) clearly illustrates the efficiency of the proposed numerical method. The approximate solution overlaps almost entirely with the exact solution, and the error curve remains extremely small—approaching zero. This behavior confirms the high precision and robust convergence of the spectral scheme based on the shifted monic orthogonal polynomials, especially when using fifth-kind Chebyshev polynomials.

Case 2: Sixth-Kind Chebyshev Polynomials.

In this case, we adopt a special instance of the generalized monic symmetric orthogonal polynomials $\mathcal{Y}_j^{(\varepsilon, r, \kappa, \rho)}(x)$ by choosing the parameters

$$\varepsilon = -1, \quad r = 1, \quad \kappa = -5, \quad \rho = 2$$

which define the sixth-kind Chebyshev polynomials and the weight function is:

$$\Theta(x) = 2(2x - 1)^2 \cdot \sqrt{x(1+x)}.$$

We approximate the unknown solution $\mathcal{U}(x)$ by:

$$\mathcal{U}(x) = \sum_{j=0}^5 \mathcal{A}_j \mathcal{Y}_j^{(-1,1,-5,2)}(x) = \mathcal{A}^T \cdot \mathcal{G}_5(x),$$

where, $\mathcal{A}$ is the coefficient vector and $\mathcal{U}(x)$ is the vector of basis functions. Substituting the approximate form into the equation yields the residual:

$$R_5(x) = \mathcal{M}^T \left[x \cdot \mathcal{B}_5^{(-1,1,-5,2)} \cdot \mathcal{A}_{(3/2)}(x) \cdot \left(\mathcal{B}_5^{(-1,1,-5,2)} \right)^{-1} \right. \\ \left. + \mathcal{B}_5^{(-1,1,-5,2)} \cdot \mathcal{A}_{(1/2)}(x) \cdot \left(\mathcal{B}_5^{(-1,1,-5,2)} \right)^{-1} + x^{\frac{3}{2}} \right] \mathcal{G}_5^T(x) - \mathcal{F}.$$

We enforce tau method $x \in [0, 1]$.

Table 2: Exact and approximate solutions using sixth-kind Chebyshev polynomials For Example 1. case 2

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00000	0.00000	1.38778×10^{-17}
0.1	-0.00900	-0.00900	3.12250×10^{-17}
0.2	-0.03200	-0.03200	6.93889×10^{-17}
0.3	-0.06300	-0.06300	8.32667×10^{-17}
0.4	-0.09600	-0.09600	9.71445×10^{-17}
0.5	-0.12500	-0.12500	6.93889×10^{-17}
0.6	-0.14400	-0.14400	2.77556×10^{-17}
0.7	-0.14700	-0.14700	2.77556×10^{-17}
0.8	-0.12800	-0.12800	5.55112×10^{-17}
0.9	-0.08100	-0.08100	2.77556×10^{-17}
1.0	0.00000	0.00000	0.00000

The absolute error is defined as:

$$\epsilon(x) = |\mathcal{U}_{\text{exact}}(x) - \mathcal{U}_{\text{approx}}(x)|.$$

As shown in Table (2), the sixth-kind Chebyshev polynomial method yields approximate solutions that closely match the exact values. The absolute errors are extremely small, verifying the high precision and numerical stability of the approach. Figure (2) illustrates that the sixth-kind Chebyshev polynomial provides results that are nearly identical to the exact solution. The absolute error remains very small throughout the domain, highlighting the method's high precision and effectiveness.

Case 3: Application Using Shifted Legendre Polynomials.

In this case, we explore the numerical solution of the multi-term fractional differential equation using Legendre polynomials as basis functions. These polynomials are obtained by shifting the classical Legendre polynomials to the interval $[0, 1]$, and they form a special case of the generalized symmetric orthogonal polynomials when the parameters are taken as:

$$\varepsilon = -1, r = 1, \kappa = -3, \rho = 0$$

The approximate solution is expressed in terms of shifted Legendre polynomials as follows:

$$\mathcal{U}(x) \approx \sum_{j=0}^5 \mathcal{A}_j \mathcal{P}_j^{(-1,1,-3,0)}(x) = \mathcal{A}^T \cdot \mathcal{G}_5(x).$$

The Caputo fractional derivative of the approximate solution can be represented as:

$${}^C D_{0+}^\alpha \mathcal{U}(x) \approx \mathcal{A}^T \cdot \mathcal{B}_5^{(-1,1,-3,0)} \cdot \mathcal{A}_{(\alpha)}(x) \cdot \left(\mathcal{B}_5^{(-1,1,-3,0)} \right)^{-1} \cdot \mathcal{G}_5(x),$$

where $\mathcal{A}_{(\alpha)}(x)$ is a diagonal matrix containing the Caputo derivatives of the monomials up to degree n .

