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Mittag-Leffler-Gegenbauer Polynomials of Two variable: Symbolic Operator Approach

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Maged G. Bin-Saad [1], Waleed K. Mohammed [2]

Abstract: In this paper, we employ the symbolic operator approach, a versatile tool for studying and generalizing special functions, to introduce a novel class of polynomials, the Two-Variable Mittag-Leffler-Gegenbauer polynomials. This family generalizes several classical polynomials, including Laguerre, Hermite, and Gegenbauer polynomials, providing a unifying framework for their analysis.We investigate the main properties of our polynomials, including series representations, generating functions, operational rules, and relations via fractional integrals and derivatives. The practical relevance is illustrated through numerical examples and graphical demonstrations. Additionally, we explore an application to fractional kinetic equations, highlighting how these polynomials naturally model memory-dependent processes and reveal new features in fractional dynamics. Overall, this work demonstrates that the combination of symbolic operators and a two-variable structure provides a powerful framework for generating, analysing, and applying new classes of special polynomials in both theoretical and applied settings.

Keywords: Symbolic operators, Mittag-Leffer function, Gegenbauer polynomials, Hermite polynomials, Legendre polynomials, Chebyshev polynomials, fractional calculus, fractional kinetic equation.

1 Introduction

Analogously to the e efficient usage of Legendre polynomials in the theory of the well-known 3-dimensional spherical harmonics, Gegenbauer polynomials play a crucial role in the theory of hyperspherical harmonics (see [1]). Notably, Legendre polynomials can be regarded as a specific case of Gegenbauer polynomials. A crucial role in the theory of symbolic method is played by the Mittag-Leffler function [2]. This function has gained popularity over the past 20 years because of its enormous potential for solving issues in the earth sciences, engineering, biomathematics, and physical sciences [3]. The purpose of the Mittag-Leffler function is to characterize the process of the analytic continuation of power series beyond the disc of their convergence, which is one of the unresolved problems in complex analysis. In this study, we emphasize the unique function of the Mittag-Leffler function with Gegenbauer polynomials and its multifaceted behavior. In addition, we conduct convolutions of the Mittag-Leffler function with Gegenbuer polynomials using suitable symbolic operators to investigate its related properties. We remember that the series definition of the Mittag-Leffler function is as follows [4, 5]:

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)},\ (Re(\alpha)>0; x\in\mathbb{C}), \tag{1.1}$$

[1] Department of Mathematics
College of Education,
University of Aden, Aden, Yemen
E-mail: mgbinsaad@yahoo.com
[2] Department of Mathematics
College of Education,
University of Aden, Aden, Yemen
E-mail: w.khadher1981@yahoo.com

where as usual notation $\Gamma(\lambda)$ for the Gamma function and $(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, n\geq 0, \lambda\neq 0,-1,-2,...$, the Pochhammer symbol [6].

Here and in the following, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ denote the sets of complex numbers, real numbers, and positive integers, respectively, and $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{R}_0^+=\mathbb{R}^+\cup\{0\}$. A generalization of (1.1) in the form [5, 7, 8]:

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)},\ (\{Re(\alpha), Re(\beta)\}>0; x\in\mathbb{C}). \tag{1.2}$$

A generalization of (1.2) is introduced in terms of the following series representation by Praphakar [9]:

$$E^{\delta}_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\alpha n+\beta)}\frac{x^n}{n!}, \tag{1.3}$$
$$(\alpha,\beta,\delta\in\mathbb{C}; \{Re(\alpha), Re(\beta), Re(\delta)\}>0).$$

In [10] the authors introduced and investigated a further generalization of the Mittag-Leffler function (1.3) defined by

$$E^{(\delta)}_{\alpha,\beta,\gamma}(x,y)=\sum_{m,n=0}^{\infty}\frac{(\delta)_{m+n}\,x^m\,y^n}{m!\,n!\,\Gamma(\alpha m+\beta n+\gamma)}. \tag{1.4}$$

Certain novel and well-known special functions can be introduced and studied effectively using the symbolic technique. The symbolic technique was developed in [11] as a result of the effort. For example, Babusci et al. [12] obtained several lacunary generating functions for the Laguerre polynomials using the symbolic technique. Dattoli et al. [12, 13] introduced a symbolic operator $\hat{c}$, which acts on the vacuum function ϕ_0 as follows:

$$\hat{c}^{\alpha}\phi_0 = \frac{1}{\Gamma(1+\alpha)}. \tag{1.5}$$

A new symbolic approach to studying special functions through the derivation of specific operators, known as symbolic operators, was introduced by Babusci et al. [14]. In their work, Dattoli et al. [4] introduced a symbolic operator denoted by $\hat{d}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathbb{R}^+$. The following equations describe the action of this operator on the vacuum function φ_0:

$$\hat{d}^{k}_{(\alpha,\beta)}\varphi_0 = \frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)} \quad \left(\alpha, \beta \in \mathbb{R}^+, k \in \mathbb{R}\right), \tag{1.6}$$

and

$$\hat{d}^{k+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\varphi_0 = \frac{\Gamma(k+\delta)}{\Gamma(\alpha k+\beta)} \quad \left(\alpha, \beta, \delta \in \mathbb{R}^+, |\delta| \leq 1, k \in \mathbb{R}\right). \tag{1.7}$$

Notably, when $k = 0$, Eq. (1.6) yields

$$\varphi_0 = \frac{1}{\Gamma(\beta)}. \tag{1.8}$$

Keep in mind that Eq. (1.7) simplifies to Eq. (1.6) if we let $\delta = 1$.

The Mittag-Leffler functions (1.3) and (1.4) may be represented symbolically using equations (1.6) and (1.7) as follows:

Lemma 1.1. *The following symbolic representations hold:*

$$E^{\delta}_{\alpha,\beta}(x) = e^{x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{1.9}$$

$$E^{\delta}_{\alpha,\beta}(x) = {}_1F_1\left[\delta;1;x\hat{d}_{(\alpha,\beta)}\right]\varphi_0, \tag{1.10}$$

$$E^{(\delta)}_{\alpha,\alpha,\beta}(x,y) = e^{(x+y)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{1.11}$$

$$E^{(\delta)}_{\alpha,\alpha,\beta}(x,y) = \Phi_1\left[\delta,1;1;\hat{D}^{-1}_x\hat{d}_{(\alpha,\beta)}, y\hat{d}_{(\alpha,\beta)}\right]\varphi_0, \tag{1.12}$$

where ${}_1F_1$ denotes the confluent hypergeometric function, and $\Phi_1[\alpha,\beta;\gamma;x,y]$ denotes the confluent hypergeometric function of two variables, defined as (see, e.g., [6]):

$${}_1F_1[\alpha,\beta;x] = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!},$$

$$\Phi_1[\alpha,\beta;\gamma;x,y] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m x^m y^n}{(\gamma)_{m+n}m!n!}.$$

Proof. We have

$$e^{x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!\,\Gamma(\delta)}\hat{d}^{\delta+n-1}_{(\alpha,\alpha(1-\delta)+\beta)}\varphi_0 = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n}{n!\,\Gamma(\alpha n+\beta)},$$

which gives the desired result (1.9).

We have

$${}_1F_1\left[\delta;1;x\hat{d}_{(\alpha,\beta)}\right]\varphi_0 = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n\,\hat{d}^{n}_{(\alpha,\beta)}\varphi_0}{(n!)^2} = \sum_{n=0}^{\infty}\frac{(\delta)_n\,x^n}{n!\,\Gamma(\alpha n+\beta)},$$

which gives the desired result (1.10). The proofs of the assertions (1.11) and (1.12) run parallel to the proofs of (1.9) and (1.10), and we skip the details. □

The generating function and series representation of the two-variable Hermite–Kampé de Fériet polynomials $H_n(x,y)$ are given as follows [15, 16, 17]:

$$\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}, \tag{1.13}$$

and

$$H_n(x,y) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2k}y^k}{k!\,(n-2k)!}, \tag{1.14}$$

respectively. For the purpose of this work, we recall the following definitions. The Mittag–Leffler–Gould–Hopper polynomials are defined by the series [18]:

$${}_EH^{(m)}_n(x,y;\alpha,\beta) = n!\sum_{k=0}^{\left\lfloor\frac{n}{m}\right\rfloor}\frac{x^{\,n-mk}\,y^{\,k}}{(n-mk)!\,\Gamma(\alpha k+\beta)}, \tag{1.15}$$

the Mittag–Leffler–Legendre polynomials ${}_ES_n(x,y;\alpha,\beta)$ are defined by [2]:

$${}_ES_n(x,y;\alpha,\beta) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^k\,y^{n-2k}}{(k!)^2\,\Gamma(\alpha(n-2k)+\beta)}, \tag{1.16}$$

the series representation of the Mittag–Leffler–Laguerre polynomials are defined by [19]:

$${}_EL^{(\alpha,\beta)}_n(x,y) = n!\sum_{k=0}^{n}\frac{(-1)^k\,x^{n-k}\,y^k}{(k!)^2\,\Gamma(\alpha(n-k)+\beta)}, \tag{1.17}$$

and the Fox-Wright function as introduced in [6]

$${}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),\ldots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\ldots,(\beta_q,B_q); & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(\alpha_l+A_l n)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j n)}\frac{z^n}{n!}. \tag{1.18}$$

Recently, by using symbolic and umbral techniques, several authors have presented and studied Mittag-Leffler functions in combination with certain polynomials, such as Laguerre polynomials, Hermite polynomials, Sheffer-type polynomials, and Konhauser polynomials (see, for example, Shahwan and Bin-Saad [20, 10], Babusci et al. [12, 21, 14], Kürt Raza and Zainab [22, 18], Subuhi Khan et al. [23, 16], and Bin-Saad [24, 25]), highlighting various properties and operational formulae that arise when Mittag-Leffler functions are combined with these polynomial families.

