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# Solving Fourth-Order Lane-Emden-Fowler Equation by the New Modified Adomian Decomposition Method

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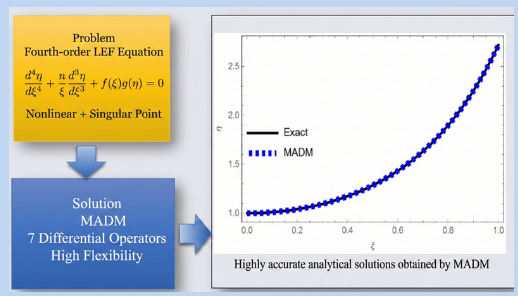
**Abstract:** This study explores the solutions of fourth-order Lane–Emden–Fowler (LEF) equations by employing a refined Modified Adomian Decomposition Method (MADM). We introduce a novel framework that features seven specialized differential operators, specifically developed and utilized to analyze the equations under specific initial and boundary conditions. Our findings demonstrate that the solutions derived from this approach not only effectively converge to the exact solutions but also offer unparalleled accuracy and reliability. A key strength of this methodology lies in its exceptional flexibility; solutions can be accurately obtained by applying at least one of these newly developed operators. This work significantly enhances our comprehension of these intricate equations and highlights the remarkable efficacy of the MADM in yielding precise solutions across diverse scenarios, thereby establishing a robust and versatile analytical tool.




**Keywords:** Fourth-Order Lane–Emden–Fowler Equation, Initial and Boundary Value Problems, Modified Adomian Decomposition Method, Analytical and Numerical Solutions.

## Introduction

The LEF equation represents an essential mathematical model for describing a wide range of nonlinear physical systems. It is a generalization of both the classical Lane–Emden (LE) and Emden–Fowler (EF) equations [1–6], combining their structural properties into a unified higher-order framework. The fourth-order LEF form, in particular, provides a more accurate representation of complex dynamical systems in astrophysics, thermodynamics and nonlinear fluid mechanics [7–17]. In general, the fourth-order LEF equation can be expressed as:

$$\frac{d^4\eta}{d\xi^4}+\frac{n}{\xi}\frac{d^3\eta}{d\xi^3}+f(\xi)g(\eta)=0, \tag{1}$$

where $\xi \in ]0,1]$, $\eta \in [0,3[$, $f(\xi)$ and $g(\eta)$ are arbitrary continuous functions, $\eta$ represents the dependent variable, $\xi$ the independent variable, and $n \geq 1$ is a shape factor. This equation arises naturally in modeling stellar interiors, radiation diffusion and thermal behavior of polytropic gas spheres. The inclusion of higher-order derivatives allows the LEF equation to capture effects neglected in lower-order formulations, such as fourth-order diffusion and nonlinear viscous stresses. Consequently, it has become a cornerstone in the theoretical investigation of self-gravitating fluids, plasma dynamics, and certain classes of quantum mechanical and relativistic systems [18–22]. Due to its strong nonlinearity and the presence of variable coefficients, analytical solutions to the LEF equation are rare and typically limited to specific parameter choices. Hence, modern analytical and semi-analytical techniques such as the Adomian Decomposition Method (ADM), homotopy analysis, and variational iteration approaches are frequently utilized to construct reliable approximate solutions. The continued development of these methods not only enhances computational efficiency but also deepens the understanding of the physical meaning behind the LEF model.

The ADM is a powerful semi-analytical approach designed to solve a wide range of linear and nonlinear differential equations without requiring linearization or small-perturbation assumptions. Initially developed by George Adomian in the late 20th century, the method decomposes a complex nonlinear problem into a rapidly convergent series of subcomponents that can be solved iteratively [23–25]. Each term of the solution is systematically determined through recursive relations, while the nonlinear terms are represented using specially constructed Adomian polynomials. This structure makes ADM highly efficient for initial value and boundary value problems across mathematics, engineering, and applied physics. Over time, various enhancements of the original ADM have been proposed to improve its convergence and computational ac-

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curacy. Among these, the MADM has emerged as one of the most effective refinements. The MADM incorporates additional correction operators and refined decomposition schemes that accelerate convergence and reduce truncation errors in the computed series. It also provides greater flexibility in handling strongly nonlinear terms and higher-order derivatives, making it particularly suitable for complex models such as the fourth-order LEF equation. A key feature of the MADM lies in its ability to maintain analytical transparency while achieving numerical precision comparable to direct computational methods. Unlike conventional perturbation or iteration techniques, MADM requires no discretization or transformation of variables, preserving the physical meaning of the problem throughout the solution process. Owing to these advantages, the MADM has been successfully applied to diverse nonlinear systems in heat transfer, fluid dynamics, astrophysics, and reaction diffusion models, demonstrating superior stability and efficiency over the traditional ADM [8–10, 16, 25–40].

Although the ADM has been widely recognized as an effective analytical approach for solving linear and nonlinear differential equations, it still suffers from several limitations that restrict its performance in certain cases. In particular, the original ADM exhibits weak performance when dealing with singular equations, since the linear operator may not be invertible at singular points, leading to loss of accuracy or even divergence of the solution. Moreover, the method often shows slow convergence for highly nonlinear problems, requiring a large number of Adomian components to achieve acceptable accuracy. These drawbacks have motivated researchers to develop various modifications and improvements of the ADM to enhance convergence, stability, and applicability. Accordingly, the present work introduces a new adaptive modification of the ADM designed to overcome these limitations and provide more accurate and rapidly convergent solutions for singular and nonlinear differential equations.

This article aims to explore and enhance the analysis of the fourth-order LEF equation by applying the MADM. This study introduces a tailored framework that comprises seven specialized operators, designed to facilitate effective solutions under a variety of initial and boundary conditions. Under different conditions, this method is characterized by its ability to solve the LEF equation using multiple operators regardless of the value of $n$. Notably, the failure of one operator to obtain the solution does not prevent the others from achieving it, which represents one of its main advantages that overcome the limitations of traditional approaches, MADM has consistently proven its efficiency and reliability in addressing both linear and nonlinear equations. It provides successive components of a solution without requiring ad hoc transformations or perturbation techniques. To demonstrate the method's capability in handling singularities and nonlinearities inherent in various models, several numerical examples, each with specified conditions, are examined.

In this study, a set of novel differential operators is introduced to efficiently solve the LEF equation. Section 2, analysis of the proposed method for the fourth-order LEF equation provides a comprehensive examination of the theoretical framework and analytical formulation of the proposed approach. Section 3, the algorithm presents a detailed description of the computational procedure used to implement the proposed method for the fourth-order LEF equation, this section outlines the operational structure and sequential computational steps of the MADM applied to various forms of the LEF equation. Section 4, numerical examples demonstrates the effectiveness of the proposed approach through four numerical examples corresponding to different values of $n$, the results are illustrated using tables and graphical representations highlighting the absolute error values and confirming the accuracy and reliability of the method. Finally, section 5 summarizes the main findings, emphasizing the precision and efficiency of the developed operators in solving nonlinear LEF equations and validating the robustness of the MADM framework. discusses the implications of our findings and concludes the paper.

## Analysis of the Proposed Method for the Fourth-Order LEF Equation

This section outlines the methodological approach employed to address the fourth-order LEF equation. Our strategy leverages the ADM adapted to effectively handle the complexities of this class of non-linear differential equations. Specifically, we introduce a set of seven distinct operators meticulously formulated to facilitate the decomposition process for fourth-order equations. These operators are:

$$L_1(\eta) = \xi^{-1}\frac{d}{d\xi}\xi^{-1}\frac{d}{d\xi}\xi^{2}\frac{d}{d\xi}\xi^{3-n}\frac{d}{d\xi}\xi^{n-3}\eta, \tag{2}$$

$$L_2(\eta) = \xi^{-2}\frac{d}{d\xi}\xi^{3-n}\frac{d}{d\xi}\xi^{n-2}\frac{d}{d\xi}\xi^{3}\frac{d}{d\xi}\xi^{-2}\eta, \tag{3}$$

$$L_3(\eta) = \xi^{-2}\frac{d}{d\xi}\xi^{2}\frac{d}{d\xi}\xi^{2-n}\frac{d}{d\xi}\xi^{n-2}\frac{d}{d\xi}\eta, \tag{4}$$

$$L_4(\eta) = \xi^{-2}\frac{d^2}{d\xi^2}\xi^{4-n}\frac{d}{d\xi}\xi^{n}\frac{d}{d\xi}\xi^{-2}\eta, \tag{5}$$

$$L_5(\eta) = \xi^{-n}\frac{d}{d\xi}\xi^{n-1}\frac{d^2}{d\xi^2}\xi^{3}\frac{d}{d\xi}\xi^{-2}\eta, \tag{6}$$

$$L_6(\eta) = \xi^{-n}\frac{d}{d\xi}\xi^{n-1}\frac{d}{d\xi}\xi^{2}\frac{d}{d\xi}\xi^{-1}\frac{d}{d\xi}\eta, \tag{7}$$

$$L_7(\eta) = \xi^{-n}\frac{d}{d\xi}\xi^{n}\frac{d^3}{d\xi^3}\eta. \tag{8}$$

Each operator $L_i$ generates a fourth-order LEF equation of the general form Eq. (1).

Starting from Eq. (2):

$$\begin{gathered}\xi^{-1}\frac{d}{d\xi}\xi^{-1}\frac{d}{d\xi}\xi^{2}\frac{d}{d\xi}\xi^{3-n}\frac{d}{d\xi}\xi^{n-3}\eta \\ = \xi^{-1}\frac{d}{d\xi}\xi^{-1}\frac{d}{d\xi}\left(\xi^{2}\frac{d^2\eta}{d\xi^2} + (n-3)\xi\frac{d\eta}{d\xi} - (n-3)\eta\right) \\ = \xi^{-1}\frac{d}{d\xi}\left(\xi\frac{d^3\eta}{d\xi^3} + (n-1)\frac{d^2\eta}{d\xi^2}\right) \\ = \xi^{-1}\left(\xi\frac{d^4\eta}{d\xi^4} + \frac{d^3\eta}{d\xi^3} + (n-1)\frac{d^3\eta}{d\xi^3}\right) = \frac{d\eta^4}{d\xi^4} + \frac{n}{\xi}\frac{d\eta^3}{d\xi^3}.\end{gathered}$$

Thus,

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \quad \xi \in ]0,1], \eta \in [0,3[, \tag{9}$$

$$\eta(0) = A, \eta(a) = B, \eta'(a) = C, \eta''(0) = D. \tag{10}$$

**Remark.** If $a = 0$, then $A = B$.

Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C, D$ and $a$ can all be nonzero. Notice that the singular point $\xi = 0$ appears only once as $\xi$ with the corresponding shape factor $n$. Moreover, this form of the equation is by the absence of the first and second-order derivative terms, $\eta'$ and $\eta''$.

When $f(\xi) = 1$, Eq. (9) reduces to the classical LE equation of the fourth order:

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \quad \xi \in ]0,1], \eta \in [0,3[, \tag{11}$$

$$\eta(0) = A, \eta(a) = B, \eta'(a) = C, \eta''(0) = D. \tag{12}$$

**Note.** If $a = 0$, then $A = B$.

From Eq. (3) and in accordance with the analysis previously detailed, get

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \quad \xi \in ]0,1], \eta \in [0,3[, \tag{13}$$

$$\eta(a) = A, \eta(b) = B, \eta'(0) = C, \eta'(b) = D. \tag{14}$$

**Remark.** The condition $a \neq b$ is imposed to guarantee the independence of the boundary conditions and consequently, the existence and uniqueness of the solution. Moreover, if $b = 0$, then $C = D$.

Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C, D$ and $a, b$ can all be nonzero. When the function $f(\xi)$ is set to $1$, Eq. (13) transforms into the fourth-order LE equation, which is expressed as

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \quad \xi \in ]0,1], \eta \in [0,3[, \tag{15}$$

$$\eta(a) = A, \eta(b) = B, \eta'(0) = C, \eta'(b) = D. \tag{16}$$

**Note.** The condition $a \neq b$ is imposed to guarantee the independence of the boundary conditions and consequently, the existence and uniqueness of the solution. Furthermore, if $b = 0$, then $C = D$.