The residual function is represented as:

$$\begin{aligned} R_5(x) = & \mathcal{A}^T \left[x \cdot \mathcal{B}_5^{(-1,1,-3,0)} \cdot \mathcal{A}_{(3/2)}(x) \cdot \left(\mathcal{B}_5^{(-1,1,-3,0)} \right)^{-1} \right. \\ & \left. + \mathcal{B}_5^{(-1,1,-3,0)} \cdot \mathcal{A}_{(1/2)}(x) \cdot \left(\mathcal{B}_5^{(-1,1,-3,0)} \right)^{-1} + x^{\frac{3}{2}} \right] \mathcal{G}_5^T(x) - \mathcal{F}. \end{aligned}$$

Table 3: Approximate Results and Absolute Errors Using Shifted Legendre Polynomials For Example 1. case 3.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.000	0.000	0.000
0.1	-0.009	-0.009	5.20417×10^{-18}
0.2	-0.032	-0.032	3.46945×10^{-17}
0.3	-0.063	-0.063	5.55112×10^{-17}
0.4	-0.096	-0.096	5.55112×10^{-17}
0.5	-0.125	-0.125	8.32667×10^{-17}
0.6	-0.144	-0.144	8.32667×10^{-17}
0.7	-0.147	-0.147	2.77556×10^{-17}
0.8	-0.128	-0.128	2.77556×10^{-17}
0.9	-0.081	-0.081	1.38778×10^{-17}
1.0	0.000	-6.93889×10^{-18}	6.93889×10^{-18}

Table (3) compares the exact and approximate solutions using shifted Legendre polynomials. The very small absolute errors confirm the high accuracy of the numerical method.

Figure (3) explain Absolute error between the exact and approximate solution using Legendre polynomials of degree $n = 5$ over the interval $x \in [0, 1]$. The error remains very small throughout the interval, with a maximum around 10^{-17} , confirming the high accuracy and numerical stability of the Legendre-based approximation method.

Case 4: Using Shifted Polynomials of own choices.

To further demonstrate the applicability and efficiency of the proposed spectral technique, we consider the case of employing the shifted polynomials of the any choices of the parameters $\varepsilon, r, \kappa, \rho$. These polynomials are derived from the generalized monic orthogonal polynomial family with the parameters (own choice):

$$\varepsilon = -1, r = 1, \kappa = -5, \rho = 4.$$

Let $\mathcal{Y}_j^{(-1,1,-5,4)}(x)$ be the corresponding basis functions. The approximate solution is expressed as

$$\mathcal{U}(x) = \sum_{j=0}^m \mathcal{A}_j \mathcal{Y}_j^{(-1,1,-5,4)}(x).$$

The transformation matrix $\mathcal{P}_5^{(-1,1,-5,4)}$, along with the operational matrices of the Caputo fractional derivatives of orders $\frac{1}{2}$ and $\frac{3}{2}$, is constructed accordingly. By applying the spectral tau method, we reduce the original fractional differential equation to a system of algebraic equations that can be solved to obtain the coefficients $\mathcal{A}_j$.

To evaluate the accuracy of the method, we compare the numerical solution $\mathcal{U}(x)$ with the exact solution $\mathcal{U}(x) = x^2$.

Table 4: Approximate Results and Absolute Errors Using Shifted Polynomials of the own proposed values For Example 1. case 4.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.000	0.000	6.19296×10^{-16}
0.1	-0.099	-0.099	7.07767×10^{-16}
0.2	-0.192	-0.192	6.93889×10^{-16}
0.3	-0.273	-0.273	7.77156×10^{-16}
0.4	-0.336	-0.336	8.32667×10^{-16}
0.5	-0.375	-0.375	8.32667×10^{-16}
0.6	-0.384	-0.384	8.32667×10^{-16}
0.7	-0.357	-0.357	8.88178×10^{-16}
0.8	-0.288	-0.288	9.43690×10^{-16}
0.9	-0.171	-0.171	9.15934×10^{-16}
1.0	0.000	0.000	8.51749×10^{-16}

The results of table (4) confirm that the proposed method yields highly accurate solutions, with absolute errors approaching machine precision, demonstrating the robustness of the spectral scheme using this specific polynomial basis.