In particular, recent studies provide a strong foundation for our approach. Raza and Zainab (see (1.15)) developed Mittag-Leffler–Gould-Hopper polynomials using a symbolic framework, demonstrating how symbolic techniques can efficiently handle generalized polynomial families and their operational properties. Shahwan and Bin-Saad (see (1.15)) developed Mittag-Leffler–Gould-Hopper polynomials using a symbolic framework, demonstrating how symbolic techniques can efficiently handle generalized polynomial families and their operational properties. Shahwan and Bin-Saad (see (1.16)) introduced and analyzed a class of two-dimensional Mittag-Leffler–Konhauser polynomials illustrating explicit series representations, recurrence relations, and operational formulae that inspire the construction of new bivariate polynomial families. Moreover, Shahwan, Bin-Saad, and Al-Hashami [10] studied fractional calculus properties of the bivariate Mittag-Leffler function, highlighting analytic and operational features that are crucial for fractional operators manipulating two-variable Mittag–Leffler–Gegenbauer polynomials. Furthetmore, recent advances in analysis and applied mathematics justify updating the theoretical background of this study. Important developments include new structural and operational properties of extended Mittag–Leffler functions [26], improved reproducing-kernel techniques for multi-dimensional integral problems [27], and enhanced decomposition methods for higher-order differential equations [28]. Additionally, clearer convergence results for double sequences [29] further contribute to a more modern analytical framework. These works collectively strengthen and update the context in which the present study is situated.

Building on these works, the present study, which is based on symbolic theory, aims to introduce and investigate a new class of two-variable Mittag–Leffler–Gegenbauer polynomials, denoted by ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$ and defined in (2.1), focusing on their structural, operational, and analytic properties. The remainder of this paper is organized to systematically develop the theory and applications of the proposed ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$ polynomials. Section 2 establishes the formal foundation for the ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$ polynomials. It begins by defining the polynomials and deriving their core properties, such as a generating function and series representation. The section then presents a numerical and comparative analysis, situating these new polynomials within the broader context of existing research. Continuing on this, Section 3 explores deeper analytical properties by formulating relationships through the lens of fractional calculus. The practical utility of these polynomials is demonstrated in Section 4 through the development and solution of a novel fractional kinetic equation, with the polynomials serving as the kernel; this section is supported by numerical simulations and an analysis of limiting cases. The paper concludes in Section 5 with a discussion of specific applications stemming from our findings and a perspective on future work.

2 Two variable Mittag-Leffler-Gegenbauer polynomials

We present and examine the Two variable Mittag-Leffler-Gegenbauer polynomials in this section. We define the Mittag-Leffler-Gegenbauer polynomials (MLGP) ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$ in accordance with Eq. (1.7) and (1.14) as:

$$\begin{aligned}&{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\\&=H_{n}\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)},y\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{\delta-1}\left\{\frac{\varphi_{0}}{\Gamma(\delta)}\right\}.\end{aligned}\tag{2.1}$$

The generating function and series formulation of the two variable Mittag-Leffler-Gegenbauer polynomials are now obtained. The generating functions of the ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$ polynomials are provided by the following result:

Theorem 2.1. *The following generating functions for the two variable Mittag-Leffler-Gegenbauer polynomials holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}\\&=e^{\left((xt+yt^{2})\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\ \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{\delta-1}\left\{\frac{\varphi_{0}}{\Gamma(\delta)}\right\},\end{aligned}\tag{2.2}$$

$$\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}=e^{xt\left(\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}E_{\alpha,\beta}^{\delta}(yt^{2}).\tag{2.3}$$

Proof. Using Eq. (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}H_{n}\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)},y\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\\&\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{\delta-1}\left\{\frac{\varphi_{0}}{\Gamma(\delta)}\right\}\frac{t^{n}}{n!},\end{aligned}\tag{2.4}$$

which on using (1.13) in the right side it becomes

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}\\&=e^{\left((xt+yt^{2})\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)}\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{\delta-1}\left\{\frac{\varphi_{0}}{\Gamma(\delta)}\right\},\end{aligned}\tag{2.5}$$

which is the desired result (2.2).

Since it is obvious that $\left[x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}t,y\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}t^{2}\right]=0.$ Therefore, using the Weyl decoupling identity [7]:

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{\frac{-k}{2}},\quad k=\left[\hat{A},\hat{B}\right],\quad(k\in\mathbb{C}),\tag{2.6}$$

in the right side of Eq. (2.5) and then using Eq. (1.9) in the resultant equation, we get assertion (2.3). □

The series definition of two variable Mittag-Leffler-Gegenbauer polynomials is provided by the following result:

Theorem 2.2. *For $\alpha(n-k)+\beta\notin\{0,-1,-2,\cdots\}$, the two-variable Mittag-Leffler-Gegenbauer polynomials have the following series definition:*

$$\begin{gathered}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\delta)_{n-k}\,x^{n-2k}\,y^{k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)},\\(x,y,\alpha,\beta,\delta\in\mathbb{C};\{\Re(\alpha),\Re(\beta),\Re(\delta)\}>0).\end{gathered}\tag{2.7}$$

Proof. Considering Eqs. (1.14) and (2.1), we have

$${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2k}\ y^{k}\ \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{n-k+\delta-1}\ \varphi_{0}}{k!\,(n-2k)!\,\Gamma(\delta)}\tag{2.8}$$

$$=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\delta)_{n-k}\,x^{n-2k}\,y^{k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)},\tag{2.9}$$

which is the desired result (2.7). □

Remark 2.1. *If we set $x\mapsto 2x$, $y\mapsto -1$, $\alpha\mapsto 0$, and $\beta\mapsto 1$ in Eq. (2.7), and then multiply the result by $1/n!$, we recover the classical Gegenbauer polynomials.*

$$C_{n}^{\delta}\left(x\right)=\sum_{k=0}^{[n/2]}\frac{(-1)^{k}\,(\delta)_{n-k}\,(2x)^{n-2k}}{k!(n-2k)!}.$$

We now focus on deriving a set of additional symbolic and non-symbolic generating functions corresponding to the polynomials ${}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)$.

Theorem 2.3. *The following generating function holds true:*

$$\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}=E_{\alpha,\alpha,\beta}^{(\delta)}\left(xt,yt^{2}\right),\tag{2.10}$$

or equivalently

$$\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}=\Phi_{1}\left[\delta,1;1;t\hat{D}_{x}^{-1}\hat{d}_{(\alpha,\beta)},yt^{2}\hat{d}_{(\alpha,\beta)}\right]\varphi_{0}.\tag{2.11}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{E}C_{n}^{\delta}(x,y;\alpha,\beta)\frac{t^{n}}{n!}&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(\delta)_{n-k}\,x^{n-2k}\,y^{k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\,t^{n}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\delta)_{n+k}\,(xt)^{n}\,\left(yt^{2}\right)^{k}}{k!\,n!\,\Gamma(\alpha n+\alpha k+\beta)},\end{aligned}\tag{2.12}$$

which by Eq. (1.4), gives (2.10). The assertion (2.11) follows from Eq. (2.10) by using Eq. (1.12). □

Another generating function for the polynomials ${}_{E}C_n^{\delta}(x,y;\alpha,\beta)$ involving Mittag-Leffler function $E_{\alpha,\beta}^{\delta}(x)$ is as follows:

Theorem 2.4. *If $\{Re(\alpha), Re(\beta), Re(\delta)\} > 0$. Then the following generating function holds:*

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x,y;\alpha,\beta)\frac{t^n}{n!} = E_{\alpha,\beta}^{\delta}\left(xt+yt^2\right), \tag{2.13}$$

or equivalently

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x,y;\alpha,\beta)\frac{t^n}{n!} = {}_1F_1\left[\delta;1;\left(xt+yt^2\right)\hat{d}_{(\alpha,\beta)}\right]\varphi_0. \tag{2.14}$$

Proof. We have

$$\begin{aligned} E_{\alpha,\beta}^{\delta}\left(xt+yt^2\right) &= \sum_{n=0}^{\infty}\frac{(\delta)_n}{n!\,\Gamma(\alpha n+\beta)}\left(xt+yt^2\right)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\delta)_n\,x^{n-k}\,y^k}{k!\,(n-k)!\,\Gamma(\alpha n+\beta)}\,t^{n+k}, \end{aligned} \tag{2.15}$$

which on letting $n=n-k$ and considering the definition (2.7), gives us the desired result (2.13). Using Eq. (1.10) in Eq. (2.13), we arrive at the result (2.14). □

Remark 2.2. *Letting $\delta=-m$, $m\in\mathbb{N}$ in Eq. (2.14) and using the relation ([8], p.395(10.38)):*

$$L_m(x) = {}_1F_1[-m;1;x], \tag{2.16}$$

we lead to the interesting relation:

$$\sum_{n=0}^{\infty} {}_{E}C_n^{-m}(x,y;\alpha,\beta)\frac{t^n}{n!} = L_m\left(\left(xt+yt^2\right)\hat{d}_{(\alpha,\beta)}\right)\varphi_0, \tag{2.17}$$

where $L_m(x)$ is the Laguerre polynomials ([30], p.200(2)).