From Eq.(4) and building on the preceding analysis, get

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \quad \xi \in ]0,1], \eta \in [0,3[, \tag{17}$$

$$\eta(a) = A, \eta'(0) = B, \eta'(b) = C, \eta''(b) = D. \tag{18}$$

**Remark.** If $b = 0$, then $B = C$.
Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C, D$ and $a, b$ can all be nonzero. For the case where $f(\xi) = 1$, Eq. (17) simplifies to the fourth-order LE equation which is expressed as

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \tag{19}$$

$$\eta(a) = A, \eta'(0) = B, \eta'(b) = C, \eta''(b) = D. \tag{20}$$

**Note.** If $b = 0$, then $B = C$.
From Eq. (5) and consistent with the analysis presented above, have

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \tag{21}$$

$$\eta(0) = A, \eta(b) = B, \eta'(0) = C, \eta''(0) = 0. \tag{22}$$

**Remark.** If $b = 0$, then $A = B$.
Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C$ and $b$ can all be nonzero. When the function $f(\xi) = 1$, Eq. (21) reduces to the fourth-order LE equation, given by

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \tag{23}$$

$$\eta(0) = A, \eta(b) = B, \eta'(0) = C, \eta''(0) = 0. \tag{24}$$

**Note.** If $b = 0$, then $A = B$.
From Eq. (6) by calculating the derivative

$$\xi^{-n}\frac{d}{d\xi}\xi^{n-1}\frac{d^2}{d\xi^2}\xi^3\frac{d}{d\xi}\xi^{-2}\eta = \xi^{-n}\frac{d}{d\xi}\xi^{n-1}\frac{d^2}{d\xi^2}\left(\xi\frac{d\eta}{d\xi} - 2\eta\right)$$

$$= \xi^{-n}\frac{d}{d\xi}\xi^{n-1}\frac{d}{d\xi}\left(\xi\frac{d\eta^2}{d\xi^2} - \frac{d\eta}{d\xi}\right) = \xi^{-n}\frac{d}{d\xi}\xi^{n-1}\left(\xi\frac{d\eta^3}{d\xi^3}\right)$$

$$= \xi^{-n}\frac{d}{d\xi}\left(\xi^n\frac{d\eta^3}{d\xi^3}\right) = \xi^{-n}\left(\xi^n\frac{d\eta^4}{d\xi^4} + n\xi^{n-1}\frac{d\eta^3}{d\xi^3}\right) = \frac{d\eta^4}{d\xi^4} + \frac{n}{\xi}\frac{d\eta^3}{d\xi^3},$$

gives,

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \tag{25}$$

$$\eta(0) = A, \eta(a) = B, \eta'(b) = C, \eta''(b) = D. \tag{26}$$

**Remark.** If $a = 0$, then $A = B$.
Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C, D$ and $a, b$ can all be nonzero. By setting $f(\xi) = 1$, Eq. (25) simplifies to the fourth-order LE equation, expressed as

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \tag{27}$$

$$\eta(0) = A, \eta(a) = B, \eta'(b) = C, \eta''(b) = D. \tag{28}$$

**Note.** If $a = 0$, then $A = B$.
From Eq. (7) and as demonstrated in the preceding analysis, obtain

$$\eta'''' + \frac{n}{\xi}\eta''' + f(\xi)g(\eta) = 0, \tag{29}$$

$$\eta(0) = A, \eta'(a) = B, \eta'(b) = C, \eta''(b) = D. \tag{30}$$

**Remark.** The condition $a \neq b$ is imposed to guarantee the independence of the boundary conditions and consequently, the existence and uniqueness of the solution.
Where $f(\xi)$ and $g(\eta)$ are given functions of $\xi$ and $\eta$, respectively, and $n \geq 1$ is called the shape factor, $A, B, C, D$ and $a, b$ can all be nonzero. When $f(\xi) = 1$, Eq. (29) becomes the LE equation of the fourth order, which is given by

$$\eta'''' + \frac{n}{\xi}\eta''' + g(\eta) = 0, \tag{31}$$

$$\eta(0) = A, \eta'(a) = B, \eta'(b) = C, \eta''(b) = D. \tag{32}$$

**Note.** The condition $a \neq b$ is imposed to guarantee the independence of the boundary conditions and consequently, the existence and uniqueness of the solution.
Finally, Eq. (8) generalized extension as defined in [14].

## The Algorithm

This section provides a detailed description of the algorithm implemented to apply the proposed method for solving the fourth-order LEF equation, the procedure is formulated to demonstrate the operational framework and computational steps of the modified ADM in addressing different forms of the LEF equation. The ADM serves as the analytical foundation of the proposed approach, being a well-established and extensively utilized technique in recent studies [16, 23–25, 34, 39, 40].
The ADM utilizes an infinite series decomposition

$$\eta(\xi) = \sum_{k=0}^{\infty}\eta_k(\xi), \tag{33}$$

to obtain the solution $\eta(\xi)$, represented by an infinite polynomial series

$$g(\eta) = \sum_{k=0}^{\infty}A_k(\eta_0, \eta_1, ..., \eta_k), \tag{34}$$

The term $g(\eta)$ signifies the nonlinear component, while $\eta_k(\xi)$ represents the recurrently determined components of the solution $\eta(\xi)$. The Adomian polynomials, $A_k$, are derived from the definitional formula detailed [24].

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[g\left(\sum_{i=0}^{k}\lambda^i\eta_i\right)\right]_{\lambda=0}, k = 0, 1, 2, ...$$

where $N\eta = g(\eta(\xi))$ is the nonlinearity. The formulas of Adomian polynomials from $A_0$ to $A_4$ as

$$A_0 = g(\eta_0),$$

$$A_1 = \eta_1 g'(\eta_0),$$

$$A_2 = \eta_2 g'(\eta_0) + \frac{1}{2!}\eta_1^2 g''(\eta_0),$$

$$A_3 = \eta_3 g'(\eta_0) + \eta_1\eta_2 g''(\eta_0) + \frac{1}{3!}\eta_1^3 g'''(\eta_0),$$

$$A_4 = \eta_4 g'(\eta_0) + (\eta_1\eta_3 + \frac{1}{2!}\eta_2^2)g''(\eta_0) + \frac{1}{2!}\eta_1^2\eta_2 g'''(\eta_0) + \frac{1}{4!}\eta_1^4 g''''(\eta_0). \tag{35}$$

## The First Operator

For Eq. (9), the proposed framework primarily involves defining the linear differential operator $L_1$ as a first-derivative expression.

$$L_1(\eta) = -f(\xi)g(\eta), \tag{36}$$

the linear differential operator $L_1$ incorporates the first four derivatives from Eq. (9), as shown in Eq. (2), to overcome the singular behavior at $\xi = 0$. Consequently, the optimal definition of the inverse operator $L_1^{-1}$ is a four-fold definite integration

$$L_1^{-1}(.) = \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \xi^{-2}\int_0^\xi \xi\int_0^\xi \xi(.) \quad d\xi d\xi d\xi d\xi. \tag{37}$$

The boundary conditions are implicitly defined by the Eq. (10) and $n > 0, n \neq 2$, have

$$L_1^{-1}(L_1\eta) = \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \xi^{-2}\int_0^\xi \xi$$
$$\int_0^\xi \left(\frac{d}{d\xi}\xi^{-1}\frac{d}{d\xi}\xi^2\frac{d}{d\xi}\xi^{3-n}\frac{d\eta}{d\xi}\xi^{n-3}\right) \quad d\xi d\xi d\xi d\xi$$
$$= \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \xi^{-2}$$
$$\int_0^\xi \xi\Big(\xi\frac{d}{d\xi}\xi^2\frac{d}{d\xi}\xi^{3-n}\frac{d\eta}{d\xi}\xi^{n-3} + \eta''(0)(1-n)\Big) \quad d\xi d\xi d\xi$$
$$= \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \xi^{-2}$$
$$\int_0^\xi \left(\xi^2\frac{d}{d\xi}\xi^2\frac{d}{d\xi}\xi^{3-n}\frac{d\eta}{d\xi}\xi^{n-3} + \eta''(0)(1-n)\xi\right) \quad d\xi d\xi d\xi$$
$$= \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \xi^{-2}\Big(\xi^2\frac{d}{d\xi}$$
$$\xi^{3-n}\frac{d\eta}{d\xi}\xi^{n-3} + \eta(0)(n-3) + \frac{\eta''(0)}{2}(1-n)\xi^2\Big) \quad d\xi d\xi$$
$$= \xi^{3-n}\int_a^\xi \xi^{n-3}\int_a^\xi \Big(\frac{d}{d\xi}\xi^{3-n}\frac{d\eta}{d\xi}$$
$$\xi^{n-3} + \eta(0)(n-3)\xi^{-2} + \frac{\eta''(0)}{2}(1-n)\Big) \quad d\xi d\xi$$
$$= \xi^{3-n}\int_a^\xi \xi^{n-3}\Big(\xi^{3-n}\frac{d\eta}{d\xi}\xi^{n-3} + \frac{u(0)}{a}(n-3)\xi^{n-3} +$$
$$\frac{a\eta''(0)}{2}(1-n) + \eta'(a) + \frac{\eta(a)}{a}(n-3)\Big) \quad d\xi$$
$$= \xi^{3-n}\int_a^\xi \Big(\frac{d\eta}{d\xi}\xi^{n-3} + \Big(\frac{\eta(0)}{a}(n-3) + \frac{a\eta''(0)}{2}(1-n) +$$
$$\eta'(a) + \frac{\eta'(a)}{a}(n-3)\Big)\xi^{n-3}\Big) \quad d\xi$$
$$= \xi^{3-n}\Big(\xi^{n-3}\eta(\xi) + \frac{1}{(n-2)}\Big(\frac{\eta(0)}{a}(n-3) + \frac{a\eta''(0)}{2}(1-n) + \eta'(a) +$$
$$\frac{\eta'(a)}{a}(n-3)\Big)\xi^{n-2} - a^{n-2}\eta(a) - a^{n-3}\eta(0) - \frac{\eta''(0)}{2}a^{n-1}$$
$$-\frac{\eta''(0)}{2(n-1)}(n-1)a^n - \frac{c + (\eta(0) - \eta'(a))(n-3)}{n-2}a^{n-2}\Big)$$
$$= \eta(\xi) - \eta(0) -$$
$$\frac{\Big(2(3-n)\eta(0) + 2(-3+n)\eta(a) - 2a\eta'(a) + a^2(1-n)\eta''(0)\Big)}{2a(-2+n)}\xi -$$
$$\frac{\eta''(0)}{2}\xi^2 - \frac{a^{-3+n}\Big(-2\eta(0) + 2\eta(a) - 2a\eta'(a) + a^2\eta''(0)\Big)}{2(-2+n)}\xi^{3-n}.$$

The four-fold integral was evaluated using repeated integration by parts. Applying the inverse operator $L_1^{-1}$ to both sides of Eq. (36) yields

$$\eta(\xi) = A + \frac{\Big(2(3-n)A + 2(-3+n)B - 2aC + a^2(1-n)D\Big)}{2a(-2+n)}\xi +$$
$$\frac{D}{2}\xi^2 + \frac{a^{-3+n}\Big(-2A + 2B - 2aC + a^2D\Big)}{2(-2+n)}\xi^{3-n} - L_1^{-1}f(\xi)g(\eta). \tag{38}$$

The solution $\eta(\xi)$ and the nonlinearity $g(\eta)$ are decomposed in accordance with Eqs. (33) and (34), respectively, allowing for the derivation of the recursive relation for the solution components

$$\eta_0 = A + \frac{\Big(2(3-n)A + 2(-3+n)B - 2aC + a^2(1-n)D\Big)}{2a(-2+n)}\xi +$$
$$\frac{D}{2}\xi^2 + \frac{a^{-3+n}\Big(-2A + 2B - 2aC + a^2D\Big)}{2(-2+n)}\xi^{3-n},$$
$$\eta_{r+1} = -L_1^{-1}(f(\xi)A_r), \quad r \geq 0. \tag{39}$$

## The Second Operator

For Eq. (13), the framework defines the second linear differential operator $L_2$ in terms of its first derivatives

$$L_2(\eta) = -f(\xi)g(\eta), \tag{40}$$

to address the singular behavior at $\xi = 0$, the linear differential operator $L_2$ contains the first four derivatives of Eq. (13), as detailed in Eq. (3). This structure allows the inverse operator $L_2^{-1}$, to be optimally defined as a four-fold definite integration

$$L_2^{-1}(.) = \xi^2\int_a^\xi \xi^{-3}\int_b^\xi \xi^{2-n}\int_0^\xi \xi^{n-3}\int_0^\xi \xi^2(.) \quad d\xi d\xi d\xi d\xi. \tag{41}$$

By the Eq. (14) and $n > 0$, get

$$L_2^{-1}(L_2\eta) = \xi^2\int_a^\xi \xi^{-3}\int_b^\xi \xi^{2-n}\int_0^\xi \xi^{n-3}$$
$$\int_0^\xi \xi^2\Big(\xi^{-2}\frac{d}{d\xi}\xi^{3-n}\frac{d}{d\xi}\xi^{n-2}\frac{d}{d\xi}\xi^3\frac{d\xi^{-2}\eta}{d\xi}\Big) \quad d\xi d\xi d\xi d\xi$$
$$= \xi^2\int_a^\xi \xi^{-3}\int_b^\xi \xi^{2-n}\int_0^\xi \Big(\frac{d}{d\xi}\xi^{n-2}\frac{d}{d\xi}\xi^3\frac{d\xi^{-2}\eta}{d\xi} + (n-2)\eta'(0)\xi^{n-3}\Big) \quad d\xi d\xi d\xi$$
$$= \xi^2\int_a^\xi \xi^{-3}\int_b^\xi \xi^{2-n}\Big(\xi^{n-2}\frac{d}{d\xi}\xi^3\frac{d\xi^{-2}\eta}{d\xi} + \eta'(0)\xi^{n-2}\Big) \quad d\xi d\xi$$
$$= \xi^2\int_a^\xi \Big(\frac{d\xi^{-2}\eta}{d\xi} + \Big(\eta'(0)\xi - b\eta'(0) - b\eta'(b) + 2\eta(b)\Big)\xi^{-3}\Big) \quad d\xi$$
$$= \xi^2\Big(\xi^{-2}\eta(\xi) - \frac{1}{2}\Big(\eta'(0)\xi - b\eta'(0) - b\eta'(b) + 2\eta(b)\Big)\xi^{-2} -$$
$$\frac{\eta(a) - \eta(b) - a\eta'(0)}{a^2} - \frac{b\eta'(0) + b\eta'(b)}{2a^2}\Big)$$
$$= \eta(\xi) - \eta(b) + \frac{b}{2}(\eta'(0) + \eta'(b)) - \eta'(0)\xi -$$
$$\frac{2(\eta(a) - \eta(b)) + (-2a + b)\eta'(0) + b\eta'(b)}{2a^2}\xi^2.$$