We set $n = 5$ and apply the tau method by enforcing the residual orthogonality conditions and satisfying the initial conditions. The resulting system of equations is solved for the unknown coefficients $\mathcal{A}$. To assess the accuracy, we compute the absolute maximum error:

$$\text{Error}_\infty = \max_{x \in [0,1]} |\mathcal{U}_{\text{exact}}(x) - \mathcal{U}_{\text{approx}}(x)|.$$

Table 5: Maximum absolute error and weight functions for different polynomial types with $n = 5$ for Example 1.

Polynomial Type	Parameters $(\varepsilon, r, \kappa, \rho)$	Weight Function $\Theta(x)$	$\ \text{Error}(x)\ _\infty$
Legendre	$(-1, 1, -3, 0)$	1	8.33×10^{-17}
Fifth Kind	$(-1, 1, -3, 2)$	$\frac{(2x-1)^2}{\sqrt{x(1-x)}}$	5.24×10^{-16}
Sixth Kind	$(-1, 1, -5, 2)$	$\frac{(2x-1)^2}{2\sqrt{x(1+x)}}$	9.71×10^{-17}
Own proposed values	$(-1, 1, -5, 4)$	$\frac{(2x-1)^4}{2\sqrt{x(1+x)}}$	9.44×10^{-16}

Table (5) presents a comparison of the maximum absolute errors and the corresponding weight functions used in the approximation for different types of orthogonal polynomials at degree $n = 5$. The Legendre polynomials exhibit the smallest error, confirming their classical efficiency. The sixth-kind Chebyshev polynomials also perform very well, slightly outperforming the fifth-kind Chebyshev in terms of accuracy. Each method uses a distinct weight function tailored to the orthogonality and structure of the respective basis.

Figure (4) presents the distribution of the absolute error across the interval $[0, 1]$ for the approximate solution obtained using the shifted Chebyshev polynomials of the proposed values. The observed error remains on the order of 10^{-16} throughout, indicating that the numerical scheme achieves near machine precision. This result confirms the reliability and high accuracy of the spectral method when applied with this polynomial basis.

Figure (5) shows the absolute error between the exact and approximate solutions using: fifth-kind Chebyshev polynomial, sixth-kind Chebyshev polynomial, Seventh-kind Chebyshev polynomial and Legendre polynomials. The results confirm that all methods are highly accurate, with the Legendre method yielding the smallest errors, followed closely by the sixth-kind Chebyshev. The fifth and the other values shows slightly higher error near the endpoints.

In all cases, Numerical results demonstrate that the proposed method yields high accuracy and spectral convergence, especially when an appropriate basis is chosen in accordance with the nature of the problem.

Table 6: Absolute Error Comparison for Different Polynomial Bases Example 1.

x	Fifth-Kind Chebyshev	Sixth-Kind Chebyshev	Own Proposed Values	Shifted Legendre
0.0	5.24×10^{-16}	0.00	6.19×10^{-16}	1.39×10^{-17}
0.1	5.38×10^{-16}	5.20×10^{-18}	7.08×10^{-16}	3.12×10^{-17}
0.2	5.90×10^{-16}	3.47×10^{-17}	6.94×10^{-16}	6.94×10^{-17}
0.3	6.25×10^{-16}	5.55×10^{-17}	7.77×10^{-16}	8.33×10^{-17}
0.4	5.55×10^{-16}	5.55×10^{-17}	8.33×10^{-16}	9.71×10^{-17}
0.5	4.44×10^{-16}	8.33×10^{-17}	8.33×10^{-16}	6.94×10^{-17}
0.6	3.05×10^{-16}	8.33×10^{-17}	8.33×10^{-16}	2.78×10^{-17}
0.7	1.39×10^{-16}	2.78×10^{-17}	8.88×10^{-16}	2.78×10^{-17}
0.8	0.00	2.78×10^{-17}	9.44×10^{-16}	5.55×10^{-17}
0.9	0.00	1.39×10^{-17}	9.16×10^{-16}	2.78×10^{-17}
1.0	1.04×10^{-17}	6.94×10^{-18}	8.52×10^{-16}	0.00