Theorem 2.5. *Then the following generating function holds:*

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x,y;\alpha,\beta)\,\frac{t^n}{n!} = \left[1-\hat{d}_{(\alpha,\beta)}\left(t+yx^{-1}t^2\right)\hat{D}_x^{-1}\right]^{-\delta}\varphi_0. \tag{2.18}$$

Proof. Denote the right side of the assertion (2.18) by I. Then

$$\begin{aligned} I &= \left[1-\hat{d}_{(\alpha,\beta)}\left(t+yx^{-1}t^2\right)\hat{D}_x^{-1}\right]^{-\delta}\varphi_0 \\ &= \sum_{n=0}^{\infty}\frac{(\delta)_n\left(xt+yt^2\right)^n\hat{d}^n_{(\alpha,\beta)}\varphi_0}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\delta)_n\,x^{-k}y^k\,\hat{D}_x^{-n}\,\hat{d}^n_{(\alpha,\beta)}\varphi_0}{k!\,(n-k)!}t^{n+k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(\delta)_{n-k}\,x^{-k}\,y^k\,\hat{D}_x^{-(n-k)}\,\hat{d}^{n-k}_{(\alpha,\beta)}\varphi_0}{k!\,(n-2k)!}\,t^n, \end{aligned} \tag{2.19}$$

which on using (1.6) and (2.7), and taking into account that $\hat{D}_x^{-\varrho}=\frac{x^{\varrho}}{\varrho!}$, we arrive at the desired result (2.18). □

Theorem 2.6. *The following generating functions are established:*

$$\sum_{n=0}^{\infty} {}_{E}C_n^{\delta}(x,y;\alpha,\beta)\,\frac{t^n}{n!} = \left[1-\left(xt+yt^2\right)\hat{c}^{\alpha}\right]^{-\delta}\hat{c}^{\beta-1}\varphi_0. \tag{2.20}$$

Proof. Denote the right side of the assertion (2.20) by I. Then

$$\begin{aligned} I &= \left[1-\left(xt+yt^2\right)\hat{c}^{\alpha}\right]^{-\delta}\hat{c}^{\beta-1}\varphi_0 \\ &= \sum_{n=0}^{\infty}\frac{(\delta)_n\left(xt+yt^2\right)^n\hat{c}^{\alpha n+\beta-1}\varphi_0}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\delta)_n\,x^{n-k}\,y^k\,\hat{c}^{\alpha n+\beta-1}\varphi_0}{k!\,(n-k)!}t^{n+k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(\delta)_{n-k}\,x^{n-k}\,y^k\,\hat{c}^{\alpha n-\alpha k+\beta-1}\varphi_0}{k!\,(n-2k)!}\,t^n, \end{aligned} \tag{2.21}$$

which on using (1.5) and (2.7), we arrive at the desired result (2.20). □

The Chebyshev polynomials of the second kind and Legendre polynomials [30] are defined in the following series forms:

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(n-k)!\,(2x)^{n-2k}}{k!\,(n-2k)!}, \tag{2.22}$$

and

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!\,x^{n-2k}}{2^n\,k!\,(n-k)!\,(n-2k)!}. \tag{2.23}$$

As an extended version of Chebyshev polynomials and Legendre polynomials, the so-called Gegenbauer polynomials are defined by the series [30]:

$$C_n^{\delta}(x) = \sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(\delta)_{n-k}(2x)^{n-2k}}{k!\,(n-2k)!}. \tag{2.24}$$

Clearly, by (2.22), (2.23), and (2.24), we have

$$C_n^{\frac{1}{2}}(x) = P_n(x)\ and\ C_n^1(x) = U_n(x). \tag{2.25}$$

Remark 2.3. *The relations in (2.25) motivate the definition of the bivariate Mittag–Leffler–Legendre polynomials ${}_{E}P_n(x,y;\alpha,\beta)$ and the bivariate Mittag–Leffler–Chebyshev polynomials ${}_{E}U_n(x,y;\alpha,\beta)$ via the following series representations:*

$${}_{E}P_n(x;\alpha,\beta) = n!\sum_{k=0}^{[\frac{n}{2}]}\frac{\left(\frac{1}{2}\right)_{n-k}x^{n-2k}\ y^k}{\Gamma(\alpha n-\alpha k+\beta)(n-2k)!\,k!}, \tag{2.26}$$

and

$${}_{E}U_n(x;\alpha,\beta) = n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(n-k)!\,x^{n-2k}\ y^k}{\Gamma(\alpha n-\alpha k+\beta)(n-2k)!\,k!}. \tag{2.27}$$

Clearly

$${}_{E}C_n^{\frac{1}{2}}(x,y;\alpha,\beta) = {}_{E}P_n(x,y;\alpha,\beta), \tag{2.28}$$

and

$${}_{E}C_n^{1}(x,y;\alpha,\beta) = {}_{E}U_n(x,y;\alpha,\beta). \tag{2.29}$$

2.1 Numerical and graphical illustration for ${}_EC_n^{\delta}(x,y;\alpha,\beta)$

Example 2.1. *Using the definition in* (2.7)*, we consider the particular case* $n=2,\ \delta=1,\ \alpha=1,\ \beta=2$*. In this case, the summation extends over* $k=0,1$*, and computing each term yields:*

$$k=0:(\delta)_2=(1)_2=2,\ \Gamma(\alpha n-\alpha k+\beta)=\Gamma(4)=6,$$

$$\Rightarrow \frac{(\delta)_2 x^2}{0!\,2!\,\Gamma(4)}=\frac{2x^2}{12}=\frac{x^2}{6},$$

$$k=1:\quad (\delta)_1=(1)_1=1,\qquad \Gamma(2-1+2)=\Gamma(3)=2,$$

$$\Rightarrow \frac{(\delta)_1 x^0 y^1}{1!\,0!\,\Gamma(3)}=\frac{y}{2}.$$

Hence, for these parameter values, the Mittag-Leffler–Gegenbauer polynomial reduces to the simple quadratic form

$${}_EC_2^1(x,y;1,2)=\tfrac{x^2}{3}+y. \tag{2.30}$$

Table 1 lists a few numerical values of ${}_EC_2^1(x,y;1,2)$.

Table 1: Numerical values of ${}_EC_2^1(x,y;1,2)=\frac{x^2}{3}+y$.

x	$y=-1$	$y=0$	$y=1$
-2	0.33	1.33	2.33
-1	-0.67	0.33	1.33
0	-1.00	0.00	1.00
1	-0.67	0.33	1.33
2	0.33	1.33	2.33

Example 2.2. *Based on* (2.7)*. Substituting* $n=3,\ \delta=1,\ \alpha=1,\ \beta=2$*, the sum runs for* $k=0,1$*:*

$$k=0:\quad (\delta)_3=(1)_3=6,\quad \Gamma(5)=24,\qquad \Rightarrow \frac{6x^3}{0!\,3!\,24}=\frac{x^3}{24},$$

$$k=1:\quad (\delta)_2=(1)_2=2,\quad \Gamma(4)=6,\qquad \Rightarrow \frac{2xy}{1!\,1!\,6}=\frac{xy}{3}.$$

After multiplying by $n!=6$*, we obtain*

$${}_EC_3^1(x,y;1,2)=6\left(\frac{x^3}{24}+\frac{xy}{3}\right)=\frac{x^3}{4}+2xy.$$

Hence,

$${}_EC_3^1(x,y;1,2)=\tfrac{x^3}{4}+2xy. \tag{2.31}$$

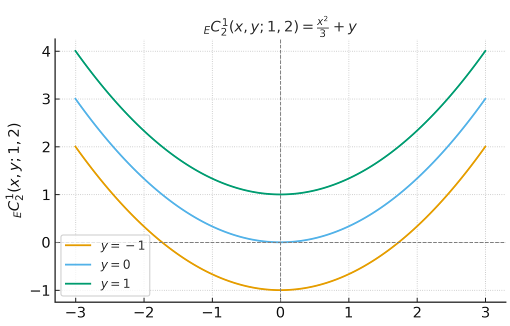

Figure 1: Graph of ${}_EC_2^1(x,y;1,2)=\frac{x^2}{3}+y$ for $y=-1,0,1$. Each curve represents a parabola shifted vertically by y.

