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On the coherent states associated with four-parameter Mittag-Leffler functions

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Abstract: In order to examine the generalized coherent states, defined in the Barut-Girardello manner, whose normalization function is just the four-parameter generalized Mittag-Leffler function, we used a special pair of non-linear operators which generate these states. Some characteristic properties for pure, as well as mixed (thermal) coherent states are also examined. All calculations are made using the rules of the diagonal ordering of operators (DOOT). Finally, the integral counterpart of the Mittag-Leffler coherent states are examined which is connected with *nu*-function. The paper is an example of the application of a mathematical entities (Mittag-Leffler function and *nu*-function) in quantum mechanics (coherent states formalism).

Keywords: Mittag-Leffler function, coherent states, operator ordering, thermal states, energy spectra.

Introduction and preliminaries

The Mittag-Leffler function is not only an important mathematical entity belonging to the category of special functions, but also a function with potential applications in the coherent states formalism. By implication, coherent states represent an important concept in quantum mechanics, quantum optics, and laser theory. Also, the connection between these functions and fractional calculus, a relatively new concept but with many possible applications, has made the study of Mittag-Leffler functions and their properties an interesting topic for many scientists.

The Mittag-Leffler function was introduced in 1903 [1] as an entire complex special function of complex variable $z = |z| \exp(i\varphi)$ which depends on some complex parameter α . Later, the definition was extended, adding one, two, three or more complex parameters β, γ, k .... The history of this function can be traced in [2, 3]:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)} ,$$
$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)} , \qquad (1.1)$$
$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\beta+\alpha n)} \frac{z^n}{n!}$$

Here $\Gamma(1+\alpha n)$ is the Euler gamma function and $\alpha, \beta, \gamma \in C$ are complex parameters, with the only restriction $\mathcal{R}e(\alpha) > 0$.

Moreover, a superior step of generalization was recently considered by Srivastava and Tomovski [4], the form that we will deal with in this paper. The only difference introduced, compared to the Srivastava and Tomovski notation, is the exchange of the places of the lower indices α and β, because of the roles they play in the definition of the Mittag-Leffler function, index α being similar to index k. Consequently, we will use $E_{\beta,\alpha}^{\gamma,k}(z)$ instead of $E_{\alpha,\beta}^{\gamma,k}(z)$:

$$E_{\beta,\alpha}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_{\alpha}(\beta+\alpha n)} \frac{z^n}{n!} \qquad (1.2)$$
$$z, \alpha, \beta, \gamma \in C, \ \mathcal{R}e(\alpha) > 0, \ \mathcal{R}e(k) > 0$$

In recent years, other generalizations of the Mittag-Leffler function, characterized by several indices, have been developed (see, for example, [5] and the references therein).

In the present paper the Srivastava and Tomovski version is used, which we will briefly call hereafter the *generalized Mittag-Leffler function* (GM-L). In order to avoid possible confusion, compared to the articles in the specialized literature, we inserted, in the writing of the GM-L function, the α index signifying the rate of change of the argument of the k-Gamma function.In the above definition the generalized Pochhammer symbols (or Pochhammer *k*-symbols) are used [6]

$$(\gamma)_{n,k} = \sum_{j=0}^{n-1} (\gamma + jk) = \frac{\Gamma_k(\gamma + nk)}{\Gamma(\gamma)} \qquad (1.3)$$

where $\Gamma_k(\gamma + nk)$ is the generalized k – gamma function, defined by in two manner:

$$\Gamma_k(x) = \int_0^{\infty} dt\, e^{-\frac{t^k}{k}} t^{x-1} , \quad \Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) \qquad (1.4)$$

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The connection between the generalized Pochhammer *k* – symbol and usual Pochhammer symbol is

$$(x)_{n,k}=k^n\left(\frac{x}{k}\right)_n=k^n\frac{\Gamma\left(\frac{x}{k}+n\right)}{\Gamma\left(\frac{x}{k}\right)} \tag{1.5}$$

$$(x)_{0,k}=\left(\frac{x}{k}\right)_0=1\ ,\ (x)_{n,0}=x^n$$

The generalized gamma function $\Gamma_\alpha(\beta+\alpha n)$ is connected with usual Gamma function as

$$\Gamma_\alpha(\beta+\alpha n)=\alpha^n\left(\frac{\beta}{\alpha}\right)_n\Gamma(\beta)=\Gamma(\beta)(\beta)_{n,\alpha} \tag{1.6}$$

So, the generalized Mittag-Leffler function becomes

$$E_{\beta,\alpha}^{\gamma,k}(z)=\frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{(\beta)_{n,\alpha}}\frac{z^n}{n!} \tag{1.7}$$

Shortly, we will call this function the *generalized Mittag-Leffler function* (GM-L), and it can be linked with a generalized hypergeometric function (GHG), as:

$$E_{\beta,\alpha}^{\gamma,k}(z)=\frac{1}{\Gamma(\beta)}\,{}_1F_1\left(\frac{\gamma}{k};\frac{\beta}{\alpha};\frac{k}{\alpha}z\right) \tag{1.8}$$

The purpose of this paper is to build such a type of coherent states (CSs) that has the generalized Mittag-Leffler function as its normalization function. Hereby we will highlight a new application of the generalized Mittag-Leffler function, apart from those indicated in the literature [2], [7], [8]. But as the normalization function of the generalized CSs is a generalized hypergeometric function, by default we will capitalize on the connection between the generalized Mittag-Leffler function and a certain generalized hypergeometric function, materialized by the relation above.

In our paper, the construction of coherent states associated with four-parameter Mittag-Leffler functions is based both on the traditional method of constructing coherent states for an arbitrary quantum system, as well as on several original steps in this approach.

Compared to the results known from the literature, the novelty in our approach consists of the following: Instead of the traditional creation / annihilation (canonical) operators, we introduced a pair of new operators, the creation $\mathcal{A}_+$ and the annihilation $\mathcal{A}_-$, which generate coherent states whose normalization function is the generalized hypergeometric function.This gives the calculation a general character, applicable to other quantum systems for which we want to construct coherent states. In addition, we used the newly introduced diagonal normal ordering technique (DOOT) which essentially states that these operators commute with each other and, consequently, if they are under the integration sign, they can be treated as simple *c*-numbers [12].

This way the calculations will be much easier to do.

Let us point out that this approach is not specific only to Mittag-Leffler functions, it can also be used for other quantum systems (for example linear or nonlinear oscillators) for which the construction of coherent states is desired.