Table (6) presents the absolute error comparison among different polynomial bases. The results indicate that all approaches yield very small errors, confirming their high computational accuracy. The fifth- and sixth-kind Chebyshev polynomials demonstrate excellent precision and numerical stability, while the proposed basis produces comparable or slightly better accuracy across several points. Although the shifted Legendre polynomials also provide satisfactory results, the Chebyshev-type and proposed bases exhibit superior performance near the boundaries, highlighting their effectiveness for solving fractional differential equations.

Table 7: Computational time (in seconds) for different polynomial bases for Example 1.

x	Fifth Kind Chebyshev	Sixth Kind Chebyshev	Own proposed values	Legendre Polynomial
Time (s)	94.328	85.313	147.905	59.562

Table (7) presents the computational time required by different polynomial bases for Example 1. It is evident that the Legendre polynomial exhibits the shortest computation time (59.562 s), while the proposed basis requires slightly more time (147.905 s) due to its higher flexibility and accuracy. The fifth- and sixth-kind Chebyshev polynomials show intermediate performance, balancing computational efficiency and precision. Overall, these results highlight the efficiency of the proposed scheme in achieving improved accuracy with a reasonable computational cost.

Example 2. Consider the following linear fractional-order initial value problem [Ahmed2023]:

$$D^2 \mathcal{U}(x) + {}^C D_{0+}^{3/2} \mathcal{U}(x) + \mathcal{U}(x) = x^3 + \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} + 5x, \quad x \in (0, 1), \quad (78)$$

subject to the initial conditions:

$$\mathcal{U}(0) = 0, \quad \mathcal{U}'(0) = 0. \quad (79)$$

To approximate the solution of the fractional differential problem, we adopt a spectral approach based on a class of shifted symmetric orthogonal polynomials defined over $[0, 1]$. These polynomials are characterized by four parameters: ε , r , κ , and ρ , which determine their structural behavior and corresponding weight function.

Let $\{\mathcal{P}_j^{(\varepsilon, r, \kappa, \rho)}(x)\}_{j=0}^n$ denote the basis set of such polynomials, ensuring orthogonality of the basis with respect to the inner product on the interval $[0, 1]$.

To proceed, we transform the differential operators acting on $\mathcal{U}(x)$ into algebraic forms using the operational matrix of the Caputo derivative, denoted by $\mathcal{A}_{(\alpha)}(x)$. These operators are implemented through similarity transformations involving the matrix $\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)}$, which relates the orthogonal polynomial basis to the standard monomial basis.

The residual of the governing equation is expressed as:

$$R_n(x) = \mathcal{A}^T \left[\cdot \mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \cdot (\mathbf{P}^T)^2 \cdot \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} + \mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \cdot \mathcal{A}_{(3/2)}(x) \cdot \left(\mathcal{B}_n^{(\varepsilon, r, \kappa, \rho)} \right)^{-1} + 1 \right] \mathcal{G}_n^T(x) - \mathcal{F},$$

where, P is the monomial basis differentiation matrix.

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}.$$

Table 8: Absolute Error Comparison for Different Polynomial Bases For Example 2.

x	Fifth Kind Chebyshev	Sixth Kind Chebyshev	Own proposed values	Legendre Polynomial
0.0	1.66533×10^{-16}	1.11022×10^{-16}	6.19296×10^{-16}	1.11022×10^{-16}
0.1	2.08167×10^{-16}	1.38778×10^{-16}	7.07767×10^{-16}	1.11022×10^{-16}
0.2	1.94289×10^{-16}	8.32667×10^{-17}	6.93889×10^{-16}	1.11022×10^{-16}
0.3	1.66533×10^{-16}	5.55112×10^{-17}	7.77156×10^{-16}	1.11022×10^{-16}
0.4	2.22045×10^{-16}	5.55112×10^{-17}	8.32667×10^{-16}	1.11022×10^{-16}
0.5	1.66533×10^{-16}	0	8.32667×10^{-16}	5.55112×10^{-17}
0.6	1.66533×10^{-16}	0	8.32667×10^{-16}	5.55112×10^{-17}
0.7	1.66533×10^{-16}	0	8.88178×10^{-16}	5.55112×10^{-17}
0.8	2.22045×10^{-16}	0	9.43690×10^{-16}	1.11022×10^{-16}
0.9	1.94289×10^{-16}	2.77556×10^{-17}	9.15934×10^{-16}	5.55112×10^{-17}
1.0	1.38778×10^{-16}	5.55112×10^{-17}	8.51749×10^{-16}	0