The four-fold integral was evaluated using repeated integration by parts. The application of the inverse operator $L_2^{-1}$ to both sides of Eq. (40) results in

$$\eta(\xi)=B-\frac{b}{2}(C+D)+C\xi+\frac{2(A-B)+(-2a+b)C+bD}{2a^2}\xi^2-L_2^{-1}f(\xi)g(\eta). \quad (42)$$

Using Eqs. (33) and (34), both the solution $\eta(\xi)$ and the nonlinearity $g(\eta)$ are decomposed, which yields the recursive relation for the solution components

$$\eta_0=B-\frac{b}{2}(C+D)+C\xi+\frac{2(A-B)+(-2a+b)C+bD}{2a^2}\xi^2,$$
$$\eta_{r+1}=-L_2^{-1}(f(\xi)A_r), \quad r\geq 0. \quad (43)$$

## The Third Operator

The third linear differential operator $L_3$ is formulated as a first-derivative expression for Eq. (17).

$$L_3(\eta)=-f(\xi)g(\eta), \quad (44)$$

as depicted in Eq. (4), the linear differential operator $L_3$ includes the first four derivatives from Eq. (17) to handle the singularity at $\xi=0$. Therefore $L_3^{-1}$ is optimally defined as a four-fold definite integration operator

$$L_3^{-1}(.)=\int_a^\xi \xi^{2-n}\int_0^\xi \xi^{n-2}\int_b^\xi \xi^{-2}\int_0^\xi \xi^2(.) \quad d\xi d\xi d\xi d\xi. \quad (45)$$

For $n>0, n\neq 1$ with boundary conditions Eq. (18), have

$$L_3^{-1}(L_3\eta)=\int_a^\xi \xi^{2-n}\int_0^\xi \xi^{n-2}\int_b^\xi \xi^{-2}\int_0^\xi (\frac{d}{d\xi}\xi^2\frac{d}{d\xi}\xi^{2-n}\frac{d}{d\xi}\xi^{n-2}\frac{d\eta}{d\xi}) \quad d\xi d\xi d\xi d\xi$$
$$=\int_a^\xi \xi^{2-n}\int_0^\xi \xi^{n-2}\int_b^\xi \left(\frac{d}{d\xi}\xi^{2-n}\frac{d}{d\xi}\xi^{n-2}\frac{d\eta}{d\xi}+(n-2)\eta'(0)\xi^{-2}\right) \quad d\xi d\xi d\xi$$
$$=\int_a^\xi \xi^{2-n}\int_0^\xi \left(\frac{d}{d\xi}\xi^{n-2}\frac{d\eta}{d\xi}-(n-2)\eta'(0)\xi^{n-3}+\frac{(n-2)(\eta'(0)-\eta'(b))-b\eta''(b)}{b}\xi^{n-2}\right) \quad d\xi d\xi$$
$$=\int_a^\xi \left(\frac{d\eta}{d\xi}-(n-2)\eta'(0)\xi^{-1}+\frac{(n-2)(\eta'(0)-\eta'(b))-b\eta''(b)}{b}\right)d\xi$$
$$=\eta(\xi)-\eta(a)-\frac{(2a(b-a)+a(a-2b)n)\eta'(0)+a^2(2-n)\eta'(b)-a^2b\eta''(b)}{2b(n-1)}-\eta'(0)\xi-\frac{(2-n)(\eta'(0)-\eta'(b))+b\eta''(b)}{2b(n-1)}\xi^2.$$

The four-fold integral was evaluated using repeated integration by parts. By applying the inverse operator $L_3^{-1}$ to both sides of Eq. (44), the following is obtained

$$\eta(\xi)=A+\frac{(2a(b-a)+a(a-2b)n)B+a^2(2-n)C-a^2bD}{2b(n-1)}+B\xi+\frac{(2-n)(B-C)+bD}{2b(n-1)}\xi^2-L_3^{-1}f(\xi)g(\eta). \quad (46)$$

The decomposition of the solution $\eta(\xi)$ and the nonlinearity $g(\eta)$ is performed using Eqs. (33) and (34), which enables the recursive relation for the solution components to be derived

$$\eta_0=A+\frac{(2a(b-a)+a(a-2b)n)B+a^2(2-n)C-a^2bD}{2b(n-1)}+B\xi+\frac{(2-n)(B-C)+bD}{2b(n-1)}\xi^2,$$
$$\eta_{r+1}=-L_3^{-1}(f(\xi)A_r), \quad r\geq 0. \quad (47)$$

## The Fourth Operator

For Eq. (21), the fourth linear differential operator $L_4$ is defined as a first-derivative term within the proposed framework.

$$L_4(\eta)=-f(\xi)g(\eta), \quad (48)$$

given its role in overcoming the singular behavior at $\xi=0$, the linear differential operator $L_4$ is designed to contain the first four derivatives of Eq. (21), as prescribed by Eq. (5). This setup leads to the definition of $L_4^{-1}$ as a four-fold definite integration operator

$$L_4^{-1}(.)=\xi^2\int_b^\xi \xi^{-n}\int_0^\xi \xi^{n-4}\int_0^\xi\int_0^\xi \xi^2(.) \quad d\xi d\xi d\xi d\xi. \quad (49)$$

Using Eq. (22) and $n>0$, have

$$L_4^{-1}(L_4\eta)=\xi^2\int_b^\xi \xi^{-n}\int_0^\xi \xi^{n-4}\int_0^\xi\int_0^\xi \left(\frac{d^2}{d\xi^2}\xi^{4-n}\frac{d}{d\xi}\xi^n\frac{d\xi^{-2}\eta}{d\xi}\right)d\xi d\xi d\xi d\xi$$
$$=\xi^2\int_b^\xi \xi^{-n}\int_0^\xi \xi^{n-4}\int_0^\xi \left(\frac{d}{d\xi}\xi^{4-n}\frac{d}{d\xi}\xi^n\frac{d\xi^{-2}\eta}{d\xi}+(n-2)\eta'(0)\right)d\xi d\xi d\xi$$
$$=\xi^2\int_b^\xi \xi^{-n}\int_0^\xi \left(\frac{d}{d\xi}\xi^n\frac{d\xi^{-2}\eta}{d\xi}+(n-2)\eta'(0)\xi^{n-3}+2(n-3)\eta(0)\xi^{n-4}\right)d\xi d\xi$$
$$=\xi^2\int_b^\xi \left(\frac{d\xi^{-2}\eta}{d\xi}+\eta'(0)\xi^{-2}+2\eta(0)\xi^{-3}\right)d\xi$$
$$=\xi^2\left(\xi^{-2}\eta(\xi)-\eta(0)\xi^{-2}-\eta'(0)\xi^{-1}+b^{-2}(\eta(0)-\eta(b)+b\eta'(0))\right)$$
$$=\eta(\xi)-\eta(0)-\eta'(0)\xi-b^{-2}(\eta(b)-\eta(0)-b\eta'(0))\xi^2.$$

The four-fold integral was evaluated using repeated integration by parts. The inverse operator $L_4^{-1}$ is applied to both sides of Eq. (48), leading to

$$\eta(\xi)=A+C\xi+b^{-2}(A-B+bC)\xi^2-L_4^{-1}f(\xi)g(\eta). \quad (50)$$

The solution $\eta(\xi)$ and the nonlinearity $g(\eta)$ are decomposed according to Eqs. (33) and (34) respectively, and derive the recursive relation for the solution components

$$\eta_0=A+C\xi+b^{-2}(A-B+bC)\xi^2,$$
$$\eta_{r+1}=-L_4^{-1}(f(\xi)A_r), \quad r\geq 0. \quad (51)$$

## The Fifth Operator

Within the proposed framework, the fifth linear differential operator $L_5$, is constructed using a first-derivative form for Eq. (25).

$$L_5(\eta) = -f(\xi)g(\eta), \tag{52}$$

where the linear differential operator $L_5$ contains the first four derivatives of Eq. (25) as Eq. (6), in order to overcome the singular behavior at $\xi = 0$. Based on Eq.(6), the optimal definition of $L_5^{-1}$ is the four-fold definite integration operator

$$L_5^{-1}(.) = \xi^2 \int_a^\xi \xi^{-3} \int_0^\xi \int_b^\xi \xi^{1-n} \int_0^\xi \xi^n (.) \quad d\xi d\xi d\xi d\xi. \tag{53}$$

From Eq. (26) with $n > 0$, have

$$L_5^{-1}(L_5\eta) = \xi^2 \int_a^\xi \xi^{-3} \int_0^\xi \int_b^\xi \xi^{1-n}$$
$$\int_0^\xi \left( \frac{d}{d\xi} \xi^{n-1} \frac{d^2}{d\xi^2} \xi^3 \frac{d\xi^{-2}\eta}{d\xi} \right) \quad d\xi d\xi d\xi d\xi$$
$$= \xi^2 \int_a^\xi \xi^{-3} \int_0^\xi \int_b^\xi \xi^{1-n} \left( \xi^{n-1} \frac{d^2}{d\xi^2} \xi^3 \frac{d\xi^{-2}\eta}{d\xi} \right) \quad d\xi d\xi d\xi$$
$$= \xi^2 \int_a^\xi \xi^{-3} \int_0^\xi \left( \frac{d}{d\xi} \xi^3 \frac{d\xi^{-2}\eta}{d\xi} - b\eta''(b) + \eta'(b) \right) \quad d\xi d\xi$$
$$= \xi^2 \int_a^\xi \left( \frac{d\xi^{-2}\eta}{d\xi} - b\eta''(b)\xi^{-2} + \eta'(b)\xi^{-2} + 2\eta(0)\xi^{-3} \right) \quad d\xi$$
$$= \eta(\xi) - \eta(0) - (\eta'(b) - b\eta''(b))\xi +$$
$$\Big( (\eta(0) - \eta(a))a^{-2} + (\eta'(b) - b\eta''(b))a^{-1} \Big) \xi^2.$$

The four-fold integral was evaluated using repeated integration by parts. Utilizing the inverse operator $L_5^{-1}$ on both sides of Eq. (52) yields

$$\eta(\xi) = A + (C - bD)\xi + \Big( (B - A)a^{-2} + (bD - C)a^{-1} \Big) \xi^2$$
$$-L_5^{-1} f(\xi) g(\eta). \tag{54}$$

To obtain the recursive relation for the solution components, both $\eta(\xi)$ and the nonlinearity $g(\eta)$ are decomposed by following Eqs. (33) and (34),

$$\eta_0 = A + (C - bD)\xi + \left( \frac{B - A}{a^{-2}} + \frac{bD - C}{a^{-1}} \right) \xi^2,$$
$$\eta_{r+1} = -L_5^{-1}(f(\xi)A_r), \quad r \geq 0. \tag{55}$$

## The Sixth Operator

For Eq. (29), the final linear differential operator $L_6$ is defined in terms of its first derivatives within this framework.

$$L_6(\eta) = -f(\xi)g(\eta), \tag{56}$$

by including the first four derivatives of Eq. (29), as shown in Eq. (7), the linea differential operator $L_6$ effectively manages the singularity at $\xi = 0$. Consequently, the inverse operator $L_6^{-1}$ is identified as a four-fold definite integration

$$L_6^{-1}(.) = \int_0^\xi \xi \int_a^\xi \xi^{-2} \int_b^\xi \xi^{1-n} \int_0^\xi \xi^n (.) \quad d\xi d\xi d\xi d\xi. \tag{57}$$

When $\eta(0) = A, \eta'(a) = B, \eta'(b) = C, \eta''(b) = D$, and $n > 0$, have

$$L_6^{-1}(L_6\eta) = \int_0^\xi \xi \int_a^\xi \xi^{-2} \int_b^\xi \xi^{1-n}$$
$$\int_0^\xi \left( \frac{d}{d\xi} \xi^{n-1} \frac{d}{d\xi} \xi^2 \frac{d}{d\xi} \xi^{-1} \frac{d\eta}{d\xi} \right) \quad d\xi d\xi d\xi d\xi$$
$$= \int_0^\xi \xi \int_a^\xi \xi^{-2} \int_b^\xi \xi^{1-n} \left( \xi^{n-1} \frac{d}{d\xi} \xi^2 \frac{d}{d\xi} \xi^{-1} \frac{d\eta}{d\xi} \right) \quad d\xi d\xi d\xi$$
$$= \int_0^\xi \xi \int_a^\xi \xi^{-2} \left( \xi^2 \frac{d}{d\xi} \xi^{-1} \frac{d\eta}{d\xi} - b\eta''(b) + \eta'(b) \right) \quad d\xi d\xi$$
$$= \int_0^\xi \left( \frac{d\eta}{d\xi} + b\eta''(b) - \eta'(b) - (\eta'(a) - \eta'(b) + b\eta''(b))a^{-1}\xi \right) \quad d\xi$$
$$= \eta(\xi) - \eta(0) - (\eta'(b) - b\eta''(b))\xi - \frac{\eta'(a) - \eta'(b) + b\eta''(b)}{2a} \xi^2.$$

The four-fold integral was evaluated using repeated integration by parts. Applying the inverse operator $L_6^{-1}$ to both sides of Eq. (56) yields

$$\eta(\xi) = A + (C - bD)\xi + \frac{B - C + bD}{2a} \xi^2 - L_6^{-1} f(\xi) g(\eta). \tag{58}$$

The decomposition of the solution $\eta(\xi)$ and the nonlinearity $g(\eta)$ is carried out according to Eqs. (33) and (34) in order to obtain the recursive relation for the solution components

$$\eta_0 = A + (C - bD)\xi + \frac{B - C + bD}{2a} \xi^2,$$
$$\eta_{r+1} = -L_6^{-1}(f(\xi)A_r), \quad r \geq 0. \tag{59}$$

### Numerical Examples

This section presents four carefully selected numerical examples to rigorously evaluate the performance, convergence, and accuracy of the proposed method. Each example considers distinct values of $n$ and specific forms of the functions $f(\xi)$ and $g(\eta)$, as cited in [7, 16, 19]. The results are systematically organized in comprehensive tables and illustrated through graphical plots highlighting the absolute errors and demonstrating the method's reliability, precision, and efficiency in solving fourth-order LEF equations.