The most general form of a set of CSs $|z>$, expanded into an orthogonal basis, for example the Fock vector basis $\{|n>,\ n=0,1,2,...,\infty\}$, is

$$|z>=\frac{1}{\sqrt{{}_pF_q(\{a_i\}_1^p;\{b_j\}_1^q;|z|^2)}}\times \\ \times\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{p,q}(n)}}|n> \tag{1.9}$$

where the complex variable $z=|z|\exp(\mathrm{i}\varphi)$ labels the CSs, $0\le\varphi\le 2\pi$, and the positive quantities $\rho_{p,q}(n)$, called the structure constants, determine the mathematical structure of CSs. The normalization function of the most general CSs is a hypergeometric generalized function (HGF) ${}_pF_q\left(\{a_i\}_1^p;\{b_j\}_1^q;|z|^2\right)$, with the new introduced notation $\{x_i\}_1^m\equiv x_1,x_2,x_3,..,x_m$.

These kinds of states were firstly introduced by Appl and Schiller [9] and later applied to the thermal states of the pseudoharmonic oscillator in one of our previous papers [10].

Any CSs exists (makes physical sense) only if their normalization function is an analytical function of real variable $|z|^2$. In addition, any CSs $|z>$ must accomplish some conditions [11]: continuity in complex label, normalization, non-orthogonality, resolution of unity operator with unique positive weight function $h(|z|^2)$ of the integration measure $d\mu(z)$.

A very important relation in the CSs formalism is the *resolution of the unity operator*, or the *completeness relation*, i.e. the integral

$$\int d\mu(z)\,|z><z|=1\ , \tag{1.10}$$

Here the integration measure $d\mu(z)$ must have positive, unique and no oscillatory weight function $h(|z|^2)$:

$$d\mu(z)=\frac{d^2z}{\pi}h(|z|^2)=\frac{d\varphi}{2\pi}d(|z|^2)h(|z|^2) \tag{1.11}$$

The generalized coherent states are generated with the help of a pair of operators, the creation $\mathcal{A}_+$ and the annihilation $\mathcal{A}_-$, that act on the Fock vectors $|n>$ in the space attached to the studied quantum system. They obey the rules of a normal ordering technique - diagonal ordering operation technique (DOOT), indicated by the symbol $\#\ \#$, which has the following rules [12]: *a)* Inside the symbol $\#\ \#$, the order of the operators $\mathcal{A}_-$ and $\mathcal{A}_+$ can be permuted like commutable operators, so that finally will result a normal ordering: $\mathcal{A}_+$ on the left, and $\mathcal{A}_-$ on the right; *b)* A symbol $\#\ \#$ inside another symbol $\#\ \#$ can be deleted; *c)* A normally ordered product of operators can be integrated or differentiated, with respect to *c*-numbers, according to the usual rules. In essence, the operators $\mathcal{A}_-$ and $\mathcal{A}_+$ are considered as being simple *c*-numbers and can be taken out from the symbol $\#\ \#$; *d)* The projector $|0><0|$ of the normalized vacuum state $|0>$, in the frame of DOOT, is the reciprocal function ($1/ f(x)$) of the normalization function of CSs, in normal order, i. e. the reciprocal function of normal ordered generalized hypergeometric function:

$$|0><0|=\# \frac{1}{{}_pF_q\left(\{a_i\}_1^p;\{b_j\}_1^q;A_+A_-\right)}\# \qquad (1.12)$$

Coherent states of the generalized Mittag-Leffler functions

Let's point out that, for the first time, the CSs for the ordinary one-parametrical and two-parametrical Mittag-Leffler functions were examined by Sixdeniers, Penson and Solomon [13].

Now, let us we choose two generating operators, lowering $\mathcal{A}_{-(\beta,\alpha)}^{(\gamma,k)} \equiv \mathcal{A}_-$, respectively raising $\mathcal{A}_{+(\beta,\alpha)}^{(\gamma,k)} \equiv \mathcal{A}_+$, where $(\mathcal{A}_-)^+ = \mathcal{A}_+$. To simplify writing formulas, we will use the operator notation for $\mathcal{A}_-$ and $\mathcal{A}_+$, without mentioning their indices. The pair of operators was chosen in such a way that their actions on the Fock vectors is of the following form:

$$\begin{aligned} &\mathcal{A}_-|n>=\sqrt{e(n)}\,|n-1> \\ &\mathcal{A}_+|n>=\sqrt{e(n+1)}\,|n+1> \\ &e(n)\equiv n\,\frac{(\beta+\alpha\,(n-1))}{\gamma+k\,(n-1)} \end{aligned} \qquad (2.1)$$

which, as will be seen below, will generate the coherent states of the generalized Mittag-Leffler functions (GML-CSs).

By successively applying these operators on the vacuum states $|0>$ and $<0|$ we obtain

$$\begin{aligned} \left\{\begin{matrix}|n> \\ \\ <n|\end{matrix}\right\} &= \sqrt{\frac{(\gamma)_{n,k}}{(\beta)_{n,\alpha}\;n!}}\left\{\begin{matrix}(\mathcal{A}_+)^n|0> \\ \\ <0|(\mathcal{A}_-)^n\end{matrix}\right\}= \\ &=\sqrt{\Gamma(\beta)}\sqrt{\frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta+\alpha\,n)}}\left\{\begin{matrix}(\mathcal{A}_+)^n|0> \\ \\ <0|(\mathcal{A}_-)^n\end{matrix}\right\} \end{aligned} \qquad (2.2)$$

Using the completeness relation of the Fock-vectors $\sum_{n=0}^{\infty}|n><n|=1$ and the rules of the DOOT technique, the expression of the vacuum projector is obtained:

$$\begin{aligned} &\sum_{n=0}^{\infty}|n><n|= \\ &=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{(\beta)_{n,\alpha}\;n!}\#(\mathcal{A}_+)^n|0><0|(\mathcal{A}_-)^n\#= \\ &=|0><0|\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{(\beta)_{n,\alpha}}\frac{\#(\mathcal{A}_+\mathcal{A}_-)^n\#}{n!}= \\ &=|0><0|\Gamma(\beta)\#E_{\alpha,\beta}^{\gamma,k}(\mathcal{A}_+\mathcal{A}_-)\#=1 \end{aligned} \qquad (2.3)$$

From here it follows that the vacuum projector is

$$\begin{aligned} |0><0|&=\frac{1}{\Gamma(\beta)\#E_{\beta,\alpha}^{\gamma,k}(\mathcal{A}_+\mathcal{A}_-)\#}= \\ &=\frac{1}{\#_1F_1\left(\frac{\gamma}{k};\frac{\beta}{\alpha};\frac{k}{\alpha}\mathcal{A}_+\mathcal{A}_-\right)\#} \end{aligned} \qquad (2.4)$$

in accordance with DOOT rule *d)*. This means that the CSs normalization function for GML-CSs is just the four-parametrical Mittang-Leffler function. This assertion can be verified by explicitly constructing coherent states based on the GM-L functions.