Similarly, for $n=4$ we have three contributing terms ($k=0,1,2$):

$$k=0:(\delta)_4=(1)_4=24,\quad \Gamma(6)=120,\qquad \Rightarrow \frac{24x^4}{0!\,4!\,120}=\frac{x^4}{120},$$

$$k=1:(\delta)_3=6,\quad \Gamma(5)=24,\qquad \Rightarrow \frac{6x^2y}{1!\,2!\,24}=\frac{x^2y}{8},$$

$$k=2:(\delta)_2=2,\quad \Gamma(4)=6,\qquad \Rightarrow \frac{2y^2}{2!\,0!\,6}=\frac{y^2}{6}.$$

Multiplying by $n!=24$, we find

$${}_EC_4^1(x,y;1,2)=24\left(\frac{x^4}{120}+\frac{x^2y}{8}+\frac{y^2}{6}\right)=\frac{x^4}{5}+3x^2y+4y^2.$$

Thus,

$${}_EC_4^1(x,y;1,2)=\tfrac{x^4}{5}+3x^2y+4y^2. \tag{2.32}$$

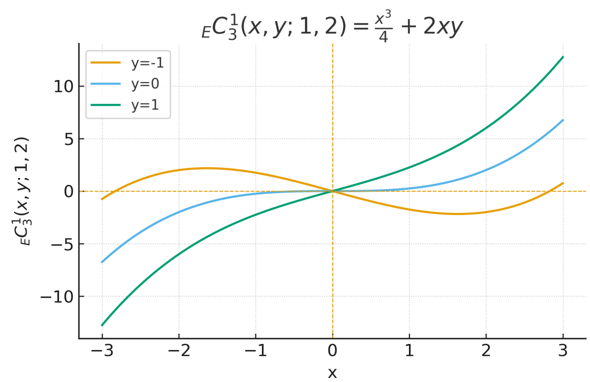

Figure 2: Graph of ${}_EC_3^1(x,y;1,2)=\frac{x^3}{4}+2xy$.

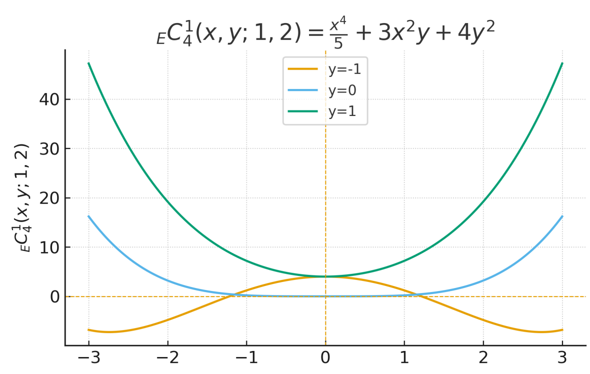

Figure 3: Graph of ${}_EC_4^1(x,y;1,2)=\frac{x^4}{5}+3x^2y+4y^2$.

Table 2: Numerical values of ${}_EC_3^1(x,y;1,2)$ and ${}_EC_4^1(x,y;1,2)$ for selected x, y.

x	${}_EC_3^1(x,y;1,2)=\frac{x^3}{4}+2xy$ $y=-1$	$y=0$	$y=1$	${}_EC_4^1(x,y;1,2)=\frac{x^4}{5}+3x^2y+4y^2$ $y=-1$	$y=0$	$y=1$
-2	-1	-2	-3	-0.8	3.2	11.2
-1	1.75	-0.25	-2.25	6.6	0.2	10.2
0	0	0	0	4	0	4
1	-2.25	0.25	2.25	10.2	0.2	6.6
2	3	2	1	11.2	3.2	-0.8

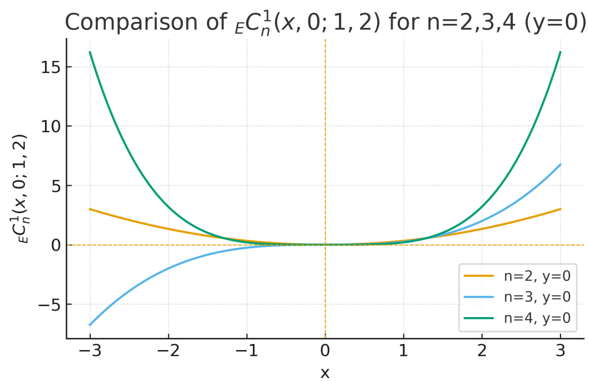

Figure 4: Comparison of ${}_EC_n^1(x,0;1,2)$ for $n=2,3,4$.

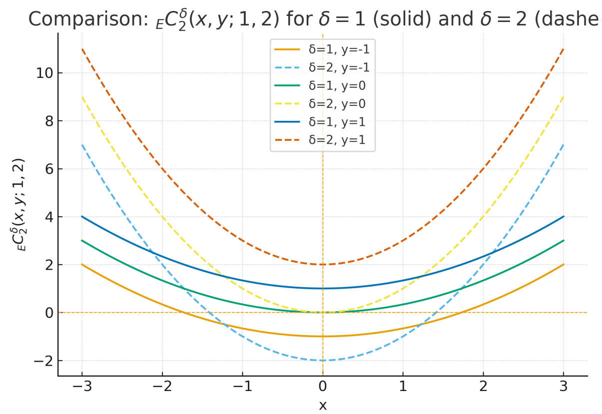

Figure 5: Plots of ${}_EC_2^\delta(x,y;1,2)$ for $\delta = 1$ and 2 at $y=-1,0,1$.

Example 2.3 (Comparison of ${}_EC_2^\delta(x,y;1,2)$ for $\delta = 1$ and $\delta = 2$). *From the series definition*

$$ {}_EC_2^\delta(x,y;1,2) = \frac{\delta(\delta+1)}{\Gamma(4)}\,x^2 + \frac{2\delta}{\Gamma(3)}\,y = \frac{\delta(\delta+1)}{6}\,x^2 + \delta\,y, \quad (2.33)$$

we obtain the explicit expressions

$$ {}_EC_2^1(x,y;1,2) = \frac{x^2}{3} + y, \qquad {}_EC_2^2(x,y;1,2) = x^2 + 2y.$$

Table 3 lists representative numerical values for $x \in \{-2,-1,0,1,2\}$ and $y \in \{-1,0,1\}$, while Fig. 5 displays their corresponding profiles. It is clear that increasing δ enhances both the curvature (in the x^2 term) and the linear y-contribution, producing steeper parabolic shapes and higher vertical scaling.

Table 3: Comparison of ${}_EC_2^\delta(x,y;1,2)$ for $\delta = 1$ and 2.

	$\delta=1$			$\delta=2$		
x	$y=-1$	$y=0$	$y=1$	$y=-1$	$y=0$	$y=1$
-2	-0.33	1.33	2.33	2	4	6
-1	0.67	0.33	1.33	1	2	3
0	-1	0	1	-2	0	2
1	0.67	0.33	1.33	1	2	3
2	-0.33	1.33	2.33	2	4	6

2.2 Comparison with some related polynomials

The motivation for comparing the Mittag–Leffler–Gegenbauer polynomials with three related known results arises from the need to establish the scope, validity, and novelty of the proposed formulation. Furthermore, this comparison verifies the correctness of the derived generating functions and operational formulas, and highlights new analytical features such as fractional–order extensions and symbolic operator structures that are not present in earlier works. In this section, we compare the definition and the derived properties of the proposed Mittag–Leffler–Gegenbauer polynomials (MLGP) with three related works, namely the Mittag–Leffler–Gould–Hopper polynomials (MLGHP) (1.15)) (see [18]), the Mittag–Leffler–Laguerre polynomials (MLLP) (1.17)[19], and the Mittag–Leffler–Legendre polynomials (MLLGP) (1.16) [2]. The comparison among the four polynomial families is based on several criteria, including their generating functions, parameter structures, operational forms, and analytical properties. These criteria enable us to identify direct connections and reductions, demonstrating that the proposed (MLGP) form a unifying generalization of the other three classes. For convenience, we denote the four polynomials under consideration by their abbreviated names in the comparisons that follow.

Case 1: Compare (MLGP) with (MLGHP)

Both Papers (MLGP) and (MLGHP) utilize the symbolic operator method as a central analytical framework for developing new classes of Mittag–Leffler-type special polynomials. While they share a common foundation in extending classical families such as Laguerre, Hermite, and Gegenbauer polynomials, their orientations diverge in scope and emphasis. Paper (MLGP) advances the theory by introducing the two-variable Mittag–Leffler–Gegenbauer polynomials, offering a more general and unified structure that integrates symbolic techniques with fractional calculus, particularly through relations involving fractional integrals, derivatives, and applications to fractional kinetic equations. This broad framework highlights the method's potential for modeling complex, memory-dependent systems. In contrast, Paper (MLGHP) focuses on presenting a rigorous development of their symbolic operational identities, summation formulas, and quasi-monomiality properties. Its contribution lies primarily in the algebraic and operational characterization of this single-variable family. Hence, while Paper (MLGP) emphasizes generalization and applicability to fractional models, Paper (MLGHP) provides a more specialized and structural exploration, making both works complementary in advancing the symbolic operator approach to special functions.