**Example 1** ***Example 4.1.*** *We begin by considering the LEF equation*

$$\eta'''' + \frac{3}{\xi} \eta''' = 96(1 - 10\xi^4 + 5\xi^8)e^{-4\eta}, \tag{60}$$

*by substituting $n = 3$ in Eqs. (9), (13), (17), (21), (25), and Eq.(29) and defining $f(\xi)g(\eta) = 96(1 - 10\xi^4 + 5\xi^8)e^{-4\eta}$, with initial conditions Eqs. (10), (22) and boundary conditions Eqs. (14), (18), (26), (30), respectively*

$$\eta(0) = \eta'(0) = \eta''(0) = 0, \quad a = 0,$$
$$\eta(0.0001) = 0, \eta(0) = \eta'(0) = 0, \quad a = 0.0001, b = 0,$$
$$\eta(0) = \eta'(0) = \eta''(0) = 0, \quad a = 0, b = 0.0001,$$
$$\eta(0) = \eta'(0) = \eta''(0) = 0, \quad b = 0,$$
$$\eta(0) = 0, \eta(0.01) = 1 * 10^{-8}, \eta'(0) = \eta''(0) = 0, \quad a = 0.01, b = 0,$$
$$\eta(0) = \eta'(0) = 0, \eta'(0.01) = 4 * 10^{-6}, \eta''(0) = 0, \quad a = 0.01, b = 0.$$

*Notice that the conditions are not all zero. Applying these conditions yields $\eta_0 = 0$ when using the operators $L_1, L_2$ and $L_4$ explained in Section 3 of the algorithm (specifically in parts 3.1, 3.2 and 3.4, respectively), while applying the remaining operators $L_3, L_5$ and $L_6$ as presented in part 3.3, 3.5 and 3.6 of the same section, resulted in different values of $\eta_0$.*
*Given the nonlinearity $e^{-4\eta}$ the Adomian polynomials are defined as follows*

$$A_0 = e^{-4\eta_0},$$
$$A_1 = -4\eta_1 e^{-4\eta_0},$$
$$A_2 = \big( -4\eta_2 + 8\eta_1^2 \big) e^{-4\eta_0},$$
$$A_3 = \big( -4\eta_3 + 16\eta_1\eta_2 - \frac{32}{3}\eta_1^3 \big) e^{-4\eta_0}, \tag{61}$$
$$...$$

*From Eq. (39), the computed solution components are*

$$\eta_0(\xi) = 0,$$

$$\eta_1(\xi) = \xi^4 - \frac{5}{14}\xi^8 + \frac{1}{33}\xi^{12},$$
$$\eta_2(\xi) = -\frac{1}{7}\xi^8 + \frac{58}{231}\xi^{12} + ...,$$
$$\eta_3(\xi) = \frac{4}{77}\xi^{12} + ...,$$
$$...$$

*The initial terms of the Taylor expansion series were used to simplify the computations for each solution component, which yielded the series solution.*

$$\eta(\xi) = \xi^4 - \frac{1}{2}\xi^8 + \frac{1}{3}\xi^{12} + ..., \quad (62)$$

*which converges to the exact solution* $\eta(\xi) =$ *ln* $(1 + \xi^4)$.
*Using Eq. (43), the calculated solution components are*

$$\eta_0(\xi) = 0,$$
$$\eta_1(\xi) = -1 * 10^{-8}\xi^2 + \xi^4 - 0.357143\xi^8 + 0.030303\xi^{12},$$
$$\eta_2(\xi) = -3.90476 * 10{-}25\xi^2 + 5.33333 * 10{-}10\xi^6 - 0.142857\xi^8$$
$$-5.333 * 10^{-9}\xi^{10} + 0.251082\xi^{12},$$
$$\eta_3(\xi) = -3.29738{*}10^{-41}\xi^2 + 2.08254{*}10^{-25}\xi^6 + 2.85714{*}10^{-17}\xi^8 -$$
$$2.16178 * 10^{-9}\xi^{10} + 0.051948\xi^{12},$$
$$...$$

*By Taylor expansion terms for each solution component were used to streamline the calculations, ultimately producing the series solution*

$$\eta(\xi) = -1 * 10^{-8}\xi^2 + \xi^4 + 5.33333 * 10^{-10}\xi^6 -$$
$$0.5\xi^8 - 7.49478 * 10^{-9}\xi^{10} + 0.333333\xi^{12} + ..., \quad (63)$$

*which closely approximates the analytical solution* $\eta(\xi) =$*ln* $(1 + \xi^4)$.
*By Eq. (47), the calculated solution components are*

$$\eta_0(\xi) = 4 * 10^{-8}\xi^2,$$
$$\eta_1(\xi) = -4 * 10^{-8}\xi^2 + \xi^4 - 2.13333 * 10^{-8}\xi^6 -$$
$$0.357143\xi^8 + 2.13333 * 10^{-8}\xi^{10} + 0.030303\xi^{12},$$
$$\eta_2(\xi) = -1.69143 * 10^{-25}\xi^2 + 2.13333 * 10^{-8}\xi^6 -$$
$$0.142857\xi^8 - 1.16622 * 10^{-8}\xi^{10} + 0.251082\xi^{12},$$
$$\eta_3(\xi) = -5.16471 * 10^{-39}\xi^2 + 9.02096 * 10^{-26}\xi^6 +$$
$$4.57143 * 10^{-16}\xi^8 - 9.67111 * 10^{-9}\xi^{10} + 0.051948\xi^{12},$$
$$...$$

*The Taylor series expansion provided the initial terms for each solution component, thereby simplifying the computations and resulting in the series solution*

$$\eta(\xi) = \xi^4 - 0.5\xi^8 - 1 * 10^{-14}\xi^{10} + 0.33333\xi^{12} + ..., \quad (64)$$

*yielding convergence to the exact solution* $\eta(\xi) =$*ln* $(1 + \xi^4)$.
*From Eq. (51), the solution is composed of the following calculated components*

$$\eta_0(\xi) = 0,$$
$$\eta_1(\xi) = -1 * 10^{-8}\xi^2 + \xi^4 - 0.357143\xi^8 + 0.030303\xi^{12},$$
$$\eta_2(\xi) = -3.90476 * 10^{-25}\xi^2 + 5.33333 * 10^{-10}\xi^6 - 0.142857\xi^8 -$$
$$5.333 * 10^{-9}\xi^{10} + 0.251082\xi^{12},$$
$$\eta_3(\xi) = -3.29738{*}10^{-41}\xi^2 + 2.08254{*}10^{-25}\xi^6 + 2.85714{*}10^{-17}\xi^8 -$$
$$2.16178 * 10^{-9}\xi^{10} + 0.051948\xi^{12},$$
$$...$$

*The initial terms of the Taylor expansion series were corresponding for each solution component to simplify the computations, yielding the series solution*

$$\eta(\xi) = -1 * 10^{-8}\xi^2 + \xi^4 + 5.33333 * 10^{-10}\xi^6$$
$$-0.5\xi^8 - 7.49478 * 10^{-9}\xi^{10} + 0.333333\xi^{12} + ..., \quad (65)$$

*which is in close agreement with the true solution* $\eta(\xi) =$*ln*$(1 + \xi^4)$.
*Taking Eq .(55), the calculated solution components are*

$$\eta_0(\xi) = 0.0001\xi^2,$$
$$\eta_1(\xi) = -0.0001\xi^2 + \xi^4 - 0.000053333\xi^6 - 0.357143\xi^8 +$$
$$0.000053333\xi^{10} + 0.030303\xi^{12},$$
$$\eta_2(\xi) = -3.90476 * 10^{-13}\xi^2 + 0.0000533333\xi^6 - 0.142857\xi^8 -$$
$$0.0000291556\xi^{10} + 0.251082\xi^{12},$$
$$\eta_3(\xi) = -3.04138 * 10^{-21}\xi^2 + 2.08254 * 10^{-15}\xi^6 +$$
$$2.85714 * 10^{-9}\xi^8 - 0.0000241778\xi^{10} + 0.51948\xi^{12},$$
$$...$$

*By Taylor series expansion provided the initial terms for each solution component, thereby simplifying the computations and resulting in the series solution*

$$\eta(\xi) = -3.90476 * 10^{-13}\xi^2 + \xi^4 + 3.00002 * 10^{-10}\xi^6$$
$$-0.5\xi^8 - 4 * 10^{-11}\xi^{10} + 0.800865\xi^{12} + ..., \quad (66)$$

*which yields a solution that precisely matches the exact solution* $\eta(\xi) =$*ln* $(1 + \xi^4)$.
*Using Eq.(59), the calculated solution components are*

$$\eta_0(\xi) = 0.0002\xi^2,$$
$$\eta_1(\xi) = -0.0002\xi^2 + \xi^4 - 0.000106667\xi^6 - 0.357143\xi^8 +$$
$$0.000106667\xi^{10} + 0.030303\xi^{12},$$
$$\eta_2(\xi) = -2.62857 * 10^{-12}\xi^2 + 0.000106667\xi^6 - 0.142857\xi^8 -$$
$$0.0000583111\xi^{10} + 0.251082\xi^{12},$$
$$\eta_3(\xi) = -6.67105 * 10^{-20}\xi^2 + 1.4019 * 10^{-12}\xi^6 + 1.14286 * 10^{-8}\xi^8$$
$$-0.0000483556\xi^{10} + 0.051948\xi^{12},$$
$$...$$

*The Taylor expansion series were used to simplify the computations for each solution component, which yielded the series solution*

$$\eta(\xi) = -2.62857 * 10^{-12}\xi^2 + \xi^4 + 1.4019 * 10^{-12}\xi^6$$
$$-0.5\xi^8 + 3 * 10^{-10}\xi^{10} + 0.33333\xi^{12} + ..., \quad (67)$$

*which is in close agreement with the true solution* $\eta(\xi) =$*ln* $(1 + \xi^4)$.

*The results obtained for Example 4.1 clearly demonstrate the high efficiency and accuracy of the proposed MADM in solving the fourth-order LEF equation. As depicted in Figure 1, the approximate solution exhibits excellent agreement with the exact analytical solution across the entire computational domain. The two curves are nearly indistinguishable, confirming the strong convergence of the method. The error distribution, also illustrated within the Figure 2, reveals that the approximate error remains extremely small and decreases monotonically as the number of terms in the decomposition series increases. This behavior confirms that the MADM achieves rapid convergence and numerical stability even in the presence of nonlinearities and boundary constraints. Overall, Figure 1 and Table 1, effectively summarizes the performance of the proposed approach, showing that the method yields results that are practically identical to the exact solution, with negligible numerical deviation. This verifies the accuracy, robustness, and reliability of the MADM in handling higher-order nonlinear problems.*

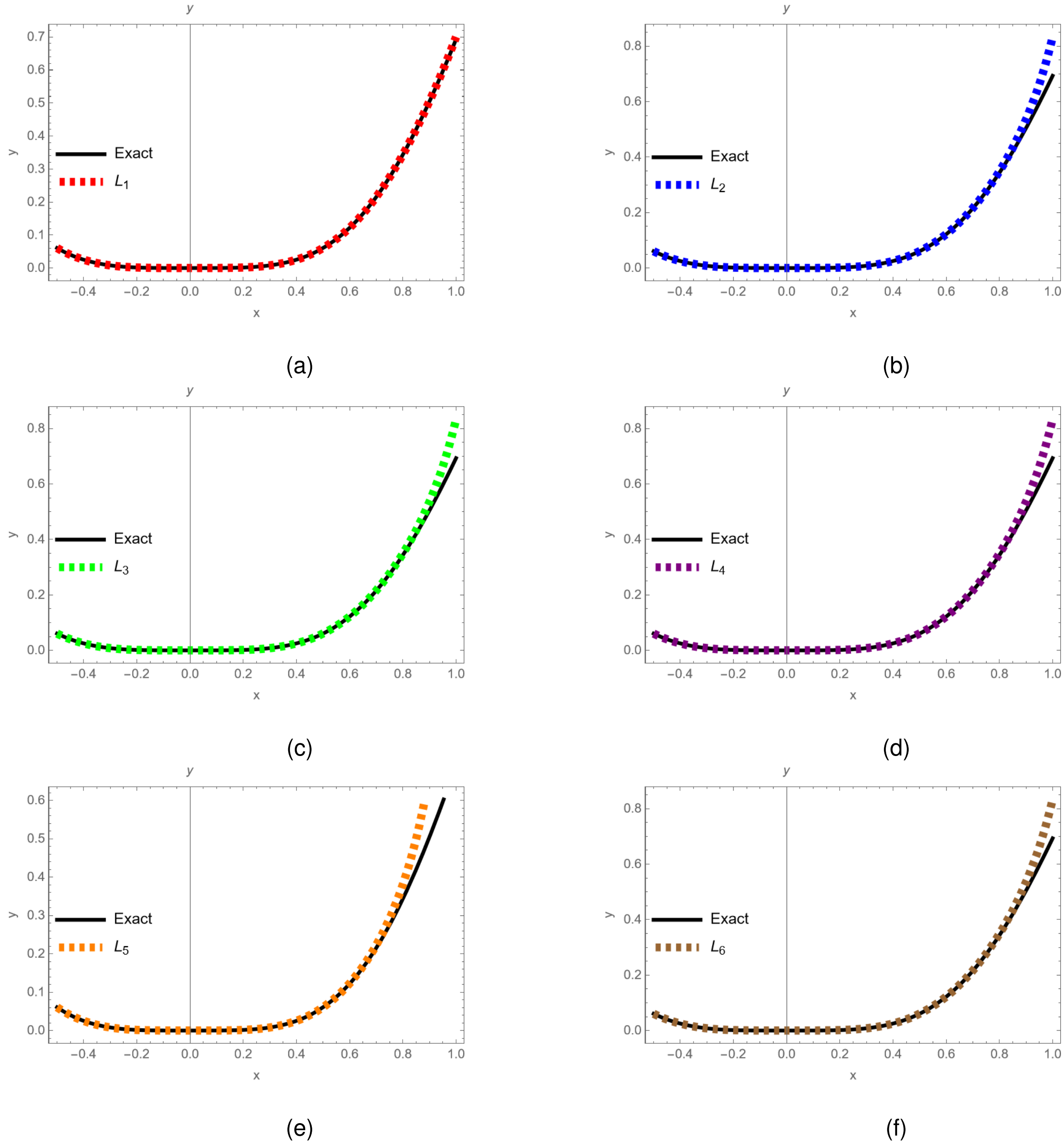



Figure 1: Comparison of the exact solution with the solutions obtained using the MADM $(L_1 - L_6)$