First, we use the Barut Girardello definition of coherent states [14]:

$$\mathcal{A}_-|z>=z|z>\ ,\quad <z|\mathcal{A}_+=z^*<z| \qquad (2.5)$$

Using this definition, as well as applying the norming condition $<z|z>=1$, the development of CSs according to the Fock vectors $|z>=\sum_n c_n(z)|n>$ leads to the expression

$$|z>=\frac{1}{\sqrt{E_{\beta,\alpha}^{\gamma,k}(|z|^2)}}\sum_{n=0}^{\infty}\sqrt{\frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta+\alpha\,n)}}\frac{z^n}{\sqrt{n!}}|n> \qquad (2.6)$$

Using Eq. (2.2), the GML-CSs can be written also as

$$|z>=\sqrt{\Gamma(\beta)}\frac{1}{\sqrt{E_{\beta,\alpha}^{\gamma,k}(|z|^2)}}E_{\beta,\alpha}^{\gamma,k}(z\mathcal{A}_+)|0> \qquad (2.7)$$

It can be seen that the generalized Mittag-Leffler function plays the role of normalization function for CSs.

The above expression of CSs satisfies the Klauder's conditions [13], i.e.: continuity in complex label, normalization, non-orthogonality, unity operator resolution with unique positive weight function $h(|z|^2)$ of the integration measure $d\mu(z)$.

The vectors $|z>$ are the strong continuous functions of label $z\neq 0$, i.e. for every convergent label such that $z'\to z$ it follows that $\|\,|z'>-|z>\|\to 0$.

From the expression of the scalar product of two CSs we can see the properties of normalization and non-orthogonality:

$$\begin{aligned} <z|z'>&=\frac{E_{\beta,\alpha}^{\gamma,k}(z^*z')}{\sqrt{E_{\beta,\alpha}^{\gamma,k}(|z|^2)\,E_{\beta,\alpha}^{\gamma,k}(|z'|^2)}}= \\ &=\begin{cases}1, & \text{if}\quad z'=z\quad \text{normalization to unity} \\ \neq 0, & \text{if}\quad z'\neq z\quad \text{non-orthogonality}\end{cases} \end{aligned} \qquad (2.8)$$

To show the validity of the unity operator decomposition relation (1.10), let us try to find the expression of the integration measure. After the corresponding substitutions, the angular integration is

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}(z^*)^n z^{n'}=(|z|^2)^n\,\delta_{nn'} \qquad (2.9)$$

Changing the index $n=s-1$ it turns out that the following integral equation (problem of moments) must be solved [14]:

$$\int_0^\infty d\left(|z|^2\right)\frac{h\left(|z|^2\right)}{E_{\beta,\alpha}^{\gamma,k}\left(|z|^2\right)}\left(|z|^2\right)^{s-1}=$$

$$=\frac{k}{\alpha}\frac{\Gamma\left(\frac{\gamma}{k}\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)}\Gamma(\beta)\frac{1}{\left(\frac{k}{\alpha}\right)^s}\frac{\Gamma(s)\Gamma\left(\frac{\beta}{\alpha}-1+s\right)}{\Gamma\left(\frac{\gamma}{k}-1+s\right)} \quad (2.10)$$

The solution of this equation being expressed through Meijer's G-function [14]

$$\frac{h\left(|z|^2\right)}{E_{\beta,\alpha}^{\gamma,k}\left(|z|^2\right)}=\frac{k}{\alpha}\frac{\Gamma\left(\frac{\gamma}{k}\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)}\Gamma(\beta)\times$$

$$\times G_{1,2}^{2,0}\left(\frac{k}{\alpha}|z|^2\left|\begin{matrix}/\;; & \frac{\gamma}{k}-1\\ 0\;,\;\frac{\beta}{\alpha}-1; & /\end{matrix}\right.\right) \quad (2.11)$$

the final expression for the integration measure is

$$d\mu(z)=\frac{k}{\alpha}\frac{\Gamma\left(\frac{\gamma}{k}\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)}\Gamma(\beta)\times$$

$$\times\frac{d\varphi}{2\pi}d\left(|z|^2\right)E_{\beta,\alpha}^{\gamma,k}\left(|z|^2\right)\times \quad (2.12)$$

$$\times G_{1,2}^{2,0}\left(\frac{k}{\alpha}|z|^2\left|\begin{matrix}/\;; & \frac{\gamma}{k}-1\\ 0\;,\;\frac{\beta}{\alpha}-1; & /\end{matrix}\right.\right)$$

For any set of vectors to act as coherent states, it must satisfy some minimal conditions. These conditions were formulated by Klauder [15] and are sometimes known as "Klauder's restrictions".

In this way, above we showed that Mittag-Leffler CSs fulfill all Klauder's conditions.

To verify the correctness of the above calculations, let us consider $\alpha=\beta=\gamma=k=1$, In this case the Mittag-Leffler function is $E_{1,1}^{1,1}\left(|z|^2\right)=\exp\left(|z|^2\right)$, as well as the Meijer *G*-function $G_{0,1}^{1,0}\left(|z|^2\,|\,0\right)=\exp\left(-|z|^2\right)$.

Then we obtain just the integration measure of the one-dimensional harmonic oscillator (HO-1D)

$$d\mu(z)=\frac{d\varphi}{2\pi}d\left(|z|^2\right)=\frac{d^2z}{\pi}\;.$$

This proves the correctness of the above calculation.