Case 2: Compare (MLGP) with (MLLP)

In both papers, the symbolic operator method has emerged as a powerful analytical framework for constructing the Mittag–Leffler–type special functions. In this context, the two papers (MLGP) and (MLLP) closely related studies introduce distinct but complementary classes of two-variable Mittag–Leffler polynomials. The first, on the (MLGP), emphasizes their unifying character among classical families such as Laguerre, Hermite, and Gegenbauer, and explores their connection with fractional integrals, derivatives, and fractional kinetic equations, thereby revealing their applicability to memory-dependent systems. In contrast, the study on the (MLLP) focuses on algebraic and differential structures, deriving generating functions, symbolic partial differential relations, multiplicative and derivative operators, and Lie-theoretic generating relations based on Tricomi functions. Moreover, it provides parameter reductions to one-variable cases and detailed graphical analyses. Together, these works demonstrate the versatility of the symbolic operator method in extending classical polynomial families to fractional, algebraic, and multivariable frameworks.

Table 4: Comparison of MLGP with MLGHP, MLLP, and MLLGP polynomials

Aspect	${}_EC_n^{\delta}$	${}_EH_n^{(m)}$	${}_EL_n^{(\alpha,\beta)}$	${}_ES_n$
Polynomial family	MLGP	MLGHP	MLLP	MLLGP
Variables	2	2	2	2
Method / Approach	Symbolic operators + fractional calculus	Symbolic / umbral operators	Symbolic operators, generating functions, Lie-theoretic techniques	Symbolic / umbral operators, Lie algebra, monomiality
Analytical focus	Series, generating functions, operational rules, fractional integrals/derivatives	Generating functions, series, operational rules, quasi-monomiality	Series, symbolic PDEs, operators, integrals, Lie-Mon relations	Series, symbolic PDEs, operational rules, Lie algebraic relations
Fractional / Applied aspects	Fractional kinetic equations	Not emphasized	Not emphasized	Not emphasized
Algebraic / Lie-theoretic focus	Limited, mainly fractional	Quasi-monomiality	Lie-Mon technique, algebraic structure	Lie algebra, symbolic-umbral relations, monomiality
Graphical / Numerical illustrations	Yes	Yes	Yes	Not specified
Scope / Generality	Multivariable fractional generalization; unifies classical polynomials	Single-variable structural study	Two-variable algebraic & differential structures; parameter reduction	Two-variable symbolic & Lie-algebraic generalization

Case 3: Compare (MLGP) with (MLLGP)

Both Paper (MLGP) and Paper (MLLGP) employ the symbolic operator (or symbolic evaluation) method as a central analytical tool for constructing and extending new families of Mittag–Leffler-based special polynomials. Each study aims to generalize classical orthogonal polynomials through symbolic and umbral techniques, highlighting the Mittag–Leffler function as a unifying kernel that connects traditional polynomial structures to fractional and operational calculus. Both papers derive generating functions, series representations, and symbolic operational rules, and discuss multiplicative and derivative operators to investigate quasi-monomiality properties. They also use symbolic operator theory to establish identities and algebraic relations, demonstrating how this framework provides a systematic way to construct and analyze hybrid families of polynomials.

Despite these shared foundations, the two works differ in mathematical focus, structure, and application scope. Paper (MLGP) introduces the Two–Variable Mittag–Leffler–Gegenbauer polynomials, emphasizing multivariable generalization and the integration of fractional operators such as Riemann–Liouville and Caputo derivatives. It further connects the theoretical framework to fractional kinetic equations, illustrating its potential in modeling memory-dependent and fractional dynamical systems. In contrast, Paper (MLLGP) develops Mittag–Leffler–Legendre polynomials, primarily exploring their symbolic, umbral, and Lie-algebraic structures. This work focuses on two distinct forms of two-variable Legendre polynomials, deriving corresponding symbolic differential equations, operational identities, and Lie algebraic relations, but without emphasizing fractional or physical applications. Together, they contribute complementary perspectives Paper (MLGP) expanding toward multivariable and fractional modeling, and Paper (MLLGP) deepening the algebraic and structural theory of symbolic special polynomials.

3 Relations via fractional integrals and derivatives

In this section, operational relations associated with the generalized Mittag–Leffler–Gegenbauer polynomials are established by employing the Riemann–Liouville and Caputo fractional integrals and derivatives with respect to ρ ($\rho > 0$). These results are obtained by applying the known power rules of fractional calculus termwise to the defining finite series representation of the polynomial. First, we recall the power rules for the Riemann–Liouville and Caputo operators (see e.g., [31]):

(Riemann–Liouville fractional integral)

$$ {}_0I_t^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha} \quad \lambda > -1, \quad , \tag{3.1}$$

(Riemann–Liouville fractional derivative),

$$ {}_0D_t^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha} \quad \lambda > -1, \tag{3.2}$$

(Caputo fractional integral),

$$ {}^{C}I_t^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha} \quad \lambda \geq 0, \tag{3.3}$$

$$ {}^{C}D_t^{\alpha} t^{\lambda} = \begin{cases} 0, & \text{if } \lambda = 0 \text{ and } \alpha > 0, \\ \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, & \lambda > \alpha - 1. \end{cases} \tag{3.4}$$

Theorem 3.1. *Let $\alpha, \beta, \delta \in \mathbb{C}; \{\Re(\alpha), \Re(\beta), \Re(\delta), \Re(\rho)\} > 0$. Then, the following identities hold:*

$$ \begin{aligned} &{}_0I_\lambda^{\rho}\left[\lambda^{\alpha n+\beta-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)\right] \\ &= \lambda^{\alpha n+\beta+\rho-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta+\rho), \end{aligned} \tag{3.5}$$

$$ \begin{aligned} &{}_0D_\lambda^{\rho}\left[\lambda^{\alpha n+\beta-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)\right] \\ &= \lambda^{\alpha n+\beta-\rho-1}\,{}_EC_n^{\delta}(\tau, \lambda^{\alpha}; \alpha, \beta-\rho), \end{aligned} \tag{3.6}$$

$$ \begin{aligned} &{}^{C}I_\lambda^{\rho}\left[\lambda^{\alpha n+\beta-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)\right] \\ &= \lambda^{\alpha n+\beta+\rho-1}\,{}_EC_n^{\delta}(\tau, \lambda^{\alpha}; \alpha, \beta+\rho), \end{aligned} \tag{3.7}$$

$$ \begin{aligned} &{}^{C}D_\lambda^{\rho}\left[\lambda^{\alpha n+\beta-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)\right] \\ &= \lambda^{\alpha n+\beta-\rho-1}\,{}_EC_n^{\delta}(\tau, \lambda^{\alpha}; \alpha, \beta-\rho). \end{aligned} \tag{3.8}$$

Proof. We prove the first identity (3.5); the others follow similarly using the corresponding power rules. Starting from the left-hand side of the assertion (3.5) and apply the Riemann–Liouville fractional integral ${}_0I_\lambda^{\rho}$, we obtain

$$ \begin{aligned} &{}_0I_\lambda^{\rho}\left[\lambda^{\alpha n+\beta-1}\,{}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)\right] \\ &= n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(\delta)_{n-k}\,\tau^{n-2k}}{k!\,(n-2k)!\,\Gamma\big(\alpha(n-k)+\beta+\rho\big)}\ \lambda^{\alpha(n-k)+\beta+\rho-1}. \end{aligned} \tag{3.9}$$

Using the definition (2.7), we obtain the right-hand side of the desired formula (3.5). Similarly, the other identities (3.6), (3.7) and (3.8) can be established by applying the corresponding rule term by term, together with the series representation of ${}_EC_n^{\delta}(\tau, \lambda^{-\alpha}; \alpha, \beta)$. □

Theorem 3.2. *Let $\alpha, \beta, \delta \in \mathbb{C}; \{\Re(\alpha), \Re(\beta), \Re(\delta), \Re(\rho), \Re(\omega)\} > 0$. Then, the following identities hold:*

$$ \begin{aligned} &{}_0I_\omega^{\rho}\left[\omega^{\beta-1}\,{}_EC_n^{\delta}(x\omega^{\alpha}, y\omega^{\alpha}; \alpha, \beta)\right] \\ &= \omega^{\beta+\rho-1}\,{}_EC_n^{\delta}(x\omega^{\alpha}, y\omega^{\alpha}; \alpha, \beta+\rho), \end{aligned} \tag{3.10}$$

$$ \begin{aligned} &{}_0D_\omega^{\rho}\left[\omega^{\beta-1}\,{}_EC_n^{\delta}(x\omega^{\alpha}, y\omega^{\alpha}; \alpha, \beta)\right] \\ &= \omega^{\beta-\rho-1}\,{}_EC_n^{\delta}(x\omega^{\alpha}, y\omega^{\alpha}; \alpha, \beta-\rho), \end{aligned} \tag{3.11}$$

$$ \begin{aligned} &{}_0D_x^{2\rho}\ {}_0I_y^{\rho}\left[{}_EC_n^{\delta}(x, y; \alpha, \beta)\right] \\ &= \frac{\Gamma(\delta+\rho)}{\Gamma(\delta)}\,{}_EC_n^{\delta+\rho}(x, y; \alpha, \beta+\alpha\rho), \end{aligned} \tag{3.12}$$

$$ \begin{aligned} &{}_0I_x^{2\rho}\ {}_0D_y^{\rho}\left[{}_EC_n^{\delta}(x, y; \alpha, \beta)\right] \\ &= \frac{\Gamma(\delta-\rho)}{\Gamma(\delta)}\,{}_EC_n^{\delta-\rho}(x, y; \alpha, \beta-\alpha\rho). \end{aligned} \tag{3.13}$$