**Example 4.2.** We now analyze the linear LEF equation

$$\eta'''' + \frac{2}{\xi}\eta''' = 4(9 - 16\xi^2 + 4\xi^4)\eta, \tag{68}$$

substituting $n = 2$ in Eqs. (9), (13), (17), (25), and Eq.(29) and defining $f(\xi)g(\eta) = 4(9 - 16\xi^2 + 4\xi^4)\eta$,
with conditions, respectively

$$\eta(0) = 1, \eta(0.01) = 1.0001, \eta''(0) = 2, \eta'(0.01) = 0.020002, \quad a = 0.01,$$
$$\eta(0) = 1, \eta(0.03) = 1.0009, \eta'(0) = 0, \quad a = 0.03, b = 0,$$
$$\eta(0) = 1, \eta'(0) = 0, \eta''(0) = 2, \quad a = b = 0,$$
$$\eta(0) = 1, \eta(0.01) = 1.0001, \eta'(0) = 0, \eta''(0) = 2, \quad a = 0.01, b = 0,$$
$$\eta(0) = 1, \eta'(0) = 0, \eta'(0.001) = 0.002, \eta''(0) = 2, \quad a = 0.001, b = 0.$$

Notice that in this case, singular at $\xi = 0$ due to the presence of the $Ln\xi$ term, the MADM handles this singularity by using a series expansion around $\xi \to 0$. The the conditions are not all zero. It is observed that, despite the variation in applied conditions, the same initial value $\eta_0 = 1 + \xi^2$ is obtained when using the operators $L_2, L_3, L_5$ and $L_6$, whereas $L_1$ serves as an exception due to its distinct mathematical formulation. This consistency among the operators highlights the internal coherence of the proposed method and confirms its capability to effectively accommodate different initial and boundary conditions without compromising numerical stability or convergence.. Subsequently, the recurrence relations Eq.(39), Eq.(43), Eq.(47), Eq.(55), and Eq.(59), respectively, are utilized to obtain the following calculated solution components.

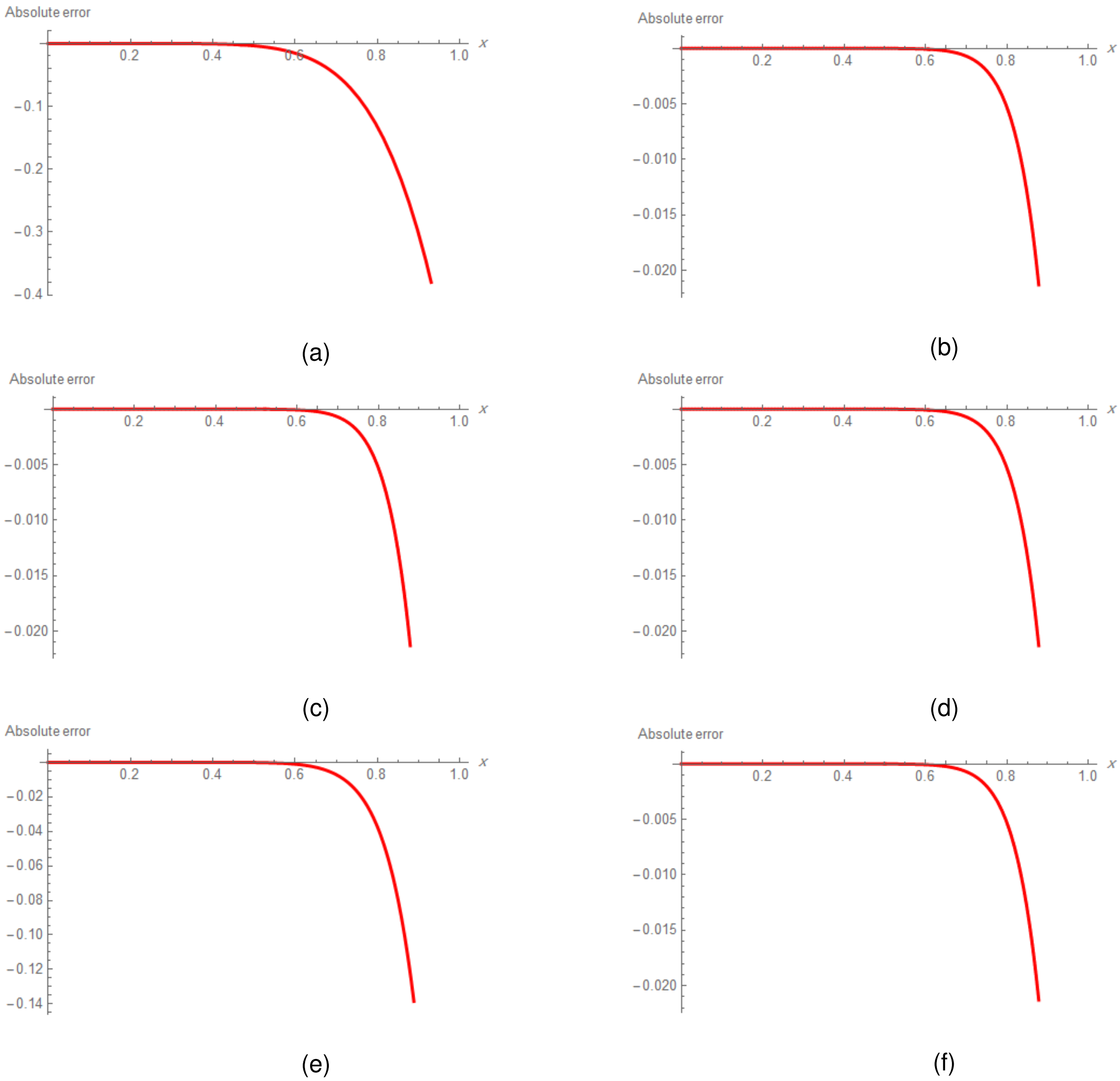



Figure 2: The obtained results exhibit strong and satisfactory convergence towards the exact solution affirming the high quality of performance and robustness of the proposed algorithm by the MADM $(L_1 - L_6)$.

From Eq. (39), the computed solution components are

$$\eta_0(\xi) = 1 + (-9.21034 * 10^{-6} + 2 * 10^{-6} Ln\xi)\xi + \xi^2,$$
$$\eta_1(\xi) = (-7.40814 * 10^{-6} - 1.50008 * 10^{-6} Ln\xi)\xi + (5.19346 * 10^{-20} + 5.63923 * 10^{-23} Ln\xi)\xi^2 + ...,$$
$$\eta_2(\xi) = (-2.27715 * 10^{-14} - 4.58612 * 10^{-15} Ln\xi)\xi + (-1.50954 * 10^{-28} + 4.645 * 10^{-31} Ln\xi)\xi^2 + ...,$$
$$\eta_3(\xi) = (-6.648 * 10^{-23} - 1.33795 * 10^{-23} Ln[\xi])\xi + (-2.01714 * 10^{-36} - 2.8246 * 10^{-37} Ln\xi)\xi^2 + ...,$$
$$...$$

Consequently, the series solution

$$\eta(\xi) = 1 + (-0.0000166185 + 4.9992 * 10^{-7} Ln\xi)\xi + (1 + 1.3323 * 10^{-22} Ln\xi)\xi^2 + ..., \quad (69)$$

which approaches the precise solution $\eta(\xi) = e^{\xi^2}$.

From Eq. (43), the resulting solution components are

$$\eta_0(\xi) = 1 + \xi^2,$$
$$\eta_1(\xi) = -0.000450135\xi^2 + 0.5\xi^4 + 0.16667\xi^6 + 0.0340136\xi^8 + 0.00246914\xi^{10},$$
$$\eta_2(\xi) = 1.63026 * 10^{-11}\xi^2 - 0.0000270081\xi^6 + 0.00764081\xi^8 + 0.0058631\xi^{10} + ...,$$
$$\eta_3(\xi) = -7.05314 * 10^{-19}\xi^2 + 9.78156 * 10^{-13}\xi^6 + 4.43608 * 10^{-13}\xi^8 - 1.50045 * 10^{-7}\xi^{10} + ...,$$
$$...$$

The resulting series solution is obtained

$$\eta(\xi) = 1 + 0.9996\xi^2 + 0.5\xi^4 + 0.166643\xi^6 + 0.0416544\xi^8 + 0.00833209\xi^{10} + ..., \quad (70)$$

which yields a solution that precisely matches the exact solution $\eta(\xi) = e^{\xi^2}$.

From Eq. (47), the derived solution components are

$$\eta_0(\xi) = 1 + \xi^2,$$

Table 1: The approximate solutions and absolute errors for Example 4.1 with n = 3.

| $\xi$ | Exact solution | MADM $L_1$ | Absolute Error | MADM $L_2$ | Absolute Error | MADM $L_3$ | Absolute Error | MADM $L_4$ | Absolute Error | MADM $L_5$ | Absolute Error | MADM $L_6$ | Absolute Error |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.00000 | 0.000000 | 0.00000 | 0.000000 | 0.00000 | 0.000000 | 0.00000 | 0.000000 | 0.00000 | 0.000000 |
| 0.1 | 0.000100 | 0.000100 | 0.000000 | 0.00010 | 0.000000 | 0.00010 | 0.000000 | 0.00010 | 0.000000 | 0.00010 | 0.000000 | 0.00010 | 0.000000 |
| 0.2 | 0.001599 | 0.001599 | 0.000000 | 0.00160 | 0.000000 | 0.00160 | 0.000000 | 0.00160 | 0.000000 | 0.00160 | 0.000000 | 0.00160 | 0.000000 |
| 0.3 | 0.008067 | 0.008067 | 0.000000 | 0.00806 | 0.008060 | 0.00807 | 0.000000 | 0.00807 | 0.000000 | 0.00807 | 0.000000 | 0.00807 | 0.000000 |
| 0.4 | 0.025278 | 0.025278 | 0.000000 | 0.02528 | 0.000002 | 0.02528 | 0.000000 | 0.02528 | 0.000000 | 0.02529 | 0.000001 | 0.02528 | 0.000000 |
| 0.5 | 0.060625 | 0.060625 | 0.000000 | 0.06063 | 0.000005 | 0.06063 | 0.000000 | 0.06063 | 0.000000 | 0.06074 | 0.000009 | 0.06063 | 0.000000 |
| 0.6 | 0.121864 | 0.121863 | 0.000001 | 0.12193 | 0.000066 | 0.12193 | 0.000003 | 0.12193 | 0.000003 | 0.12295 | 0.001096 | 0.12193 | 0.000003 |
| 0.7 | 0.215192 | 0.215192 | 0.000000 | 0.21589 | 0.000698 | 0.21589 | 0.000003 | 0.21589 | 0.000003 | 0.22236 | 0.007168 | 0.21589 | 0.000003 |
| 0.8 | 0.343306 | 0.343354 | 0.000048 | 0.34862 | 0.005314 | 0.34862 | 0.005314 | 0.34862 | 0.005314 | 0.38075 | 0.037443 | 0.34862 | 0.005314 |
| 0.9 | 0.504465 | 0.505277 | 0.000812 | 0.53501 | 0.030545 | 0.53501 | 0.030545 | 0.53501 | 0.030545 | 0.66705 | 0.162589 | 0.53501 | 0.030545 |
| 1.0 | 0.693147 | 0.700167 | 0.007200 | 0.83333 | 0.140183 | 0.83333 | 0.140183 | 0.83333 | 0.140183 | 1.30087 | 0.607723 | 0.83333 | 0.140183 |

$$\eta_1(\xi) = \frac{1}{2}\xi^4 + \frac{1}{6}\xi^6 + \frac{5}{147}\xi^8 + \frac{1}{405}\xi^{10},$$

$$\eta_2(\xi) = \frac{3}{392}\xi^8 + \frac{19}{3240}\xi^{10} + \frac{731}{533610}\xi^{12} + \frac{2719}{15651090}\xi^{14} + ...,$$

$$\eta_3(\xi) = \frac{9}{474320}\xi^{12} + \frac{3091}{125208720}\xi^{14} + ...,$$

$$...$$

Leads to the series solution.