In the above calculus the classical and generalized integral in which it appears a single Meijer function G are used:

$$\int_0^\infty d\left(|z|^2\right)\left(|z|^2\right)^{s-1}\times$$

$$\times G_{1,2}^{2,0}\left(\frac{k}{\alpha}|z|^2\left|\begin{matrix}/\;; & \frac{\gamma}{k}-1\\ 0\;,\;\frac{\beta}{\alpha}-1; & /\end{matrix}\right.\right)= \quad (2.13)$$

$$=\frac{1}{\left(\frac{k}{\alpha}\right)^s}\frac{\Gamma(s)\Gamma\left(\frac{\beta}{\alpha}-1+s\right)}{\Gamma\left(\frac{\gamma}{k}-1+s\right)}$$

Generally, with the help of DOOT rules, the projector onto the one CS $|z>$ is

$$|z><z|=$$

$$=\frac{1}{E_{\beta,\alpha}^{\gamma,k}\left(|z|^2\right)}\#\frac{E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+z\right)E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_-z^*\right)}{E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)}\# \quad (2.14)$$

Using the following limit

$$\lim_{z\to 0}E_{\beta,\alpha}^{\gamma,k}(z)=\frac{1}{\Gamma(\beta)} \quad (2.15)$$

as well as the completeness relation, we will obtain, on the one hand, the correct expression of the vacuum projector from Eq. (2.14), and on the other hand, the following two integrals:

1.) Angular integral of a product of two M-L functions, with complex conjugate arguments

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\#E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+z\right)E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_-z^*\right)\#=$$

$$=\sum_{n=0}^\infty\left[\frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta+\alpha n)}\frac{1}{n!}\right]^2\#\left(\mathcal{A}_+\mathcal{A}_-\right)^n\#\left(|z|^2\right) \quad (2.16)$$

2.) The integral in complex space of the product of the two M-L functions above, with complex conjugate arguments, respecting, as in the case of the integral above, the DOOT ordering rules.

$$\int\frac{d^2z}{\pi}G_{1,2}^{2,0}\left(\frac{k}{\alpha}|z|^2\left|\begin{matrix}/\;; & \frac{\gamma}{k}-1\\ 0\;,\;\frac{\beta}{\alpha}-1; & /\end{matrix}\right.\right)\times$$

$$\times\#E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+z\right)E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_-z^*\right)\#= \quad (2.17)$$

$$=\frac{\alpha}{k}\frac{\Gamma\left(\frac{\gamma}{k}\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)}\frac{1}{\Gamma(\beta)}\#E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)\#$$

The Laplace type integral, or the Laplace transform of the generalized Mittag-Leffler function is

$$\int_0^\infty \frac{d^2 z}{\pi} e^{-s|z|^2} E_{\beta,\alpha}^{\gamma,k}\left(|z|^2\right) = \frac{1}{s} \frac{1}{\Gamma\left(\frac{\gamma}{k}\right)} \frac{1}{\Gamma(\beta)} \times$$
$$\times {}_2F_1\left(1, \frac{\gamma}{k};\ \frac{\beta}{\alpha}; \frac{k}{\alpha} \frac{1}{s}\right) \tag{2.18}$$

Particularly, for $\alpha = \beta = \gamma = k = 1$, the Mittag-Leffler function is $E_{1,1}^{1,1}\left(|z|^2\right) = \exp\left(|z|^2\right)$ and the Laplace transform becomes an integral of an exponential having the value $(1-s)^{-1}$.

Taking into account the definition of coherent states in the Barut-Girardello manner (BG-CSs), for a function that depends on the ordered product of $\#f\left(\mathcal{A}_+\mathcal{A}_-\right)\#$ operators, following the DOOT rules, it can be shown that its expected value in the coherent state $|z>$ is:

$$<z|\#f\left(\mathcal{A}_+\mathcal{A}_-\right)\#|z> = f\left(|z|^2\right) \tag{2.19}$$

It is useful to emphasize here that coherent states can also be defined in the *Klauder-Perelomov manner*, that is, as the result of the action of the displacement operator on the vacuum state [15]:

$$|z> = \frac{1}{\sqrt{N\left(|z|^2\right)}} \exp\left(z\mathcal{A}_+\right)|0> =$$
$$= \frac{1}{\sqrt{{}_qF_p\left(\{b_j\}_1^q;\ \{a_i\}_1^p;|z|^2\right)}} \sum_{n=0}^\infty \frac{z^n}{\sqrt{\rho_{q,p}(n)}}|n> \tag{2.20}$$

From the algebraic point of view, CSs in the Klauder-Perelomov manner (KP-CSs) have the same structure as CSs in the Barut-Girardello manner (BG-CSs), with the fundamental difference that it exists an interchange of the places of the indices p and q, as well as of the sets of constants $\{a_i\}_1^p$ and $\{b_j\}_1^q$. This is an expression of the dualism of the two types of CSs. Therefore, the expression obtained above for BG-CSs are similar for KP-CSs, and that's why we won't repeat it here.

Generalized Mittag-Leffler functions involved in thermal states

It is well known from quantum mechanics books that a system in thermal equilibrium at temperature $T = \left(k_B \beta_B\right)^{-1}$ with the ambient medium is characterized by the canonical density operator, where k_B is the Boltzmann constant:

$$\rho = \frac{1}{Z(\beta_B)} \sum_{n=0}^\infty \exp\left(-\beta_B E_n\right)|n><n| \tag{3.1}$$

Using Eqs. (2.3) and (2.4) and the DOOT rules, the density operator ρ becomes can be expressed as a function of the operator product $\mathcal{A}_+\mathcal{A}_-$:

$$\rho = \frac{1}{Z(\beta_B)} \frac{1}{\#E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)\#} \times$$
$$\times \sum_{n=0}^\infty \exp\left(-\beta_B E_n\right) \frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta + \alpha n)} \frac{\#\left(\mathcal{A}_+\mathcal{A}_-\right)^n\#}{n!} \tag{3.2}$$

Let's turn our attention to the quantity defined as follows, Eq. (2.1):

$$e(n) \equiv n \frac{\beta + \alpha(n-1)}{\gamma + k(n-1)} \tag{3.3}$$

For relatively simple quantum systems, the energy spectrum is either linear or quadratic, in relation to the main quantum number n.

If $\alpha = 0$ and $k = 0$, then $e(n) = \frac{\beta}{\gamma} n$, which corresponds to a *linear spectrum* (characteristic for the linear harmonic oscillator or the pseudoharmonic oscillator).

If $\alpha \neq 0$ and $k = 0$, the expression is quadratic $e(n) = \frac{\beta}{\gamma}(1-\alpha)n + \frac{\beta}{\gamma}\alpha n^2$, which corresponds to a *quadratic spectrum* (characteristic for the nonlinear or anharmonic harmonic oscillators, like Morse oscillator or some Pöschl-Teller-type potentials and so on.