Proof. To establish (3.10), we first invoke (2.7) in the right-hand side of (3.10) and interchange the order of summation and fractional integration. Such an interchange is justified under the assumptions specified in the theorem. Subsequently, by employing the definition provided in (3.1), we obtain

$$ \begin{aligned} &{}_0I_\omega^{\rho}\left[\omega^{\beta-1}\,{}_EC_n^{\delta}(x\omega^{\alpha}, x\omega^{\alpha}; \alpha, \beta)\right] \\ &= n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(\delta)_{n-k}\,x^{n-2k}\,y^{k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\ {}_0I_\omega^{\rho}\left(\omega^{\alpha n+\alpha k+\beta-1}\right) \\ &= n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(\delta)_{n-k}\,x^{n-2k}\,y^{k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta+\rho)} \omega^{\alpha n+\alpha k+\beta+\rho-1}, \end{aligned} \tag{3.14}$$

which gives us the desired result (3.10).
For the proof of (3.11), we refer to the proof of (3.11) with replacing (3.1) by (3.2).
Again, to establish (3.12), we first invoke (2.7) in the right-hand side of (3.12) and interchange the order of summation and fractional integration. Such an interchange is justified under the assumptions specified in the theorem. Subsequently, by employing the definitions provided in (3.1) and (3.2), we obtain

$$ \begin{aligned} &{}_0D_x^{2\rho}\ {}_0I_y^{\rho}\left[{}_EC_n^{\delta}(x, y; \alpha, \beta)\right] \\ &= n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(\delta)_{n-k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\ {}_0D_x^{2\rho}\left(x^{n-2k}\right){}_0I_y^{\rho}\left(y^{k}\right) \\ &= n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(\delta)_{n-k}\,x^{n-2k-2\rho}\,y^{k+\rho}}{(k+\rho)!\,(n-2k-2\rho)!\,\Gamma(\alpha n-\alpha k+\beta+\rho)}. \end{aligned} \tag{3.15}$$

Replacing k by $k-\rho$ and applying (2.7) yields (3.12). The proof of (3.13) proceeds analogously to that of (3.12) and is therefore omitted. □

We now derive several fractional integral relations for the polynomials ${}_EC_n^{\delta}(x,t;\alpha,\beta)$ using the fractional integral operators introduced by Saigo [32].

$$\left(I_{0,y}^{\mu,\nu,\eta}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(\lambda)\Gamma(\lambda-\nu+\eta)}{\Gamma(\lambda-\nu)\Gamma(\lambda+\mu+\eta)}y^{\lambda-\nu-1}, \tag{3.16}$$

and

$$\left(I_{y,\infty}^{\mu,\nu,\eta}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(1-\lambda+\nu)\Gamma(1-\lambda+\eta)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\mu+\nu+\eta)}y^{\lambda-\nu-1}. \tag{3.17}$$

If we take $\nu=-\mu$ in Eqs. (3.16) and (3.17), we have

$$\left(\mathcal{R}_{0,y}^{\mu}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}y^{\lambda+\mu-1}, \tag{3.18}$$

and

$$\left(\mathcal{W}_{y,\infty}^{\mu}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)}y^{\lambda+\mu-1}, \tag{3.19}$$

where $\mathcal{R}_{0,y}^{\mu}$ and $\mathcal{W}_{y,\infty}^{\mu}$ denoted the Riemann-Liouville and the Erdélyil-Kober fractional integral operators (see [32]) and if we take $\nu=0$ in Eqs. (3.16) and (3.17), we have

$$\left(\mathcal{E}_{0,y}^{\mu,\eta}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(\lambda+\eta)}{\Gamma(\lambda+\mu+\eta)}y^{\lambda-1}, \tag{3.20}$$

and

$$\left(\mathcal{K}_{y,\infty}^{\mu,\eta}\, t^{\lambda-1}\right)(y)=\frac{\Gamma(1-\lambda+\eta)}{\Gamma(1-\lambda+\mu+\eta)}y^{\lambda-1}, \tag{3.21}$$

where $\mathcal{E}_{0,y}^{\mu,\eta}$ and $\mathcal{K}_{y,\infty}^{\mu,\eta}$ denoted the Weyl type and the Erdélyil-Kober fractional integral operators (see [32]).

Theorem 3.3. *Let $\alpha\in\mathbb{N},\mu,\nu,\eta,\lambda,\beta,\delta\in\mathbb{C},\{x,y\}>0$ and n be a non-negative integer. Then*

$$\left(I_{0,y}^{\mu,\nu,\eta}\left[t^{\lambda-1}{}_EC_n^{\delta}(x,t;\alpha,\beta)\right]\right)(y)=\frac{n!\,x^n y^{\lambda-\nu-1}}{\Gamma(\delta)}$$
$$\times\,{}_3\Psi_4\left[\begin{array}{c}(\delta+n,-1),(\lambda,1),(\lambda-\nu+\eta,1);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda-\nu,1),(\lambda+\mu+\eta,1);\end{array}\ \frac{y}{x^2}\right]. \tag{3.22}$$

Proof. Seeking simplicity, we denote the left side of (3.22) by Ω. Then, by using (2.7), interchanging the order of integration and summation, we conclude.

$$\Omega=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(\delta)_{n-k}\,x^{n-2k}}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)}\left(I_{0,y}^{\mu,\nu,\eta}\,t^{\lambda+k-1}\right)(y). \tag{3.23}$$

Employing relation (3.16), we arrive at

$$\Omega=x^n y^{\lambda-\nu-1}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(\delta)_{n-k}\,\Gamma(\lambda+k)\,\Gamma(\lambda-\nu+\eta+k)\;y^k}{k!\,(n-2k)!\,\Gamma(\alpha n-\alpha k+\beta)\,\Gamma(\lambda-\nu+k)}$$
$$\times\frac{x^{-2k}}{\Gamma(\lambda+\mu+\eta+k)}$$
$$=\frac{n!\,x^n y^{\lambda-\nu-1}}{\Gamma(\delta)}\times\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\Gamma(\delta+n-k)\,\Gamma(\lambda+k)\,\Gamma(\lambda-\nu+\eta+k)}{\Gamma(n+1-2k)\,\Gamma(\alpha n+\beta-\alpha k)\,\Gamma(\lambda-\nu+k)}$$
$$\times\frac{\left(\frac{y}{x^2}\right)^k}{\Gamma(\lambda+\mu+\eta+k)\,k!}. \tag{3.24}$$

Finally, using definition (1.18), we lead to the formula (3.22). ◻

Corollary 3.1. *As a consequence of (3.18) and Theorem 3.3 with $\nu=-\mu$, we obtain*

$$\left(\mathcal{R}_{0,y}^{\mu}\left[t^{\lambda-1}{}_EC_n^{\delta}(x,t;\alpha,\beta)\right]\right)(y)=\frac{n!\,x^n y^{\lambda+\mu-1}}{\Gamma(\delta)}$$
$$\times\,{}_2\Psi_3\left[\begin{array}{c}(\delta+n,-1),(\lambda,1);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda+\mu,1);\end{array}\ \frac{y}{x^2}\right]. \tag{3.25}$$

Corollary 3.2. *As a consequence of (3.20) and Theorem 3.3 with $\nu=0$, we obtain*

$$\left(\mathcal{E}_{0,y}^{\mu,\eta}\left[t^{\lambda-1}{}_EC_n^{\delta}(x,t;\alpha,\beta)\right]\right)(y)=\frac{n!\,x^n y^{\lambda-1}}{\Gamma(\delta)}$$
$$\times\,{}_2\Psi_3\left[\begin{array}{c}(\delta+n,-1),(\lambda+\eta,1);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(\lambda+\mu+\eta,1);\end{array}\ \frac{y}{x^2}\right]. \tag{3.26}$$

Theorem 3.4. *Let $\alpha\in\mathbb{N}$, $\mu,\nu,\eta,\lambda,\beta,\delta\in\mathbb{C}$, $x,y>0$, and n be a non-negative integer. Then*

$$\left(I_{y,\infty}^{\mu,\nu,\eta}\left[t^{\lambda-1}{}_EC_n^{\delta}\left(x,\frac{1}{t};\alpha,\beta\right)\right]\right)(y)=\frac{n!\,x^n y^{\lambda-\nu-1}}{\Gamma(\delta)}$$
$$\times\,{}_3\Psi_4\left[(\delta+n,-1),(1-\lambda+\nu,1),(1-\lambda+\eta,1);(n+1,-2),\right.$$
$$\left.(\alpha n+\beta,-\alpha),(1-\lambda,1),(1-\lambda+\mu+\nu+\eta,1);\frac{1}{x^2y}\right]. \tag{3.27}$$

Proof. By considering the operator (3.17) and proceeding in the same way as in the proof of Theorem 3.3, we can easily prove Theorem 3.4. ◻