Table (8) displays the absolute errors in approximating the exact solution $\mathcal{U}(x) = x^3 - x^2$ using four different polynomial bases For example 2. The Legendre and sixth-kind Chebyshev methods yield the smallest errors, typically on the order of 10^{-17} . In comparison, the fifth- and seventh-kind Chebyshev polynomials show slightly higher errors, with the reaching a maximum of approximately 9.44×10^{-16} .

Each method in figure (6) approximates the exact solution $\mathcal{U}(x) = x^3 - x^2$, and the resulting errors are plotted over the interval $x \in [0, 1]$. Highlights the relative accuracy and performance of each polynomial type in capturing the solution behaviour.

The figure (7) demonstrates that while all approaches achieve high accuracy, there are distinctions in performance. The Legendre and sixth-kind Chebyshev polynomials show superior precision with maximum absolute errors on the order of 10^{-17} , indicating strong agreement with the exact solution. In contrast, the fifth- and own values exhibit slightly higher error magnitudes, typically around 10^{-16} , but still within an acceptable range. This comparison emphasizes the effectiveness of the generalized orthogonal polynomial framework and highlights how selecting an appropriate polynomial basis can influence numerical accuracy when solving fractional differential equations.

Example 3. By considering the following nonlinear fractional differential equation, as presented in [Kosunalp2024, Sadri2024, Mirshojaei2025]:

$$x^{\frac{7}{2}} {}_C D_{0+}^{3/2} \mathcal{U}(x) + D^2 \mathcal{U}(x) + G^2(x) = 2 + \frac{4x^4}{\sqrt{\pi}} + x^4, \quad x \in [0, 1],$$

subject to the initial conditions $\mathcal{U}(0) = 0, \mathcal{U}'(0) = 0$.

As presented before by approximate the solution using the proposed method, the following results obtain.

Table 9: Absolute Error Comparison for Different Polynomial Bases For Example 3.

x	Fifth Kind Chebyshev	Sixth Kind Chebyshev	own proposed values	Legendre Polynomial
0.0	1.75×10^{-15}	1.54×10^{-15}	1.95×10^{-15}	1.11×10^{-16}
0.1	1.75×10^{-15}	1.53×10^{-15}	1.91×10^{-15}	1.11×10^{-16}
0.2	1.65×10^{-15}	1.45×10^{-15}	1.87×10^{-15}	1.11×10^{-16}
0.3	1.54×10^{-15}	1.37×10^{-15}	1.69×10^{-15}	1.11×10^{-16}
0.4	1.36×10^{-15}	1.25×10^{-15}	1.50×10^{-15}	1.11×10^{-16}
0.5	1.17×10^{-15}	1.08×10^{-15}	1.30×10^{-15}	5.55×10^{-17}
0.6	9.99×10^{-16}	8.88×10^{-16}	1.11×10^{-15}	5.55×10^{-17}
0.7	7.22×10^{-16}	7.77×10^{-16}	8.33×10^{-16}	5.55×10^{-17}
0.8	5.55×10^{-16}	4.44×10^{-16}	4.44×10^{-16}	1.11×10^{-16}
0.9	2.22×10^{-16}	3.33×10^{-16}	2.22×10^{-16}	5.55×10^{-17}
1.0	0.00	1.11×10^{-16}	0.00	0.00

Table (9) shows the absolute errors for approximating $\mathcal{U}(x) = x^2$ in Example 3 using four orthogonal polynomial methods. All approaches show high accuracy, with errors mostly within 10^{-15} , 10^{-16} . Among them, the Legendre polynomial method achieves the lowest errors indicating superior stability and convergence.

Figure (8) compares the absolute errors of four polynomial methods in Example 3. All methods yield accurate results, with the Legendre and sixth-kind Chebyshev polynomials showing the highest precision. The fifth- and seventh-kind methods exhibit slightly larger errors, particularly near the left endpoint.