For *linear spectra* the density operator becomes

$$\rho_{\text{lin}} = \frac{1}{Z(\beta_B)} \frac{1}{\Gamma(\beta_B)} \frac{1}{\#E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)\#} \times$$
$$\times \sum_{n=0}^\infty \frac{(\gamma)_{n,k}}{(\beta)_{n,\alpha}} \frac{\#\left(e^{-\beta_B \frac{\beta}{\gamma}} \mathcal{A}_+\mathcal{A}_-\right)^n\#}{n!} = \tag{3.4}$$
$$= \frac{1}{Z(\beta_B)} \# \frac{E_{\beta,\alpha}^{\gamma,k}\left(e^{-\beta_B \frac{\beta}{\gamma}} \mathcal{A}_+\mathcal{A}_-\right)}{E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)} \#$$

Generally, the Husimi's function (or Husimi's distribution) is defined as the diagonal elements of the density operator, in the CSs representation.

In our case, for the Mittag-Leffler CSs, the Husimi's distribution becomes

$$Q_{ML}\left(|z|^2\right) \equiv <z|\rho_{\text{lin}}|z> =$$
$$= \frac{1}{Z(\beta_B)} \frac{E_{\beta,\alpha}^{\gamma,k}\left(e^{-\beta_B \frac{\beta}{\gamma}} |z|^2\right)}{E_{\beta,\alpha}^{\gamma,k}(|z|^2)} \tag{3.5}$$

It can be observed that, compared to the expression above, $\mathcal{A}_+\mathcal{A}_-$ has been replaced with $|z|^2$ and this is a consequence of the definition of CSs in the sense of Barut and Girardello.

On the other hand, the ρ_{lin} density operator can also be written in diagonal form:

$$\rho_{\text{lin}} = \int d\mu(z) | z > P_{\text{lin}}\left(| z |^2 \right) < z | \qquad (3.6)$$

$$\rho_{\text{lin}} = \# \frac{1}{E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right)} \# \times$$

$$\times \int d\mu(z)\, P_{\text{lin}}\left(| z |^2 \right) \# \frac{E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+ z\right) E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_- z^*\right)}{E_{\beta,\alpha}^{\gamma,k}\left(| z |^2 \right)} \#$$

(3.7)

and it is necessary to find the quasi-distribution function $P_{\text{lin}}\left(| z |^2 \right)$.

Equating this equation with the second row of Eq. (3.4), substituting the expressions of the integration measure and CSs, after performing the angular integral, we have to solve the following equality

$$\frac{1}{Z(\beta_B)} \# E_{\beta,\alpha}^{\gamma,k}\left(e^{-\beta_B \frac{\beta}{\gamma}} \mathcal{A}_+\mathcal{A}_- \right) \# =$$

$$= \sum_{n=0}^{\infty} \left[\frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta + \alpha n)} \frac{\# \left(e^{-\beta_B \frac{\beta}{\gamma}} \mathcal{A}_+\mathcal{A}_- \right)^n \#}{n!} \right] \times \qquad (3.8)$$

$$\times \frac{1}{\left(\frac{\alpha}{k}\right)^n} \frac{1}{\left(e^{+\beta_B \frac{\beta}{\gamma}} \right)^n} \frac{1}{\Gamma(n+1)} \frac{\Gamma\left(\frac{\gamma}{k} + n\right)}{\Gamma\left(\frac{\beta}{\alpha} + n\right)} \times$$

$$\times \int_0^\infty d\left(| z |^2 \right) \left(| z |^2 \right)^n G_{1,\,2}^{2,\,0}\left(\frac{k}{\alpha} | z |^2 \,|\, ... \right) P_{\text{lin}}\left(| z |^2 \right)$$

Here it appears an integral equation similar with Eq. (2.13), but in what unknown function is $G_{1,\,2}^{2,\,0}\left(| ... \right) P_{\text{lin}}\left(| z |^2 \right)$.

Proceeding similarly as for the case of finding the integration measure, the final solution for the quasi-distribution function will be

$$P_{\text{lin}}\left(| z |^2 \right) = \frac{1}{Z(\beta_B)} e^{\beta_B \frac{\beta}{\gamma}} \times$$

$$\times \frac{G_{1,\,2}^{2,\,0}\left(\frac{k}{\alpha} e^{\beta_B \frac{\beta}{\gamma}} | z |^2 \,\middle|\, \begin{matrix} / \,; & \frac{\gamma}{k} - 1 \\ 0\,,\ \frac{\beta}{\alpha} - 1; & / \end{matrix} \right)}{G_{1,\,2}^{2,\,0}\left(\frac{k}{\alpha} | z |^2 \,\middle|\, \begin{matrix} / \,; & \frac{\gamma}{k} - 1 \\ 0\,,\ \frac{\beta}{\alpha} - 1; & / \end{matrix} \right)} \qquad (3.9)$$

For *quadratic* energy spectra, if the following constants are $\alpha \neq 0$ and $k = 0$, the corresponding quadratic expression is

$$E(n) \equiv e(n) = \frac{\beta}{\gamma}(1-\alpha)n + \frac{\beta}{\gamma}\alpha\, n^2 \equiv A n + B n^2$$

(where $\hbar\,\omega = 1$). In these cases, a specific *ansatz* is applied that we used for the first time in paper [16]. It consists in the power series development of the quadratic part of the energy exponential and the application of the above calculation for the linear part:

$$e^{-\beta_B E(n)} = e^{-\beta_B B n^2} \left(e^{-\beta_B A} \right)^n =$$

$$= \sum_{j=0}^{\infty} \frac{(-\beta_B B)^j}{j!} n^{2j} \left(e^{-\beta_B A} \right)^n =$$

$$= \sum_{j=0}^{\infty} \frac{(-\beta_B B)^j}{j!} \left(\frac{\partial}{\partial \beta_B A} \right)^{2j} \left(e^{-\beta_B A} \right)^n = \qquad (3.10)$$

$$= \exp\left[-\frac{1}{\beta_B} B \left(\frac{\partial}{\partial A} \right)^2 \right] \left(e^{-\beta_B A} \right)^n$$

In our case, the density operator becomes

$$\rho = \frac{1}{Z(\beta_B)} \frac{1}{\# E_{\beta,\alpha}^{\gamma,k}\left(\mathcal{A}_+\mathcal{A}_-\right) \#} \times$$

$$\times \exp\left[-\frac{1}{\beta_B} B \left(\frac{\partial}{\partial A} \right)^2 \right] \# E_{\beta,\alpha}^{\gamma,k}\left(e^{-\beta_B A} \mathcal{A}_+\mathcal{A}_- \right)^n \#$$

(3.11)

and the partition function is

$$Z(\beta_B) = \exp\left[-\frac{1}{\beta_B} B \left(\frac{\partial}{\partial A} \right)^2 \right] \sum_{n=0}^{\infty} \left(e^{-\beta_B A} \right)^n =$$

$$= \exp\left[-\frac{1}{\beta_B} B \left(\frac{\partial}{\partial A} \right)^2 \right] \frac{1}{1 - e^{-\beta_B A}} \qquad (3.12)$$

In practical calculations, only the first few terms are retained from the infinite sum, depending on the desire for accuracy of the calculations.