$$\eta(\xi) = 1+\xi^2+\frac{1}{2}\xi^4+\frac{1}{6}\xi^6+\frac{1}{24}\xi^8+\frac{1}{120}\xi^{10}+\frac{1}{720}\xi^{12}+\frac{1}{5040}\xi^{14}+..., \quad (71)$$

which is in close agreement with the true solution $\eta(\xi) = e^{\xi^2}$.
From Eq. (55), the obtained solution components are

$$\eta_0(\xi) = 1 + \xi^2,$$

$$\eta_1(\xi) = -0.00005\xi^2+0.5\xi^4+0.16667\xi^6+0.0340136\xi^8+0.00246914\xi^{10},$$

$$\eta_2(\xi) = 2.234871296{*}10^{-14}\xi^2-3{*}10^{-6}\xi^6+0.00765\xi^8+0.0058641\xi^{10}+...,$$

$$\eta_3(\xi) = -1.1932718 * 10^{-23}\xi^2 + 1.3409228 * 10^{-15}\xi^6 + 6.081 * 10^{-16}\xi^8 - 1.6667 * 10^{-8}\xi^{10} + ...,$$

$$...$$

The series solution is established

$$\eta(\xi) = 1+0.99995\xi^2+0.5\xi^4+0.166664\xi^6+0.0416653\xi^8+0.00833319\xi^{10}+..., \quad (72)$$

yielding the exact solution $\eta(\xi) = e^{\xi^2}$.
From Eq. (59), the determined solution components are

$$\eta_0(\xi) = 1 + \xi^2,$$

$$\eta_1(\xi) = -1{*}10^{-6}\xi^2+0.5\xi^4+0.16667\xi^6+0.0340136\xi^8+0.0024691\xi^{10},$$

$$\eta_2(\xi) = 1.49388{*}10^{-19}\xi^2-6{*}10^{-8}\xi^6+0.007653\xi^8+0.0058642\xi^{10}+...,$$

$$\eta_3(\xi) = -2.533 * 10^{-23}\xi^2 + 8.9632755 * 10^{-20}\xi^6 + 4.06498 * 10^{-23}\xi^8 - 3.333335 * 10^{-10}\xi^{10} + ...,$$

$$...$$

The series solution is yielded

$$\eta(\xi) = 1+0.9999\xi^2+0.5\xi^4+0.1667\xi^6+0.0416666\xi^8+0.0083\xi^{10}+..., \quad (73)$$

convergence to the exact solution $\eta(\xi) = e^{\xi^2}$.

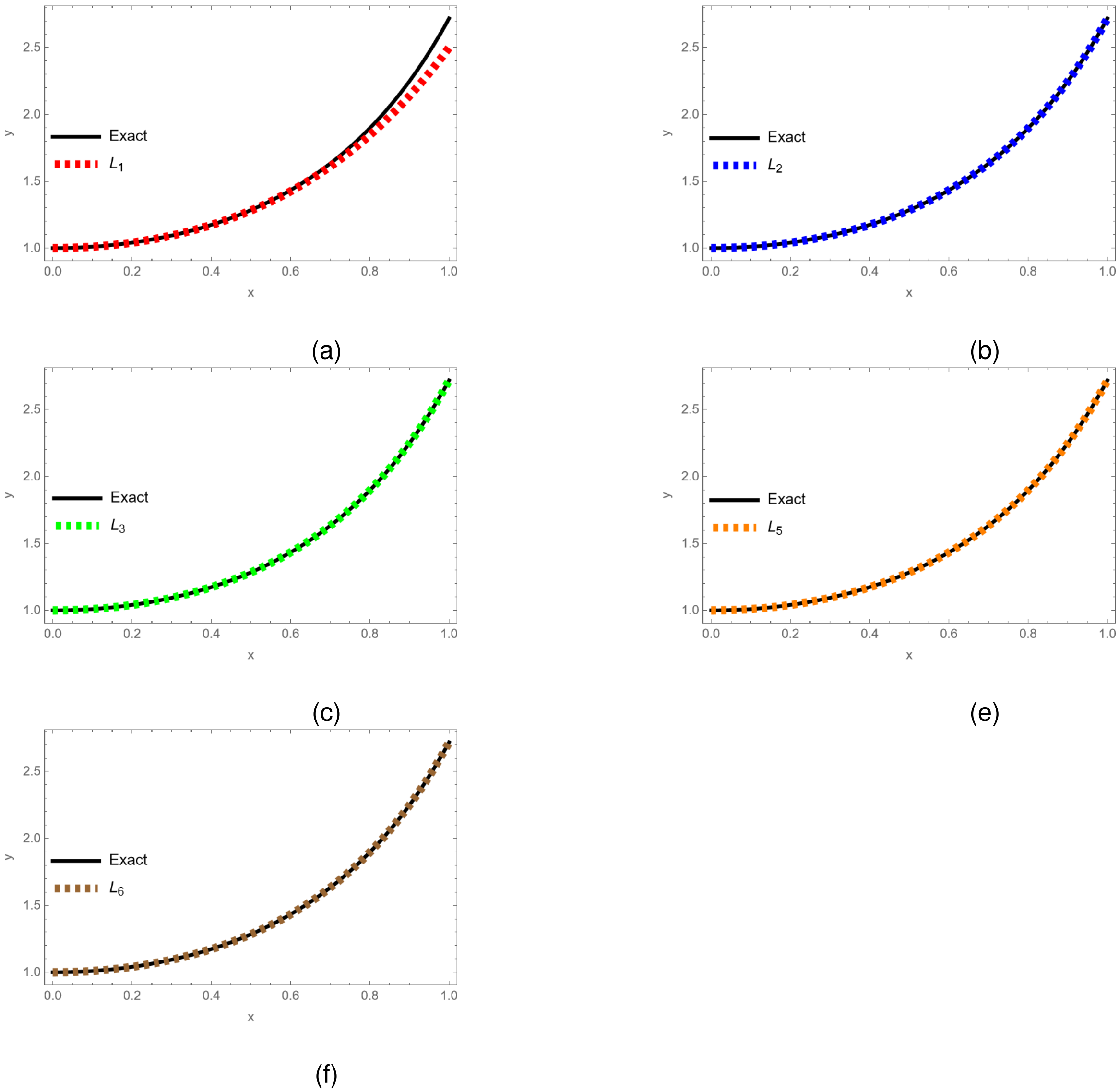



Figure 3: A comparative analysis between the exact solution and the solutions derived from the MADM $(L_1, L_2, L_3, L_5, L_6)$

Table 2 clearly demonstrates the convergence between the exact and approximate solutions under the conditions.

**Example 4.3.** We next consider the LEF equation

$$\eta'''' + \frac{4}{\xi}\eta''' = 15\eta^5(3 - 7\xi^2\eta^2)(1 - \xi^2\eta^2), \tag{74}$$

by substituting $n = 4$ in Eqs. (13), (17), (21), (25), and Eq.(29) and by defining $f(\xi)g(\eta) = 15\eta^5(3 - 7\xi^2\eta^2)(1 - \xi^2\eta^2)$,
with conditions, respectively

$$\eta(1) = 0.447, \eta(0) = 0.5, \eta'(0) = 0, \quad a = 1, b = 0,$$

$$\eta(1) = 0.447, \eta'(0) = 0, \eta'(0.1) = -0.0125, \eta''(0.1) = -0.124,$$
$$a = 1, b = 0.1,$$

$$\eta(0) = 0.5, \eta(0.01) = 0.499994, \eta'(0) = 0, \eta''(0) = 0, \quad b = 0.01,$$

$$\eta(0) = 0.5, \eta(1) = 0.447, \eta'(0) = 0, \eta''(0) = -0.125, \quad a = 1, b = 0,$$

$$\eta(0) = 0.5, \eta'(0) = 0, \eta'(0.02) = -0.0025, \eta''(0) = -0.125,$$
$$a = 0.02, b = 0.$$

Given the nonlinearity $\eta^5, \eta^2$ the Adomian polynomials are defined as Eq. (35)
From Eq. (43), the calculated solution components are

$$\eta_0(\xi) = 0.5 - 0.052794\xi^2,$$
$$\eta_1(\xi) = -0.00979147\xi^2 + 0.0117188\xi^4 - 0.00227892\xi^6 + 0.000406088\xi^8 - 0.0000613389\xi^{10} + ...,$$
$$\eta_2(\xi) = 0.0000797423\xi^2 - 0.00016392\xi^6 + 0.000126849\xi^8 - 0.0000556863\xi^{10} + ...,$$
$$...$$

The series solution is

$$\eta(\xi) = 0.5 - 0.0625057\xi^2 + 0.0117188\xi^4 - 0.00244284\xi^6 + 0.000532937\xi^8 - 0.000117025\xi^{10} + ..., \tag{75}$$

which closely approximates the analytical solution $\eta(\xi) = \frac{1}{\sqrt{4+\xi^2}}$.
From Eq. (47), the calculated solution components are

$$\eta_0(\xi) = 0.509714 - 0.0625\xi^2,$$

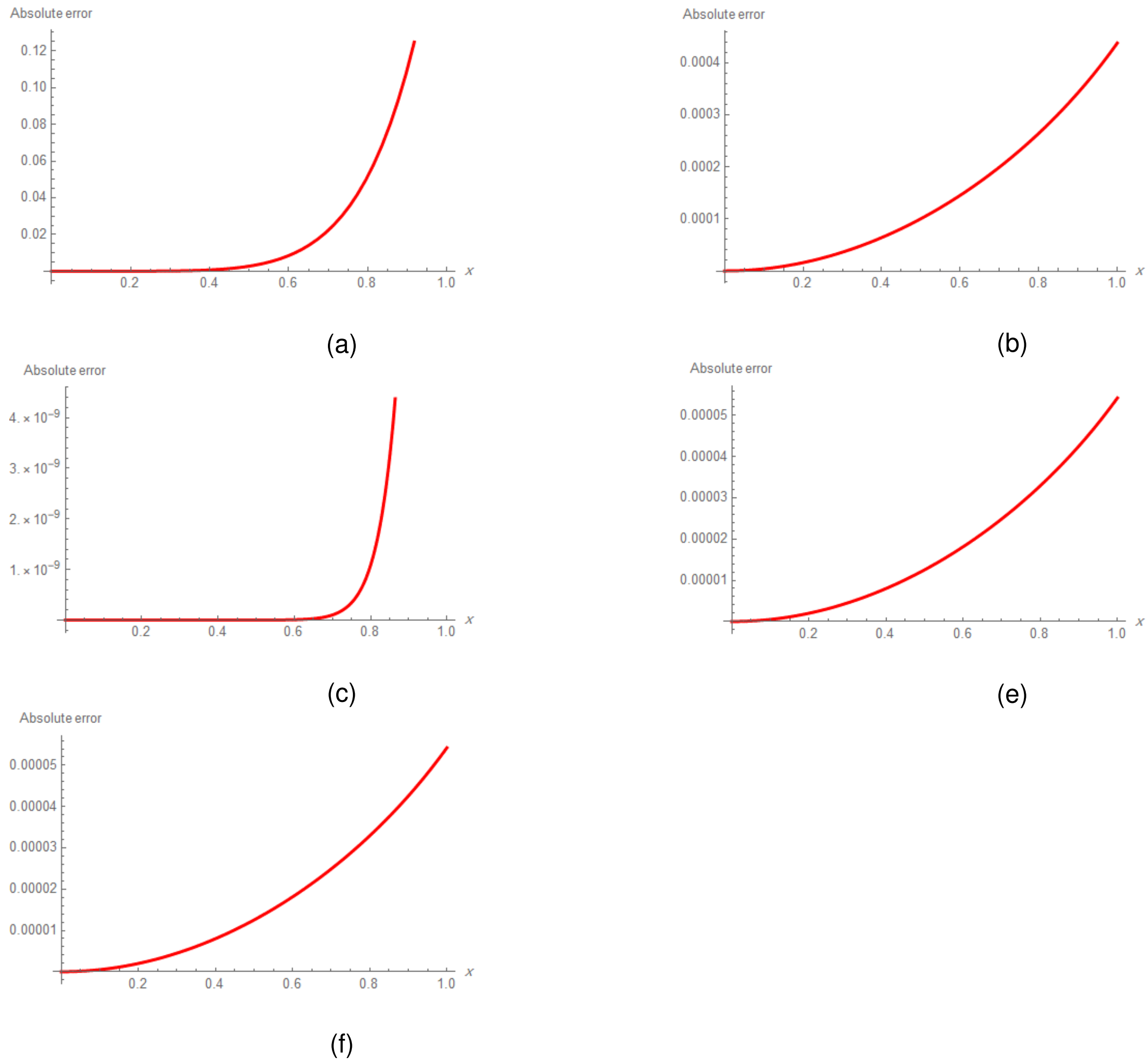



Figure 4: An exceptional convergence of the solution towards the exact solution is observed, which demonstrates the high reliability and efficacy of the MADM $(L_1, L_2, L_3, L_5, L_6)$

$$\eta_1(\xi) = -0.0106302 - 4.3005*10^{-6}\xi^2 + 0.0129022\xi^4 - 0.00272627\xi^6 + 0.000538199\xi^8 - 0.0000910358\xi^{10} + ...,$$

$$\eta_2(\xi) = 0.00104419 + 4.48441*10^{-7}\xi^2 - 0.00134539\xi^4 + 0.00032722\xi^6 - 2.73842*10^{-6}\xi^8 - 0.0000359778\xi^{10} + ...,$$

$$...$$

Yielding the series solution

$$\eta(\xi) = 0.500128 - 0.0625039\xi^2 + 0.0115568\xi^4 - 0.00239905\xi^6 + 0.000535461\xi^8 - 0.000127014\xi^{10} + ..., \quad (76)$$

which converges to the exact solution $\eta(\xi) = \frac{1}{\sqrt{4+\xi^2}}$.