Conclusion

This study introduced a unified spectral method based on shifted monic orthogonal polynomials including Legendre and Chebyshev polynomials of various kinds for solving fractional differential equations. the operational matrices of ordinary and fractional derivatives are presented for the symmetric polynomials which represented as a unified operational matrices are given for various types of known polynomials. the tau spectral approach is used with symmetric polynomials as a computational matrix technique to treat both linear and non-linear FDEs. Numerical results demonstrated that all methods are accurate, with the Legendre and several-kind of Chebyshev polynomials offering the best performance in terms of stability and precision. The proposed framework is flexible and effective, and it can be extended to more complex problems in future research.

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflicts of Interest

The authors declare that no conflict of interest.

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Author's contribution

Yasser Amer: Supervision, & Conceptualization. Kamal Raslan: Data Curation, Formal Analysis, Writing & Review Mohamed A. Abd El Salam: Methodology, Validation, Visualization, Writing – Review & Editing, & Writing – Original Draft. Aziza Mahdy: Methodology, Validation, Editing, & Writing – Original Draft. All authors reviewed the results and approved the final version of the manuscript.

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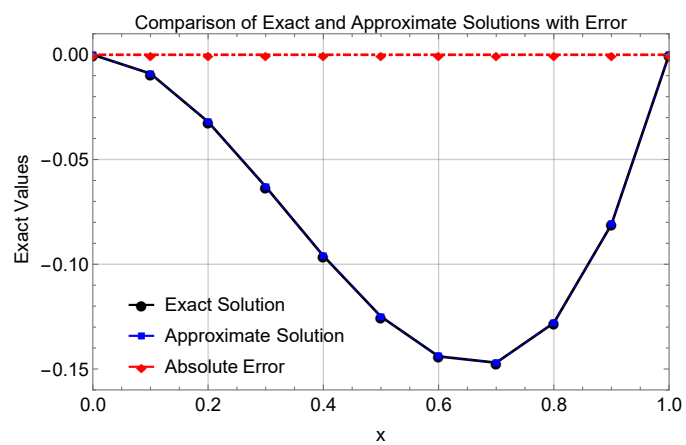


Figure 1: Comparison of Exact and Approximate Solutions with Error For Example 1. case 1.

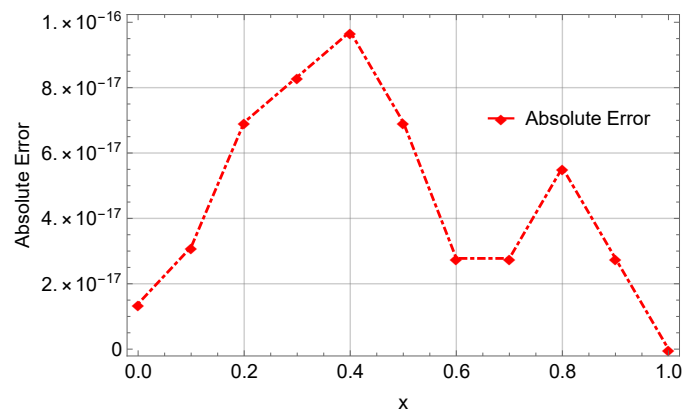


Figure 2: x values and absolute error For Example 1. case 2

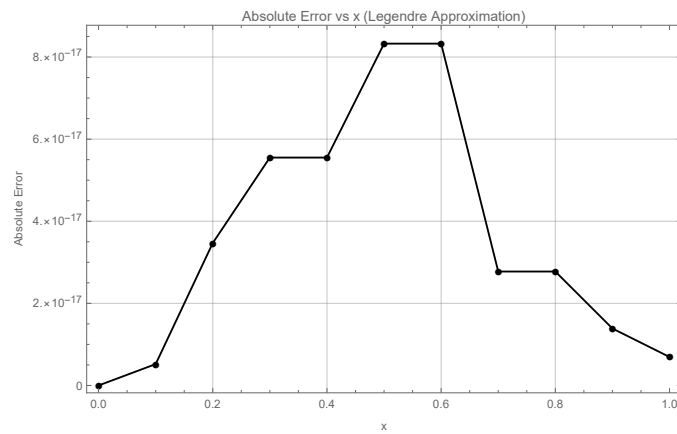


Figure 3: Comparison of x and absolute error with legender polynomial For Example 1. case 3.