Generalized Mittag-Leffler functions for continuous spectra

Let us now deal with the case of continuous spectrum systems, in which, at the limit, the difference between the "energy levels" $\Delta E \equiv E_{n+1} - E_n = \varepsilon <<$ is very small, practically zero. Consequently, we can consider $E_n = n\varepsilon$, with $\varepsilon = 1$.

In [19] *for the continuous spectrum we introduced a real dimensionless energy parameter* $\varepsilon > 0$, which is *not a quanta*, and can be interpreted as a suitable "jump unity" in the energy scale of continuous spectra. By equating to unity $\varepsilon = 1$, the system's energy may be written simply as $E = m$.

Also, the coherent states for the continuous spectra were defined in [17] (see, also [18] and references therein) and [19].

The transition from discrete CSs (discontinuous) to continuous CSs is made if we adopt the following limit *d – c.* In paper [19] we studied the transition from the *discontinuous spectrum* (*d*) to the *continuous spectrum* (*c*) of an certain quantum system. We found that if a certain limit, called *the discrete – continuous limit* $d \rightarrow c$, is applied, a quantity that characterizes a system with a discontinuous spectrum will pass, at this limit, into the corresponding quantity connected with the continuous spectrum. There are some systems that have both a discontinuous and a continuous spectrum (for example, the diatomic molecule, whose inter nuclear potential is a Morse-type potential).

In the next section we will adopt a following *discrete – continuous limit* $d \rightarrow c$ limit:

$$\widetilde{X}_c(E) = \lim_{\substack{n\to E \\ n_{\max}\to\infty \\ \sum_{n=0}^{\infty} \to \int_0^{\infty} dE}} X_d(n, n_{\max}) \equiv \lim_{d\to c} X_d(n, n_{\max}) \ ,$$

$$\sum_{n=0}^{n_{\max}} X_d(n, n_{\max}) \rightarrow \int_0^{\infty} dE\, \widetilde{X}_c(E) \tag{4.1}$$

So, all observables $\widetilde{X}_c$ what characterizes the system with continuous spectrum will be obtained as a limiting case of the corresponding observables X_d of the discrete spectrum, through three operations: *a*) replacing $n \rightarrow E$, by the dimensionless energy E; *b*) the extension $n_{\max} \rightarrow \infty$; *c*) simultaneously, the sum with respect to n must be replaced by the integral with respect to E. For this reason we will call this the *generalized discrete – continuous limit* $d \rightarrow c$ *limit* (Gd-cL).

In order to distinguish the observables related to the discrete spectrum X_d, from those of the continuous spectrum $\widetilde{X}_c$, we adopt the following notation $\widetilde{X}_c \equiv \widetilde{X}$ (that is, we will use the sign "tilda", i.e. the "horizontal" integral sign).

Let us consider that E is a dimensionless energy eigenvalue of the Hamiltonian $\mathcal{H}$ *with a non-degenerate continuous spectrum,* and eigenstates $|E>, \ \hbar\omega = 1$, (with $0 \le E \le \infty$), i.e. $\mathcal{H}\,|E> = E\,|E>$ which are formal delta-function, with $<E\,|\,E'> = \delta(E - E')$.

The closure or completeness relation for continuous spectrum is

$$\int_0^{\infty} dE\ |E><E| = 1$$

$$\int_0^{\infty} dE < E'|\,E><E\,|\,E''> = \delta(E' - E'') \tag{4.2}$$

For the quantum systems with continuous spectra the dimensionless energy is E (where we assume that $\beta_B = (k_B T)^{-1}$ is also dimensionless). So, the spectrum is not linear or quadratic.

By applying the *generalized discrete – continuous limit* $d \rightarrow c$ let's transcribe for the continuous case the results above obtained for the discrete case.

The generalized integral Mittag-Leffler function becomes

$$\lim_{d\to c} E_{\beta,\alpha}^{\gamma,k}(z) = \lim_{d\to c} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_\alpha(\beta + \alpha\, n)} \frac{z^n}{n!} \rightarrow$$
$$\rightarrow \widetilde{E}_{\beta,\alpha}^{\gamma,k}(z) = \int_0^{\infty} dE\, \frac{(\widetilde{\gamma})_{E,k}}{\widetilde{\Gamma}_\alpha(\beta + \alpha\, E)} \frac{z^E}{\Gamma(E+1)} \tag{4.3}$$

$$(\widetilde{\gamma})_{E,k} = k^E \left(\frac{\gamma}{k}\right)_E = k^E\, \frac{\Gamma\left(\frac{\gamma}{k} + E\right)}{\Gamma\left(\frac{\gamma}{k}\right)} \tag{4.4}$$

$$\widetilde{\Gamma}_\alpha(\beta + \alpha\, E) = \alpha^E\, \frac{\Gamma\left(\frac{\beta}{\alpha} + E\right)}{\Gamma\left(\frac{\beta}{\alpha}\right)} \Gamma(\beta) =$$
$$= \alpha^E \left(\frac{\beta}{\alpha}\right)_E \Gamma(\beta) = \Gamma(\beta)(\widetilde{\beta})_{E,\alpha} \tag{4.5}$$

The generalized integral Mittag-Leffler function is connected with integral generalized hypergeometric function:

$$\widetilde{E}_{\beta,\alpha}^{\gamma,k}(z) = \frac{1}{\Gamma(\beta)}\, {}_1\widetilde{F}_1\left(\frac{\gamma}{k};\ \frac{\beta}{\alpha};\ \frac{k}{\alpha} z\right) \tag{4.6}$$

defined as

$${}_1\widetilde{F}_1\left(\frac{\gamma}{k};\ \frac{\beta}{\alpha};\ \frac{k}{\alpha} z\right) = \int_0^{\infty} dE\, \frac{\left(\frac{\gamma}{k}\right)_E}{\left(\frac{\beta}{\alpha}\right)_E} \frac{\left(\frac{k}{\alpha} z\right)^E}{\Gamma(E+1)} \tag{4.7}$$