Corollary 3.3. *As a consequence of (3.19) and Theorem 3.4 with $\nu=-\mu$, we obtain*

$$\left(\mathcal{W}_{y,\infty}^{\mu}\left[t^{\lambda-1}{}_EC_n^{\delta}\left(x,\frac{1}{t};\alpha,\beta\right)\right]\right)(y)=\frac{n!\,x^n y^{\lambda+\mu-1}}{\Gamma(\delta)}$$
$$\times\,{}_2\Psi_3\left[\begin{array}{c}(\delta+n,-1),(1-\lambda-\mu,1);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(1-\lambda,1);\end{array}\ \frac{1}{x^2y}\right]. \tag{3.28}$$

Corollary 3.4. *As a consequence of (3.21) and Theorem 3.4 with $\nu=0$, we obtan*

$$\left(\mathcal{K}_{y,\infty}^{\mu,\eta}\left[t^{\lambda-1}{}_EC_n^{\delta}\left(x,\frac{1}{t};\alpha,\beta\right)\right]\right)(y)=\frac{n!\,x^n y^{\lambda-1}}{\Gamma(\delta)}$$
$$\times\,{}_2\Psi_3\left[\begin{array}{c}(\delta+n,-1),(1-\lambda+\eta,1);\\ (n+1,-2),(\alpha n+\beta,-\alpha),(1-\lambda+\mu+\eta,1);\end{array}\ \frac{1}{x^2y}\right]. \tag{3.29}$$

4 Application to fractional kinetic equation

In this section, we justify the relevance of the MLGPs ${}_{E}C_{n}^{\delta}(\tau,\zeta;\alpha,\beta)$ by formulating a fractional kinetic equation that incorporates the new ${}_{E}C_{n}^{\delta}(\tau,\zeta;\alpha,\beta)$ polynomials in its kernel. The fractional kinetic equations introduced by Saxena and Kalla are defined as follows (see [33]; see also [34]):

$$\mathcal{N}(\tau)-\mathcal{N}_0 f(\tau)=-\varepsilon^{\nu}\,{}_0\hat{D}_{\tau}^{-\nu}\mathcal{N}(\tau),\quad (Re(\nu)>0), \tag{4.1}$$

where $\mathcal{N}(\tau)$ is the number density of a given species at time τ and ε is a constant. When $\tau=0$, then $\mathcal{N}_0=\mathcal{N}(0)$. Consider $f\in L(0,\infty)$ and Riemann-Liouville integral operator ${}_0\hat{D}_{\tau}^{-\nu}$ (see [31, 35]) as follows

$${}_0\hat{D}_{\tau}^{-\nu}f(\tau)=\frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-s)^{\nu-1}f(s)ds,\quad (Re(\nu)>0). \tag{4.2}$$

Theorem 4.1. *If $\mu>0$ and $\nu>0$, then the equation*

$$\mathcal{N}(\tau)-\mathcal{N}_0\,\tau^{\mu+\frac{n}{2}-1}\,{}_{E}C_{n}^{\delta}\left(-\xi^{\nu}\tau^{\frac{\nu}{2}},\zeta;\nu,\mu\right)=-\omega^{\nu}\,{}_0\hat{D}_{\tau}^{-\nu}\mathcal{N}(\tau) \tag{4.3}$$

has the solution

$$\mathcal{N}(\tau)=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\zeta^{k}\,(\delta)_{n-k}}{k!\,(n-2k)!}\,\tau^{\nu n-\nu k+\mu-1}\,E_{\nu,\nu n-\nu k+\mu}\left(-(\omega\tau)^{\nu}\right), \tag{4.4}$$

where $E_{\nu,\mu}(x)$ is the Mittag-Leffler function (1.2).

Proof. The Laplace transform involving the Riemann–Liouville fractional integral operator, as introduced in [3], is given by:

$$L\left[{}_0\hat{D}_{t}^{-\nu}f(\tau):p\right]=p^{-\nu}F(p), \tag{4.5}$$

where

$$F(p)=\int_0^{\infty}e^{-p\tau}f(\tau)\,d\tau,\quad (Re(p)>0). \tag{4.6}$$

Now, by applying the Laplace transform to (4.3), we obtain

$$\begin{aligned}\mathcal{N}(p)&=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\,\zeta^{k}\,(\delta)_{n-k}\;p^{-(\nu k-\nu n-\mu)}}{k!\,(n-2k)!}-\omega^{\nu}p^{-\nu}\mathcal{N}(p),\\ &=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\,\zeta^{k}\,(\delta)_{n-k}}{k!\,(n-2k)!\,p^{-(\nu n-\nu k+\mu)}}\left(1+\omega^{\nu}p^{-\nu}\right)^{-1}.\end{aligned} \tag{4.7}$$

Computing the Laplace inverse of (4.7) and using

$$L^{-1}\left[p^{-\nu}:\tau\right]=\frac{\tau^{\nu-1}}{\Gamma(\nu)},$$

we obtain that

$$L^{-1}\{\mathcal{N}(p)\}=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\,\zeta^{k}\,(\delta)_{n-k}}{k!\,(n-2k)!}\times\sum_{r=0}^{\infty}(-1)^{r}\omega^{\nu r}L^{-1}\left\{p^{-(\nu n-\nu k+\nu r+\mu)}\right\}, \tag{4.8}$$

which can be rewritten as

$$\begin{aligned}\mathcal{N}(\tau)&=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\,\zeta^{k}\,(\delta)_{n-k}}{k!\,(n-2k)!}\times\sum_{r=0}^{\infty}\frac{(-1)^{r}\omega^{\nu r}\tau^{\nu n-\nu k+\nu r+\mu-1}}{\Gamma(\nu n-\nu k+\nu r+\mu)},\\ &=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{\nu(n-2k)}\,\zeta^{k}\,(\delta)_{n-k}}{k!\,(n-2k)!}\tau^{\nu n-\nu k+\mu-1}\times\sum_{r=0}^{\infty}\frac{(-1)^{r}\omega^{\nu r}\tau^{\nu r}}{\Gamma(\nu r+\nu n-\nu k+\mu)},\end{aligned} \tag{4.9}$$

which on using (1.2), we get the desired result (4.4). □

Remark 4.1. *The appearance of the Mittag–Leffler–Gegenbauer polynomials in Eq. (4.3) reflects their role as a bridge between fractional dynamics and classical polynomial structures. These polynomials encode both the oscillatory (via Gegenbauer-type terms) and memory (via the Mittag–Leffler function) effects inherent in fractional systems. The solution (4.4) shows that the temporal evolution of $\mathcal{N}(\tau)$ is governed by a superposition of modes weighted by the Mittag–Leffler function $E_{\nu,\nu n-\nu k+\mu}(-(\omega\tau)^{\nu})$, which describes subdiffusive or relaxation-type behaviors depending on ν. Hence, the new Mittag–Leffler–Gegenbauer polynomials provide a natural fractional extension of classical polynomial solutions to kinetic equations, capturing both local and nonlocal effects in time. The following figure (Figure 6) provides a comprehensive illustration of the behavior of the Mittag–Leffler–Gegenbauer polynomials in the fractional kinetic equation, showing the effects of varying both ν and δ.*

Remark 4.2. *A particular case of the series representation (2.7) yields*

$${}_{E}C_{n}^{1}(x;1,1)=H_n(x),$$

where $H_n(x)$ denotes the classical Hermite polynomials. This special case provides a useful guideline for deriving the following corollary.

Corollary 4.1. *The equation*

$$\mathcal{N}(\tau)-\mathcal{N}_0\,\tau^{\frac{n}{2}}H_n\left(-\xi\tau^{\frac{1}{2}}\right)=-\omega_0\hat{D}_{\tau}^{-1}\mathcal{N}(\tau) \tag{4.10}$$

has the solution

$$\mathcal{N}(\tau)=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{n-2k}\,(1)_{n-k}}{k!\,(n-2k)!}\,\tau^{n-k}E_{1,n-k+1}\left(-(\omega\tau)\right), \tag{4.11}$$

Proof. By letting $\nu=\mu=\delta=1$ in Theorem 4.1, we obtain corollary 4.1. □

In Theorem 4.1, we derived the solution of the kinetic equation involving the (MLGP) ${}_{E}C_{n}^{\delta}(\tau,\zeta;\alpha,\beta)$ with respect to the variable τ. In the following theorem, we establish the corresponding solution with respect to ζ.