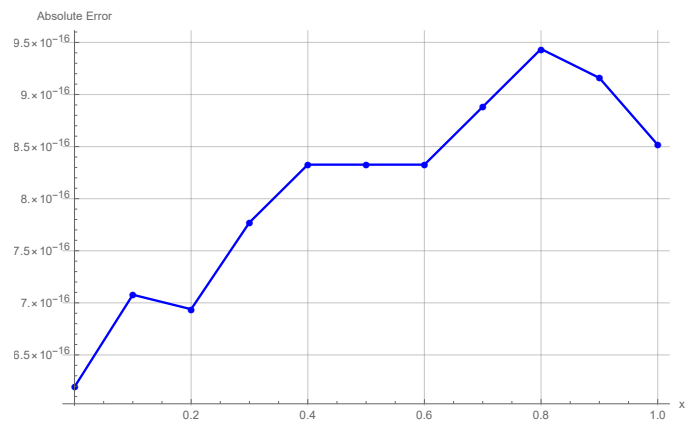


Figure 4: Absolute Error and x for the approximate solution using shifted polynomials of the proposed values For Example 1. case 4.

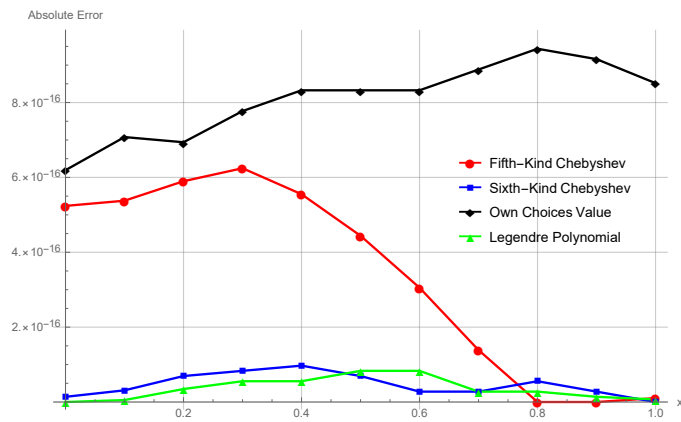


Figure 5: Absolute Error Comparison Of Four Polynomial Method For Example 1.