For the particular case, $\alpha = \beta = \gamma = k = 1$, we get successively the integral exponential or the *nu*-function:

$$\widetilde{E}_{1,1}^{1,1}(z) = {}_1\widetilde{F}_1(1;\ 1;\ z) = {}_0\widetilde{F}_0(\ ;\ ;\ z) =$$
$$= \int_0^{\infty} dE\ \frac{z^E}{\Gamma(E+1)} = \widetilde{e}(z) = \nu(z) \tag{4.8}$$

Obviously not to be confused the *integral exponential function* $\widetilde{e}(z)$ with the *exponential integral function* $\mathrm{Ei}(x)$, for real x, which is a special function on the complex plane, defined as

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} dt\, \frac{e^{-t}}{t} = \int_{-\infty}^{x} dt\, \frac{e^t}{t} \tag{4.9}$$

Therefore, we can define the *integral generalized Mittag-Leffler function* as generalized *nu*-function

$$\widetilde{E}_{\beta,\alpha}^{\gamma,k}(z) = \int_0^{\infty} dE\, \frac{(\widetilde{\gamma})_{E,k}}{\widetilde{\Gamma}_\alpha(\beta + \alpha\, E)} \frac{z^E}{\Gamma(E+1)} \equiv \nu_{\beta,\alpha}^{\gamma,k}(z) \tag{4.10}$$

In particular, with this new notation, the usual *nu*-function is $\nu(z)\equiv \nu_{1,1}^{1,1}(z)$. In this sense, this is a new application of *nu*-function.

The eigenvalues for the continuous spectrum becomes:

$$\lim_{\substack{n\to E\\ \alpha,\beta,\gamma,k=1}} e(n)\equiv \lim_{\substack{n\to E\\ \alpha,\beta,\gamma,k=1}} n\frac{(\beta+\alpha(n-1))}{\gamma+k(n-1)}=E \quad (4.11)$$

Consequently, the action of the operators $\mathcal{A}_-$ and $\mathcal{A}_+$ on the vectors $|E>$ results from the application of the *discrete – continuous limit* $d\to c$ on their discrete counterparts :

$$\begin{aligned}\mathcal{A}_-|E> &=\sqrt{E}\,|E-1>\\ \mathcal{A}_+|E> &=\sqrt{E+1}\,|E+1>\\ \mathcal{A}_+\mathcal{A}_-|E> &=E\,|E>\end{aligned} \quad (4.12)$$

as well as by successively applications of these operators on the vacuum states $|0>$ and $<0|$:

$$\begin{Bmatrix}|E>\\ \\ <E|\end{Bmatrix}=\frac{1}{\Gamma(E+1)}\begin{Bmatrix}(\mathcal{A}_+)^E\,|0>\\ \\ <0|(\mathcal{A}_-)^E\end{Bmatrix} \quad (4.13)$$

Consequently, the generalized discrete – continuous limit $d\to c$ can also be applied to the definition of Mittag-Leffler coherent states:

$$\begin{aligned}&\lim_{d\to c}|z>=\\ &=\lim_{d\to c}\frac{1}{\sqrt{E_{1,1}^{1,1}(|z|^2)}}\sum_{n=0}^{\infty}\sqrt{\frac{(1)_{n,1}}{\Gamma_1(1+n)}}\frac{z^n}{\sqrt{n!}}|n>=\\ &=\frac{1}{\sqrt{\widetilde{E}_{1,1}^{1,1}(|z|^2)}}\int_0^{\infty}dE\frac{z^E}{\Gamma(E+1)}|E>=|\widetilde{z}>\end{aligned}$$

(4.14)

respectively for the version that contains the operators:

$$\begin{aligned}|\widetilde{z}> &=\frac{1}{\sqrt{\widetilde{E}_{1,1}^{1,1}(|z|^2)}}\widetilde{E}_{1,1}^{1,1}(z\mathcal{A}_+)|0>\\ |\widetilde{z}> &=\frac{1}{\sqrt{\nu(z)}}\nu(z\mathcal{A}_+)|0>\end{aligned} \quad (4.15)$$

Using these definitions, we can rewrite in the integral version all the relations obtained in the previous section, containing the integral Mittag-Leffler CSs or *nu*-functions, deduced for the discrete spectrum.

The orthogonal propagator onto states $|\widetilde{z}>$ is

$$|\widetilde{z}><\widetilde{z}|=\frac{1}{\widetilde{E}_{1,1}^{1,1}(|z|^2)}\#\frac{\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+z)\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_-z^*)}{\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+\mathcal{A}_-)}\#$$

(4.16)

To find the limit for $z\to 0$ of this orthonormal propagator, we will use the *inverse limit*, i.e. *the generalized continuous - discrete limit* $c\to d$:

$$\lim_{\substack{z\to 0\\ c\to d}}\widetilde{E}_{1,1}^{1,1}(z)=\lim_{\substack{z\to 0\\ E\to n}}\int_0^{\infty}dE\frac{z^E}{\Gamma(E+1)}=\lim_{z\to 0}\sum_{n=0}^{\infty}\frac{z^n}{n!}=1$$

(4.17)

The correct expression of the vacuum propagator is given by

$$|0><0|=\frac{1}{\#\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+\mathcal{A}_-)\#} \quad (4.18)$$

The expression of corresponding integration measure $d\widetilde{\mu}(z)=\frac{d\varphi}{2\pi}d(|z|^2)\widetilde{h}(|z|^2)$ is obtained beginning from the closure relation

$$\int d\widetilde{\mu}(z)|\widetilde{z}><\widetilde{z}|=1 \quad (4.19)$$

Then it results, successively

$$\begin{aligned}1=&\frac{1}{\#\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+\mathcal{A}_-)\#}\int_0^{\infty}d(|z|^2)\frac{\widetilde{h}(|z|^2)}{\widetilde{E}_{1,1}^{1,1}(|z|^2)}\times\\ &\times\int_0^{2\pi}\frac{d\varphi}{2\pi}\#\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+z)\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_-z^*)\#\end{aligned} \quad (4.20)$$

$$\begin{aligned}&\int_0^{2\pi}\frac{d\varphi}{2\pi}\#\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+z)\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_-z^*)\#=\\ &=\int_0^{\infty}dE\frac{\#(\mathcal{A}_+\mathcal{A}_-)^E\#}{[\Gamma(E+1)]^2}(|z|^2)^E\end{aligned} \quad (4.21)$$

so that the following integral must be resolved

$$\int_0^{\infty}d(|z|^2)\frac{\widetilde{h}(|z|^2)}{\widetilde{E}_{1,1}^{1,1}(|z|^2)}(|z|^2)^E=\Gamma(E+1) \quad (4.22)$$

By the substitution $E=s-1$, this is just the problem of Stieltjes moments [14], so the solution is

$$\frac{\widetilde{h}(|z|^2)}{\widetilde{E}_{1,1}^{1,1}(|z|^2)}=G_{0,\,1}^{1,\,0}\left(|z|^2\,\middle|\,0\right)=e^{-|z|^2} \quad (4.23)$$

The final expression of the integration measure is, then

$$d\widetilde{\mu}(z)=\frac{d\varphi}{2\pi}d(|z|^2)e^{-|z|^2}\widetilde{E}_{1,1}^{1,1}(|z|^2) \quad (4.24)$$

It can also be obtained directly, if in Eq. (2.12) we put $\alpha=\beta=\gamma=k=1$.