From Eq. (51), the calculated solution components are

$$\eta_0(\xi) = 0.5 - 0.0527864\xi^2,$$

$$\eta_1(\xi) = -1.1719*10^{-6}\xi^2 + 0.01172\xi^4 - 0.00228\xi^6 + 0.00041\xi^8 - 0.000061325\xi^{10} + ...,$$

$$\eta_2(\xi) = 1.4168*10^{-16}\xi^2 - 1.9619*10^{-8}\xi^6 + 0.000055\xi^8 - 0.000037111\xi^{10} + ...,$$

$$...$$

The resulting series solution is obtained

$$\eta(\xi) = 0.5 - 0.05279\xi^2 + 0.0117198\xi^4 - 0.002279\xi^6 + 0.00046054\xi^8 - 0.00009844\xi^{10} + ..., \quad (77)$$

which is in close agreement with the true solution $\eta(\xi) = \frac{1}{\sqrt{4+\xi^2}}$.

Using Eq. (55), the calculated solution components are

$$\eta_0(\xi) = 0.5 + 0.0528\xi^2,$$

$$\eta_1(\xi) = -0.0116\xi^2 + 0.01172\xi^4 - 0.0005112\xi^6 + 0.0004061\xi^8 - 0.00005717\xi^{10} + ...,$$

$$\eta_2(\xi) = 0.00013\xi^2 - 0.000194\xi^6 + 0.00009452\xi^8 - 0.00004\xi^{10} + ....,$$

$$...$$

The series solution is

$$\eta(\xi) = 0.5 - 0.06424\xi^2 + 0.01172\xi^4 - 0.000705\xi^6 + 0.000501\xi^8 - 0.000096\xi^{10} + ..., \quad (78)$$

which converges to the exact solution $\eta(\xi) = \frac{1}{\sqrt{4+\xi^2}}$.

By Eq. (59), the calculated solution components are

$$\eta_0(\xi) = 0.5 - 0.0625\xi^2,$$

$$\eta_1(\xi) = -9.37383*10^{-6}\xi^2 + 0.01172\xi^4 - 0.0024\xi^6 +$$

Table 2: An evaluation of the convergence between the exact solution and the MADM-generated solutions for Example 2 with n = 2.

| $\xi$ | Exact solution | MADM $L_1$ | Absolute Error | MADM $L_2$ | Absolute Error | MADM $L_3$ | Absolute Error |
|---|---|---|---|---|---|---|---|
| 0.1 | 1.01005 | 1.01005 | 0.00000 | 1.01005 | 0.00000 | 1.01005 | 0.00000 |
| 0.2 | 1.04081 | 1.04080 | 0.00001 | 1.04079 | 0.00002 | 1.04081 | 0.00000 |
| 0.3 | 1.09417 | 1.09404 | 0.00013 | 1.09414 | 0.00003 | 1.09417 | 0.00000 |
| 0.4 | 1.17351 | 1.17279 | 0.00072 | 1.17345 | 0.00006 | 1.17351 | 0.00000 |
| 0.5 | 1.28403 | 1.28124 | 0.00279 | 1.28392 | 0.00011 | 1.28403 | 0.00000 |
| 0.6 | 1.43333 | 1.42479 | 0.00854 | 1.43318 | 0.00015 | 1,43333 | 0.00000 |
| 0.7 | 1.63232 | 1.61004 | 0.02228 | 1.63212 | 0.00020 | 1.63232 | 0.00000 |
| 0.8 | 1.89648 | 1.84479 | 0.05169 | 1.89622 | 0.00026 | 1.89648 | 0.00000 |
| 0.9 | 2.24791 | 2.13803 | 0.10988 | 2.24757 | 0.00034 | 2.24791 | 0.00000 |
| 1.0 | 2.71828 | 2.49948 | 0.21880 | 2.71784 | 0.00044 | 2.71828 | 0.00000 |

| MADM $L_5$ | Absolute Error | MADM $L_6$ | Absolute Error |
|---|---|---|---|
| 1.01005 | 0.00000 | 1.01005 | 0.00000 |
| 1.04081 | 0.00000 | 1.04081 | 0.00000 |
| 1.09417 | 0.00000 | 1.09417 | 0.00000 |
| 1.17350 | 0.00001 | 1.17350 | 0.00001 |
| 1.28401 | 0.00002 | 1.28400 | 0.00003 |
| 1.43331 | 0.00002 | 1.43324 | 0.00009 |
| 1.63229 | 0.00003 | 1.63227 | 0.00005 |
| 1.89645 | 0.00003 | 1.89642 | 0.00006 |
| 2.24787 | 0.00004 | 2.24783 | 0.00008 |
| 2.71823 | 0.00005 | 2.71818 | 0.00001 |

$$0.00048\xi^8 - 0.0001\xi^{10} + ...,$$

$$\eta_2(\xi) = 6.1361 * 10^{-14}\xi^2 - 1.5693 * 10^{-7}\xi^6 +$$

$$0.00005457\xi^8 - 0.000039\xi^{10} + ...,$$

$$...$$

Yielding the series solution

$$\eta(\xi) = 0.5\ - 0.06251\xi^2 + 0.01172\xi^4 - 0.002442\xi^6 + 0.00053\xi^8 - 0.00012\xi^{10} + ..., \quad (79)$$

which converges to the exact solution $\eta(\xi) = \frac{1}{\sqrt{4+\xi^2}}$.

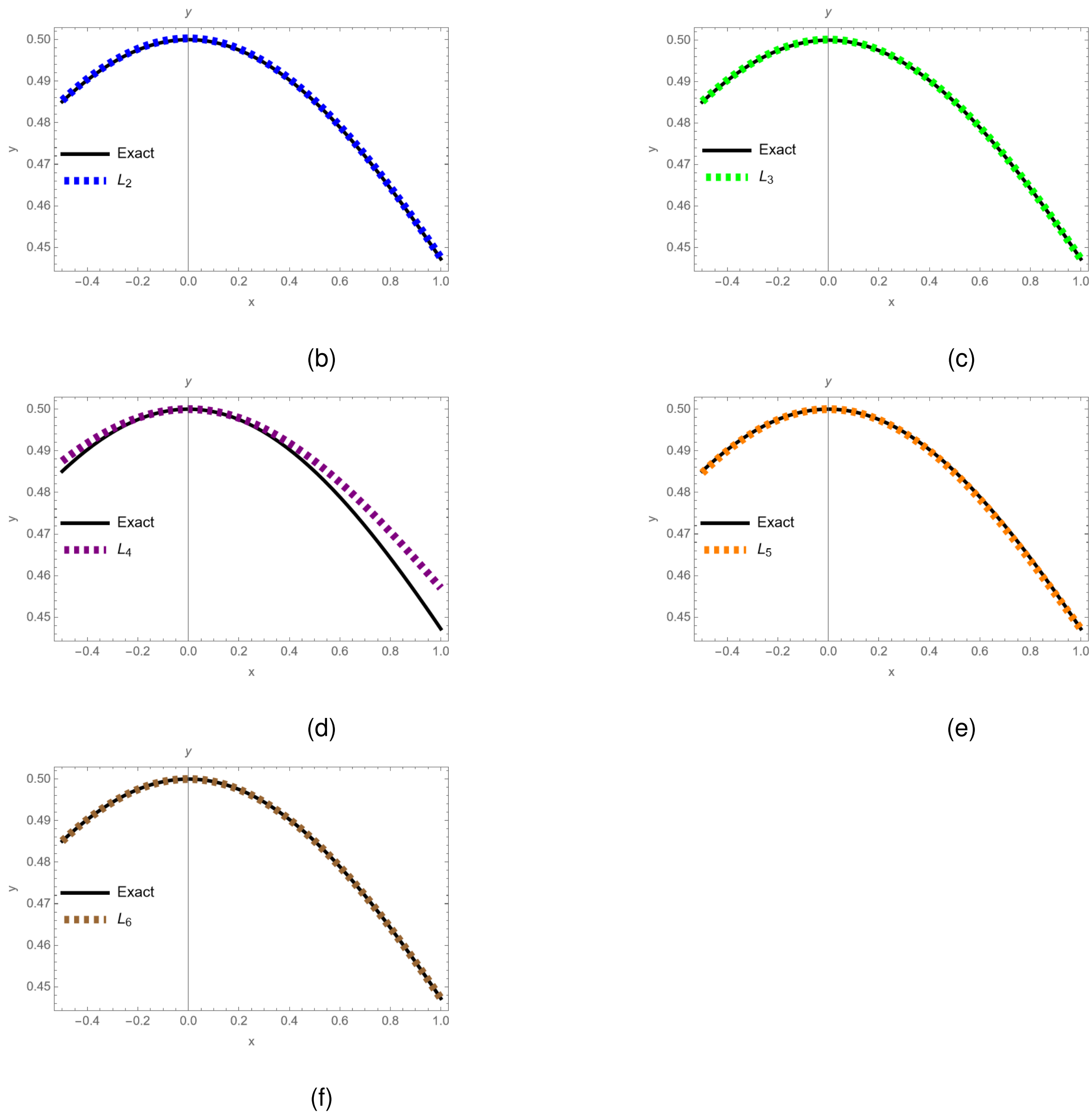



Figure 5: A comparative of the exact solution and the solutions produced by the MADM $(L_2, L_3, L_4, L_5, L_6)$

**Example 4.4.** Finally, we consider the LEF equation

$$\eta'''' + \frac{4}{x}\eta''' = -\eta^m, \tag{80}$$

by substituting $n = 4$ in Eqs. (11), (15), (23), and Eq.(27) and by defining $g(\eta) = \eta^m$,
with conditions, respectively

$$\eta(0) = 1, \eta'(0) = 0, \eta''(0) = 0, \quad a = 0,$$

$$\eta(0) = \eta(0.001) = 1, \eta'(0) = 0, \quad a = 0.001, b = 0,$$

$$\eta(0) = \eta(0.001) = 1, \eta'(0) = 0, \eta''(0) = 0 \quad b = 0.001,$$

$$\eta(0) = \eta(0.01) = 1, \eta'(0) = \eta''(0) = 0, \quad a = 0.01, b = 0,$$

Given the nonlinearity $-\eta^m$ the Adomian polynomials are defined as follows

$$A_0 = -\eta_0^m,$$

$$A_1 = -m\eta_1\eta_0^{m-1},$$

$$A_2 = -(m\eta_2\eta_0^{m-1} + \frac{m(m-1)}{2}\eta_1^2\eta_0^{m-2}), \tag{81}$$

...

From Eq. (39), the calculated solution components are

$$\eta_0(\xi) = 1,$$

$$\eta_1(\xi) = -\frac{1}{120}\xi^4,$$

$$\eta_2(\xi) = \frac{m}{362880}\xi^8,$$

$$\eta_3(\xi) = -\frac{m(-63 + 68m)}{31135104000}\xi^{12},$$

...

The series solution is

$$\eta(\xi) = 1 - \frac{1}{120}\xi^4 + \frac{m}{362880}\xi^8 - \frac{m(-63 + 68m)}{31135104000}\xi^{12} + ..., \tag{82}$$

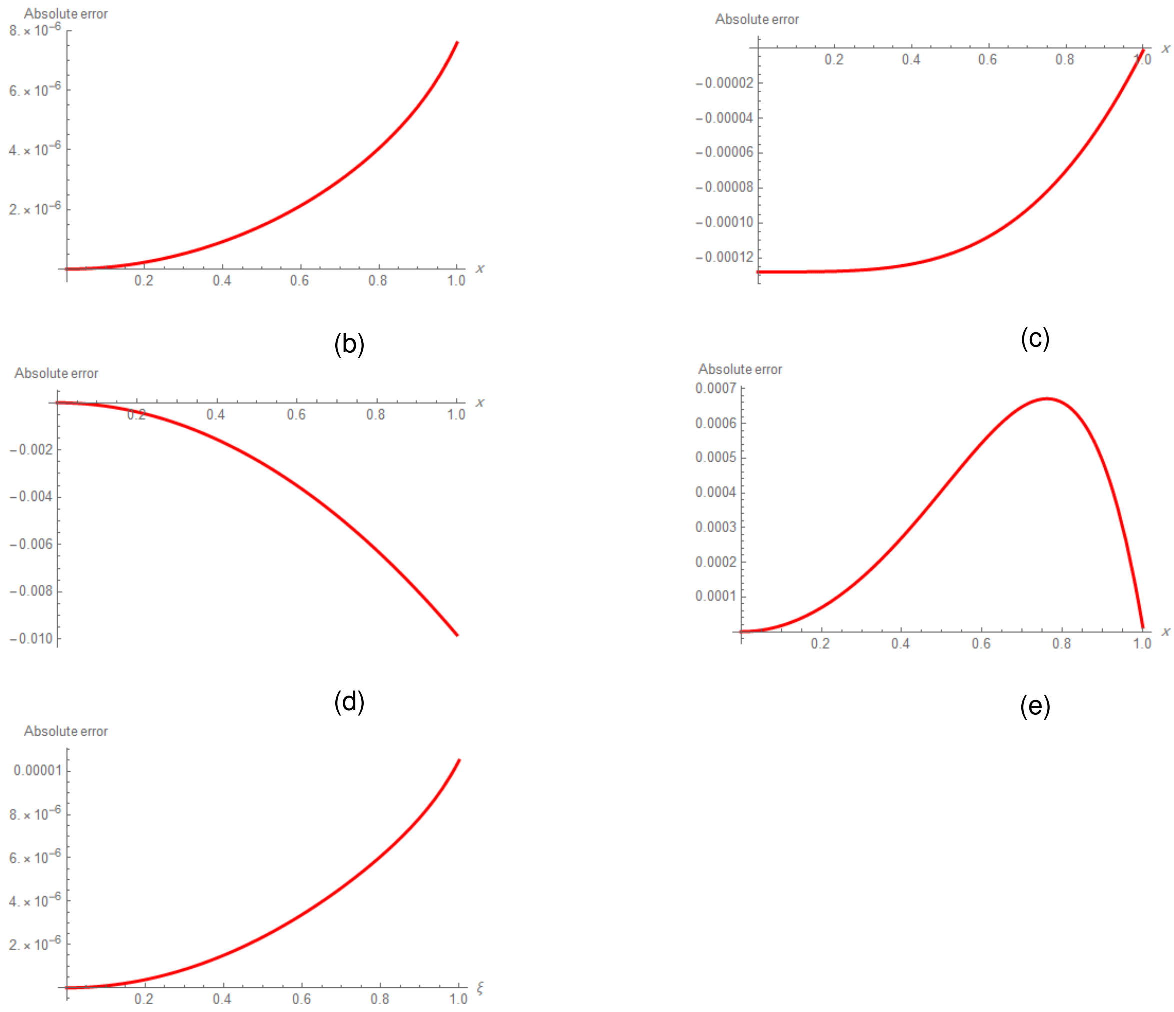
at $m = 0$, the exact solution is given as $\eta(\xi) = 1 - \frac{1}{120}\xi^4$.