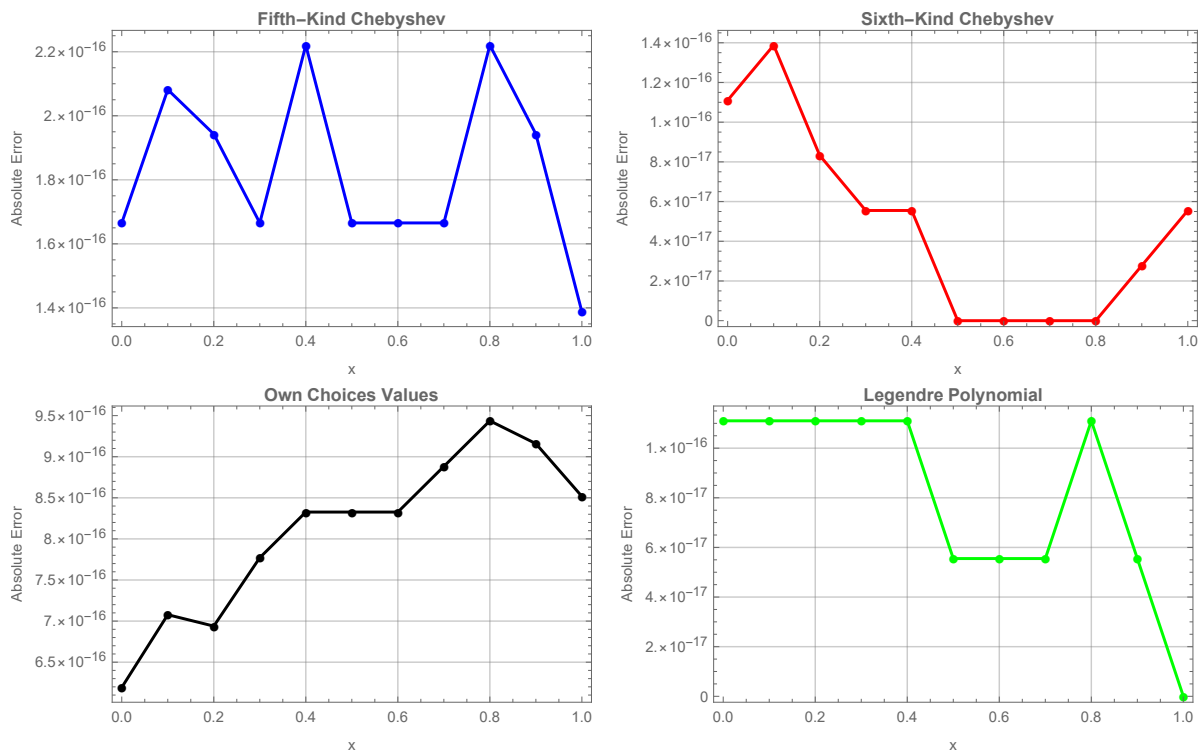


Figure 6: Absolute errors for Example 2 using different polynomial basis functions.

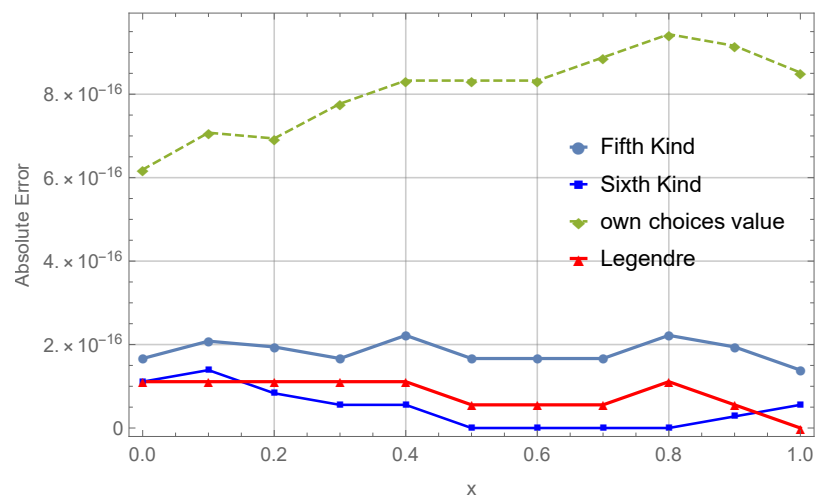


Figure 7: Absolute error comparison of four polynomials For Example 2.

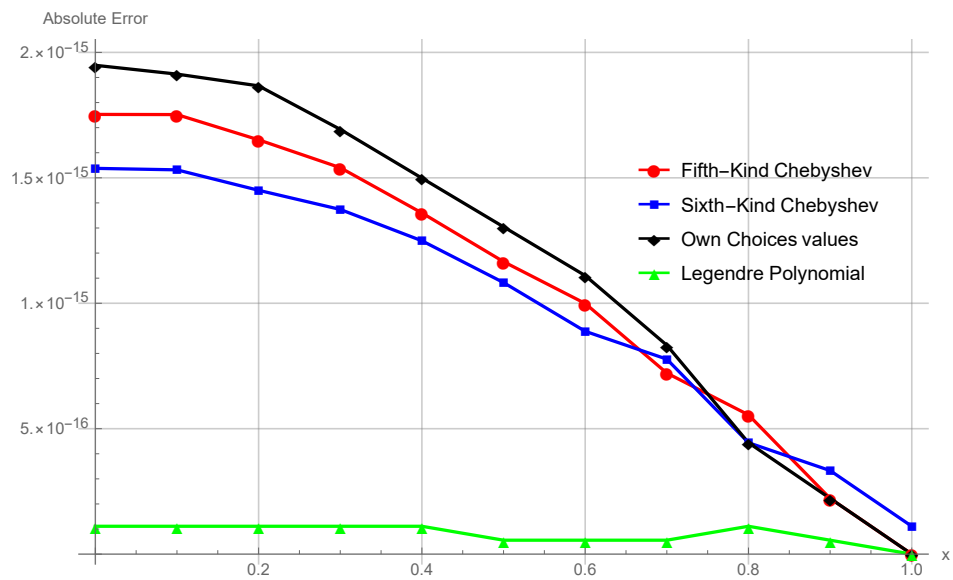


Figure 8: Comparison of approximate solutions and absolute errors using different bases for Example 3.