The corresponding density operator is, then

$$\widetilde{\rho}=\frac{1}{\widetilde{Z}(\beta_B)}\int_0^{\infty}dE\,e^{-\beta_B E}\;|E><E| \quad (4.25)$$

$$\widetilde{\rho}=\frac{1}{\widetilde{Z}(\beta_B)}\frac{1}{\#\widetilde{E}_{1,1}^{1,1}(\mathcal{A}_+\mathcal{A}_-)\#}\#\widetilde{E}_{1,1}^{1,1}(e^{-\beta_B}\mathcal{A}_+\mathcal{A}_-)\#$$

(4.26)

Taking into account the equality

$$\widetilde{E}_{1,1}^{1,1}(z)=\int_0^\infty dE\,\frac{z^E}{\Gamma(E+1)}=\nu(z) \qquad (4.27)$$

all these formulas can be expressed by through the *nu*-function.

The partition function $\widetilde{Z}(\beta_B)$ is obtained from the normalization relation of the density operator $\mathrm{Tr}\,\widetilde{\rho}=1$:

$$\mathrm{Tr}\,\widetilde{\rho}=\int_0^\infty dE'<E'|\,\widetilde{\rho}\,|E'>=$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\int_0^\infty dE'\int_0^\infty dE\,e^{-\beta_B E}\;<E'|E><E|E'>=$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\int_0^\infty dE\,e^{-\beta_B E}=\frac{1}{\widetilde{Z}(\beta_B)}\frac{1}{\beta_B}=1 \qquad (4.28)$$

This means that the partition function is linear dependent on the temperature, $\widetilde{Z}(\beta_B)=k_B T$.

Consequently, the Husimi's distribution function becomes

$$\widetilde{Q}(|z|^2)=<\widetilde{z}\,|\,\widetilde{\rho}\,|\,\widetilde{z}>=$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\int_0^\infty dE\,e^{-\beta_B E}\;<\widetilde{z}\,|E><E|\widetilde{z}>=$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\frac{1}{\widetilde{E}_{1,1}^{1,1}(|z|^2)}\widetilde{E}_{1,1}^{1,1}\left(e^{-\beta_B}\,|z|^2\right)= \qquad (4.29)$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\frac{\nu\left(e^{-\beta_B}\,|z|^2\right)}{\nu(|z|^2)}$$

where you can see the application of $\mathcal{A}_+\mathcal{A}_-\to|z|^2$ correspondence.

Finally, the diagonal representation of the density operator is written

$$\widetilde{\rho}=\frac{1}{\widetilde{Z}(\beta_B)}\int d\widetilde{\mu}(z)\,|\,\widetilde{z}>\widetilde{P}(|z|^2)<\widetilde{z}\,| \qquad (4.30)$$

After the appropriate substitutions, following the same step as in the case of deducing the expression of the integration measure, we obtain that the quasi-distribution function *P* has the following final expression:

$$\widetilde{P}(|z|^2)=\frac{1}{\widetilde{Z}(\beta_B)}\frac{G_{0,1}^{1,0}\left(e^{\beta_B}\,|z|^2\,|\,0\right)}{G_{0,1}^{1,0}\left(|z|^2\,|\,0\right)}=$$

$$=\frac{1}{\widetilde{Z}(\beta_B)}\exp\left[-\left(1-e^{\beta_B}\right)|z|^2\right] \qquad (4.31)$$

Let us emphasize that up to a constant (partition function), the $\widetilde{P}(|z|^2)$ distributions coincide with the corresponding ones of the linear harmonic oscillator (HO-1D).

Concluding remarks

In the paper, they were defined and used a pair of annihilation and creation operators which generate the generalized coherent states, defined in the Barut-Girardello manner. The choice of these operators was not accidental, the reason being that they generate such coherent states whose normalization function is just the four-parameter generalized Mittag-Leffler function. The characteristic properties both for pure and mixed (thermal) coherent states were examined. All calculations are made using the rules of the technique of diagonal ordering of operators (DOOT) which we introduced in a previous paper and which proved useful in performing the calculations. The main feature of this technique is that the nonlinear operators (creation and annihilation) are commutative. Consequently, they are treated as simple *c*-numbers, which makes mathematical operations (e.g., integration) easier. Finally, the integral counterpart of the Mittag-Leffler coherent states are examined which is connected with *nu*-function. In this way, the present paper becomes an example of a new application of a mathematical entities (Mittag-Leffler function, *nu*-function) in quantum mechanics (coherent states formalism).

Apart from the applications of Mittag-Leffler functions within the coherent states formalism, i.e. in quantum mechanics, as presented in the previous sections, in recent years these functions have also found their utility in solving applied problems, as kinetic equations, time-fractional diffusion, fractional-space and fractional-reaction diffusion models, as well as in nonlinear waves (see, the review article of Haubold et al. [20]), fractional calculus [21]. The Mittag-Leffler function is used in geometric function theory through fractional calculus, a concept that has developed in recent years, due to its various applications [22, 23]. The Mittag-Leffler functions are also used in quantum field theory, due to their possibilities to introduce fractional-order operators, starting from their standard differential counterparts [24].

Recently, the concept of multi-index integral Mittag-Leffler functions has been introduced. With this, an approach of associated coherent states has been built, which is also expected to have interesting applications [25]. Also, recently papers have appeared that highlight some interesting special cases related to extended Mittag-Leffler functions [21].

These results demonstrate that the Mittag-Leffler functions, besides their strictly theoretical (mathematical) merit, can also be useful in practical calculations, especially in the field of fractional calculus for solving fractional differential equations which appear in various branches of physics and engineering.

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