From Eq. (43), the calculated solution components are

$$\eta_0(\xi) = 1,$$

$$\eta_1(\xi) = 8.33333 * 10^{-9}\xi^2 - 0.00833333\xi^4,$$

$$\eta_2(\xi) = 7.1649 * 10^{-24}m\xi^2 - 9.92063 * 10^{-12}m\xi^6 +$$




(b) (c) (d) (e) (f)

Figure 6: A comprehensive visual analysis of the absolute error plots for increasing terms $L_2$ to $L_6$ decisively demonstrates the high fidelity and robustness of the MADM.

$$2.75573 * 10^{-6} m\xi^8,$$
$$\eta_3(\xi) = (4.73741 * 10^{-29} m + 2.70021 * 10^{-29} m^2)\xi^2 -$$
$$8.52964 * 10^{-27} m^2 \xi^6 + (-1.14822 * 10^{-20} m +$$
$$1.14822 * 10^{-20} m^2)\xi^8 + ...,$$
$$...$$

Yielding the series solution

$$\eta(\xi) = 1 + 8.33333 * 10^{-9}\xi^2 + 7.1649 * 10^{-24} m\xi^2 +$$
$$(4.73741 * 10^{-29} m + 2.70021 * 10^{-29} m^2)\xi^2 - 0.00833333\xi^4 -$$
$$9.92063 * 10^{-12} m\xi^6 - 8.52964 * 10^{-27} m^2\xi^6 + 2.75573 * 10^{-6} m\xi^8 +$$
$$(-1.14822 * 10^{-20} m + 1.1482 * 10^{-20} m^2)\xi^8 + ..., \quad (83)$$

when $m = 0$, which converges to the exact solution $\eta(\xi) = 1 - \frac{1}{120}\xi^4$. Using Eq. (47), the calculated solution components are

$$\eta_0(\xi) = 1,$$
$$\eta_1(\xi) = 8.33333 * 10^{-9}\xi^2 - 0.00833333\xi^4,$$
$$\eta_2(\xi) = 7.1649 * 10^{-24} m\xi^2 - 9.92063 * 10^{-12} m\xi^6 +$$
$$2.75573 * 10^{-6} m\xi^8,$$
$$\eta_3(\xi) = (4.73741 * 10^{-29} m + 1.2175 * 10^{-28} m^2)\xi^2 -$$
$$8.52964 * 10^{-25} m^2\xi^6 +$$
$$(1.14822 * 10^{-20} m - 1.14822 * 10^{-20} m^2)\xi^8 + ...,$$
$$...$$

The series solution is

$$\eta(\xi) = 1 + 8.33333 * 10^{-9}\xi^2 + 7.1649 * 10^{-24} m\xi^2 +$$
$$(-4.73741 * 10^{-29} m + 1.2175 * 10^{-28} m^2)\xi^2 - 0.00833333\xi^4 -$$
$$9.92063 * 10^{-12} m\xi^6 - 8.52964 * 10^{-25} m^2\xi^6 + 2.75573 * 10^{-6} m\xi^8 + ..., \quad (84)$$

note that $m = 0$, which converges to the exact solution $\eta(\xi) = 1 - \frac{1}{120}\xi^4$.
By Eq. (55), the calculated solution components are

$$\eta_0(\xi) = 1,$$
$$\eta_1(\xi) = 8.3333 * 10^{-7}\xi^2 - 0.0083333\xi^4,$$
$$\eta_2(\xi) = 7.16487 * 10^{-24} m\xi^2 - 9.9206 * 10^{-12} m\xi^6 + 2.75572 * 10^{-6} m\xi^8,$$
$$\eta_3(\xi) = 1.2175 * 10^{-38}(-0.389108 + m)m\xi^2 -$$
$$8.52961 * 10^{-27} m^2\xi^6 - 1.14821 * 10^{-20}(-1 + m)m\xi^8 + ...,$$
$$...$$

Table 3: The approximate solutions and absolute errors for Example 4.3 with n = 4.

| $\xi$ | Exact solution | MADM $L_2$ | Absolute Error | MADM $L_3$ | Absolute Error | MADM $L_4$ | Absolute Error | MADM $L_5$ | Absolute Error | MADM $L_6$ | Absolute Error |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.500000 | 0.500000 | 0.000000 | 0.500028 | 0.000028 | 0.500000 | 0.000000 | 0.500000 | 0.000000 | 0.500000 | 0.000000 |
| 0.1 | 0.499376 | 0.499376 | 0.000000 | 0.499354 | 0.000022 | 0.499473 | 0.000097 | 0.499375 | 0.000001 | 0.499376 | 0.000000 |
| 0.2 | 0.497519 | 0.497519 | 0.000000 | 0.497500 | 0.000019 | 0.497907 | 0.000388 | 0.497449 | 0.000007 | 0.497518 | 0.000001 |
| 0.3 | 0.494468 | 0.494468 | 0.000000 | 0.494452 | 0.000016 | 0.495342 | 0.000874 | 0.494313 | 0.000155 | 0.494467 | 0.000001 |
| 0.4 | 0.490290 | 0.490289 | 0.000001 | 0.490301 | 0.000011 | 0.491845 | 0.001555 | 0.490100 | 0.000190 | 0.490289 | 0.000001 |
| 0.5 | 0.485071 | 0.485070 | 0.000001 | 0.485100 | 0.000029 | 0.487502 | 0.002431 | 0.485001 | 0.000070 | 0.485069 | 0.000002 |
| 0.6 | 0.478913 | 0.478911 | 0.000002 | 0.478907 | 0.000006 | 0.482416 | 0.003503 | 0.478910 | 0.000003 | 0.478910 | 0.000003 |
| 0.7 | 0.471929 | 0.471926 | 0.000003 | 0.472027 | 0.000002 | 0.476704 | 0.004777 | 0.471927 | 0.000002 | 0.471925 | 0.000004 |
| 0.8 | 0.464238 | 0.464234 | 0.000004 | 0.464308 | 0.000007 | 0.470487 | 0.006249 | 0.464230 | 0.000008 | 0.464232 | 0.000006 |
| 0.9 | 0.455961 | 0.455955 | 0.000006 | 0.456001 | 0.000004 | 0.463889 | 0.007928 | 0.455950 | 0.000011 | 0.455953 | 0.000008 |
| 1.0 | 0.447214 | 0.447206 | 0.000008 | 0.447215 | 0.000004 | 0.457032 | 0.009818 | 0.447200 | 0.000014 | 0.447203 | 0.000011 |

Yielding the series solution

$$\eta(\xi) = 1 + 8.333 * 10^{-7}\xi^2 + 7.16487 * 10^{-24} m\xi^2 +$$

$$1.2175 * 10^{-38}(-0.389108 + m)m\xi^2 - 0.0083333\xi^4 -$$

$$9.9206 * 10^{-11} m\xi^6 - 8.52961 * 10^{-27} m^2\xi^6 + 2.75572 * 10^{-6} m\xi^8 + ..., \quad (85)$$

when $m = 0$, which is in close agreement with the true solution $\eta(\xi) = 1 - \frac{1}{120}\xi^4$.

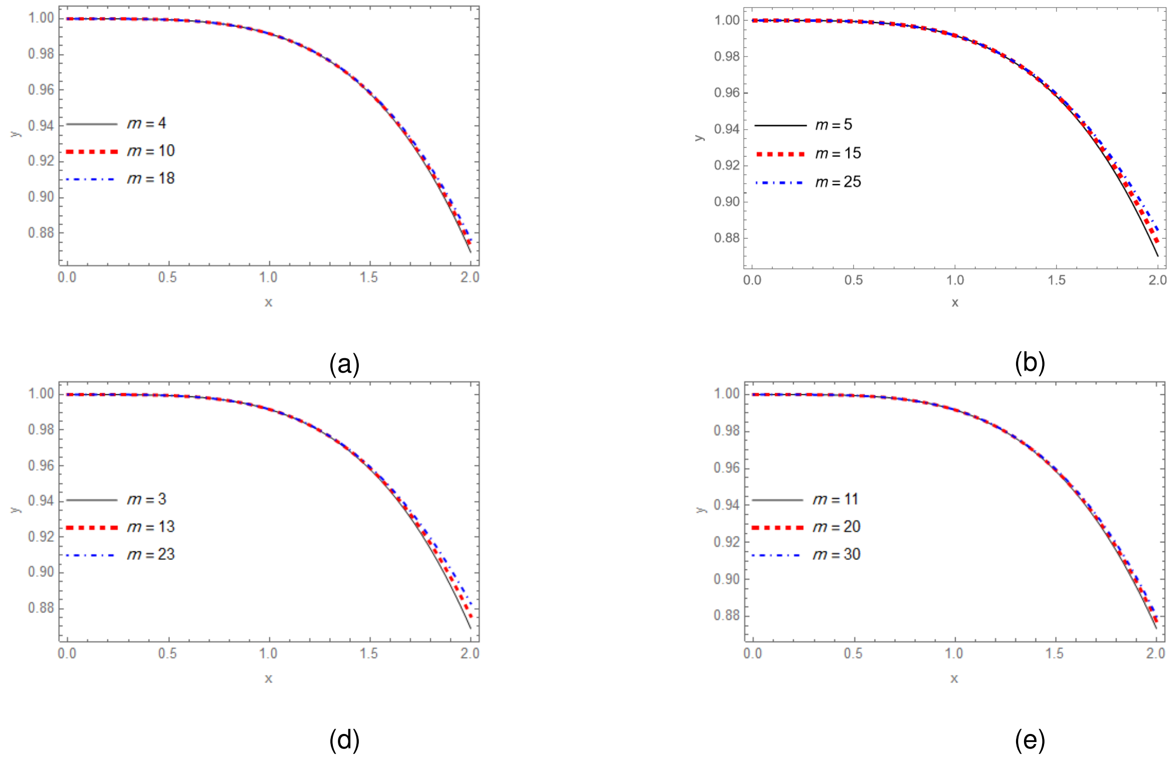



Figure 7: Numerical solutions for Example 4.4 at different values by $(L_1, L_2, L_4, L_5)$

## Conclusion

In this research, a novel modified version of the MADM was successfully developed and applied to various forms of the fourth-order LEF equations, the effectiveness of the proposed approach was comprehensively validated through four distinct examples involving both linear and nonlinear cases under diverse initial and boundary conditions. The numerical results presented in Examples 4.1 - 4.4 clearly demonstrate the strong agreement between the approximate and exact analytical solutions as confirmed by the corresponding figures and tables. The MADM exhibited rapid convergence exceptional numerical stability and minimal absolute error across all tested operators $L_1$ - $L_6$. In particular, the method maintained its high accuracy even in the presence of strong nonlinearities and singular points at $x = 0$ proving its robustness and computational reliability. Moreover, the constructed series solutions using the proposed operators successfully reproduced the exact analytical forms such as $\eta(\xi) = \ln(1 + \xi^4)$, $\eta(\xi) = e^{\xi^2}$, and $\eta(\xi) = 1/\sqrt{4 + \xi^2}$. These results confirm the method's capability to handle higher-order nonlinear problems with remarkable precision. Therefore, the MADM can be considered a powerful and flexible analytical tool for solving a wide range of linear and nonlinear differential equations with singularities.

## Ethics approval and consent to participate

Not applicable

## Consent for publication

Not applicable

## Availability of data and materials

The raw data required to reproduce these findings are available in the body and illustrations of this manuscript.

## Author's contribution

The authors confirm contribution to the paper as follows: Study conception and design: Zainab Ali Abdu AL-Rabahi; Theoretical calculations and modeling: Zainab Ali Abdu AL-Rabahi; Data analysis and validation: Zainab Ali Abdu AL-Rabahi, Yahya Qaid Hasan; Draft manuscript preparation: Zainab Ali Abdu AL-Rabahi. Both authors reviewed the results and approved the final version of the manuscript.

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## Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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