Theorem 4.2. *If $\mu>0$ and $\nu>0$, then the equation*

$$\mathcal{N}(\zeta)-\mathcal{N}_0\,\zeta^{\nu n+\mu-1}\,{}_{E}C_{n}^{\delta}\left(\tau,-\xi^{\nu}\zeta^{-\nu};\nu,\mu\right)=-\omega^{\nu}\,{}_0\hat{D}_{\zeta}^{-\nu}\mathcal{N}(\zeta) \tag{4.12}$$

has the solution

$$\mathcal{N}(\zeta)=\mathcal{N}_0\sum_{r=0}^{\infty}\omega^{r\nu}\zeta^{nr+\nu r+\mu-1}\;{}_{E}C_{n}^{\delta}\left(\tau,-\xi^{\nu}\zeta^{-\nu};\nu,\mu+r\nu\right). \tag{4.13}$$

Proof. Applying the Laplace transform to (4.12), we obtain

$$\begin{aligned}\mathcal{N}(p)&=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{k\nu}\,\tau^{n-2k}\,(\delta)_{n-k}\;p^{-(n\nu-k\nu-\mu)}}{k!\,(n-2k)!}-\omega^{\nu}p^{-\nu}\mathcal{N}(p),\\ &=\mathcal{N}_0\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(-\xi)^{k\nu}\,\tau^{n-2k}\,(\delta)_{n-k}\;p^{-(n\nu-k\nu-\mu)}}{k!\,(n-2k)!}\left(1+\omega^{\nu}p^{-\nu}\right)^{-1}.\end{aligned} \tag{4.14}$$

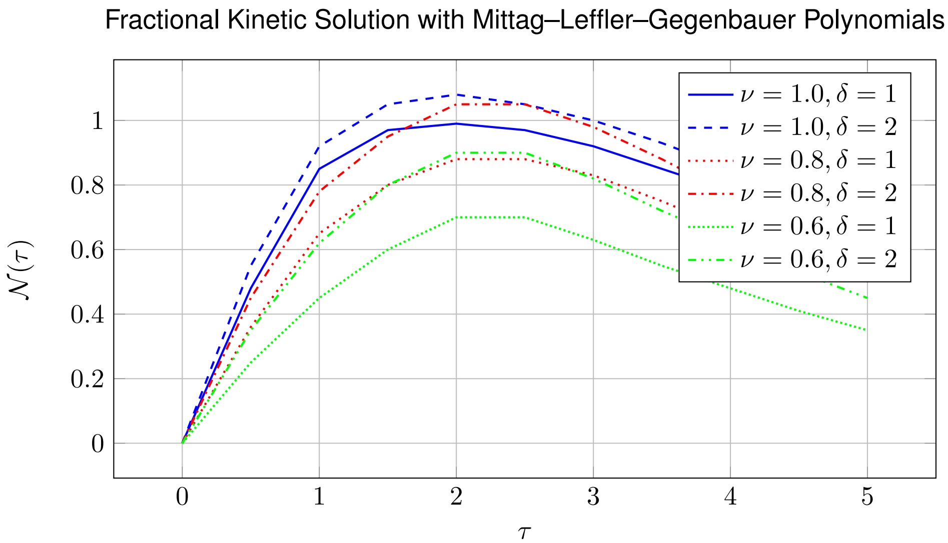

Figure 6: Illustration of the fractional kinetic solution $\mathcal{N}(\tau)$ for multiple fractional orders ν and polynomial parameters δ. Decreasing ν enhances memory effects, slowing the decay, while increasing δ increases the amplitude due to higher polynomial weighting.

Computing the Laplace inverse of (4.14) and simplify the resulting expression, we obtain

$$\mathcal{N}(\zeta) = \mathcal{N}_0 \sum_{r=0}^{\infty} \omega^{r\nu} \, \zeta^{n\nu + r\nu + \mu - 1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-\xi)^k \, \tau^{\, n-2k} \, (\delta)_{n-k} \, \zeta^{-\nu}}{k! \, (n-2k)! \, \Gamma(n\nu + r\nu - k\nu + \mu)}, \tag{4.15}$$

which, upon using the definition given in (2.7), yields the desired result (4.13). □

From Theorem 4.2 and relation (2.28), we immediately obtain the following corollary.

Corollary 4.2. *If $\mu > 0$ and $\nu > 0$, then the equation*

$$\mathcal{N}(\zeta) - \mathcal{N}_0 \, \zeta^{\nu n + \mu - 1} {}_E P_n \left(\tau, -\xi^\nu \zeta^{-\nu}; \nu, \mu\right) = -\omega^\nu {}_0\hat{D}_\zeta^{-\nu} \mathcal{N}(\zeta) \tag{4.16}$$

has the solution

$$\mathcal{N}(\zeta) = \mathcal{N}_0 \sum_{k=0}^{\infty} \omega^{k\nu} \zeta^{nr + \nu r + \mu - 1} \; {}_E P_n \left(\tau, -\xi^\nu \zeta^{-\nu}; \nu, \mu + r\nu\right). \tag{4.17}$$

Proof. Setting $\delta = \frac{1}{2}$ in Theorem 4.2 immediately gives Corollary 4.2. □

Example 4.1 (Numerical illustration for kinetic equation)**.** *Consider Corollary 4.1 with the parameters*

$$n = 3, \quad \xi = 0.5, \quad \omega = 1, \quad \mathcal{N}_0 = 1, \quad \tau = 1.$$

The solution formula (4.11) *is*

$$\mathcal{N}(\tau) = \mathcal{N}_0 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!(-\xi)^{\, n-2k} (1)_{\, n-k}}{k!(n-2k)!} \, \tau^{\, n-k} E_{1,n-k+1}(-\omega\tau).$$

For $n = 3$, the summation runs over $k = 0, 1$:

$$\begin{aligned}
\textit{For } k = 0: \quad & \frac{3!(-0.5)^3 (1)_3}{0!(3-0)!} \tau^{3-0} E_{1,4}(-1) \\
&= \frac{6 \cdot (-0.125) \cdot 6}{1 \cdot 6} \cdot 1 \cdot E_{1,4}(-1) = -0.75 \, E_{1,4}(-1), \\
\textit{For } k = 1: \quad & \frac{3!(-0.5)^1 (1)_2}{1!(3-2)!} \tau^{3-1} E_{1,3}(-1) \\
&= \frac{6 \cdot (-0.5) \cdot 2}{1 \cdot 1} \cdot 1 \cdot E_{1,3}(-1) = -6 \, E_{1,3}(-1).
\end{aligned}$$

Approximating the Mittag-Leffler functions with the first few terms:

$$E_{1,4}(-1) \approx 0.1333, \qquad E_{1,3}(-1) \approx 0.375.$$

Thus,

$$\begin{aligned}
\mathcal{N}(1) &\approx -0.75 \cdot 0.1333 - 6 \cdot 0.375 \\
&\approx -0.100 - 2.25 \\
&\approx -2.35.
\end{aligned}$$

Hence, for the chosen parameters, the solution of the fractional kinetic equation (4.10) *is approximately*

$$\mathcal{N}(1) \approx -2.35.$$

This demonstrates the practical relevance of Corollary 4.1, showing how the classical Hermite polynomials and the Mittag-Leffler function combine to provide an explicit solution.

5 Conclusion and future work

In this paper, we introduced the Two-Variable Mittag–Leffler–Gegenbauer polynomials using the symbolic operator approach, generalizing classical polynomials such as Laguerre, Hermite, and Gegenbauer. We studied their main properties, including series representations, generating functions, operational rules, and fractional integral relations, and illustrated their practical relevance with numerical examples and applications to fractional kinetic equations.

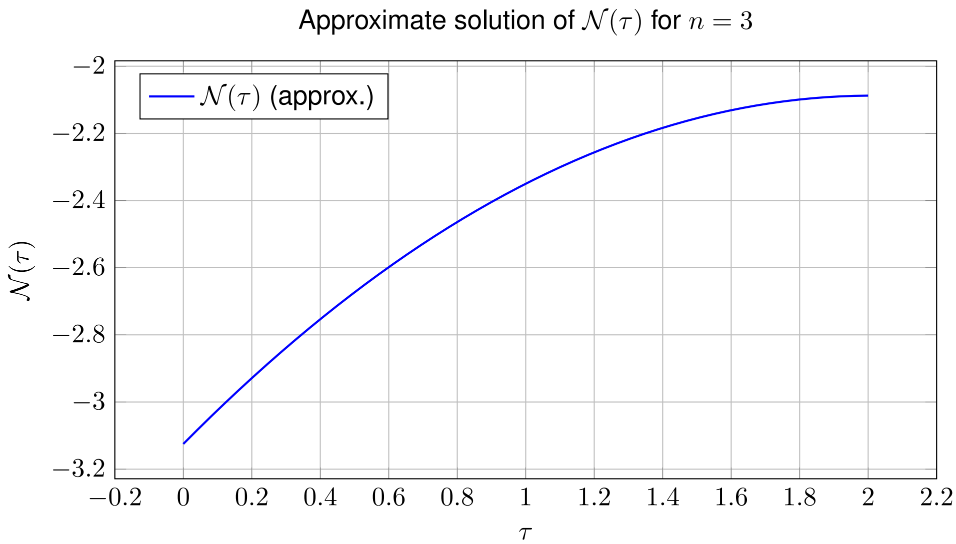

Figure 7: Plot of the approximate solution $\mathcal{N}(\tau)$ versus τ for $n = 3$, $\xi = 0.5$, $\omega = 1$, $\mathcal{N}_0 = 1$.

The results demonstrate that polynomials provide a flexible framework for both theoretical analysis and applied problems in fractional calculus and related areas. Future work includes exploring orthogonality properties, higher-dimensional generalizations, further applications in fractional differential equations and stochastic processes, and the development of efficient computational methods for practical evaluations of these polynomials.

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable

Availability of data and materials

Not applicable.

Author's contribution

The authors confirm their contribution to the paper as follows: study conception and design: Maged Bin-Saad, theoretical calculations and modeling: Waleed Mohammed; data analysis and validation: Maged Bin-Saad, Waleed Mohammed, manuscript preparation: Maged Bin-Saad, Waleed Mohammed. All authors reviewed the results and approved the final version of the manuscript.

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