



Fuzzy Radial Basis Function Approach for Solving Linear Volterra Integral Equation Systems of the Second Kind

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Abstract: In this paper, we extend the numerical solution of linear Volterra integral equation systems of the second kind by incorporating fuzzy numbers into the model. While the classical approach relies on compactly supported radial basis functions (RBFs) of Wendland type, our contribution reformulates the problem under uncertainty by representing the input data and kernels as fuzzy quantities. Using the α -cut representation, the fuzzy Volterra integral system is transformed into a family of crisp integral equations, which are then approximated by the RBF variational method. The fuzzy solution is reconstructed from the left and right bounds of these crisp solutions across all α -levels. Convergence in the fuzzy framework is analyzed through the Hausdorff metric, ensuring that the proposed method inherits the convergence properties of the classical scheme. Numerical experiments with triangular fuzzy numbers demonstrate the applicability of the method and show that it provides reliable interval-type solutions that capture the inherent uncertainty of the system. This approach provides a promising numerical framework for Volterra integral equation systems involving uncertain or imprecise data. The presented numerical examples demonstrate the construction of fuzzy solution bands and the decay of the endpoint errors as the number of Wendland centers increases. Future work should investigate adaptive center selection, nonlinear fuzzy Volterra systems, Volterra–Fredholm extensions, and applications to engineering and biological models involving uncertain data.

Keywords: Wendland functions; α -cut representation; Hausdorff metric; uncertainty modeling; meshless approximation.

Research Highlights

- Extends the Wendland radial basis function framework to linear Volterra systems with fuzzy data.
- Uses the α -cut representation to construct lower and upper deterministic endpoint systems.
- Establishes convergence of the reconstructed fuzzy solution in the Hausdorff metric.
- Numerical experiments demonstrate fuzzy solution bands and decreasing endpoint L^2 -errors.

Introduction

Volterra integral equations (VIEs) of the second kind have been the subject of extensive research since the early contributions of Volterra in 1913. These equations naturally model hereditary, cumulative, and memory-dependent processes and thus appear in numerous applied sciences, including viscoelasticity, seismology, epidemiology, quantum mechanics, fluid dynamics, and economics [1, 2]. The single-equation form on $[0, T]$ reads

$$x(t) = f(t) + \int_0^t k(t, s) x(s) ds, \quad 0 \leq s \leq t \leq T, \quad (1.1)$$

where the solution $x(t)$ depends on its past values. Because closed-form solutions are rarely available, numerical methods are indispensable.

Classical numerical approaches include collocation methods, polynomial approximation schemes, block-pulse and triangular functions, and quadrature rules [3–5]. Wavelet-based techniques, beginning with Haar wavelets and extending to multi-resolution approaches, have also been employed to approximate Volterra equations efficiently [6, 7]. Other works have explored semi-analytical methods such as the Adomian decomposition method and the homotopy perturbation method [8, 9]. Despite their utility, these schemes can suffer from stability and computational difficulties in high dimensions or with irregular kernels.

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In recent decades, meshless methods have emerged as powerful alternatives. Radial basis functions (RBFs), which approximate functions using isotropic kernels depending only on the Euclidean distance to centers, are especially attractive due to their flexibility and dimension independence [10]. Compactly supported RBFs, particularly Wendland functions, provide smooth, positive-definite basis functions with local support, resulting in sparse system matrices and reduced computational cost [11]. Recently, González-Rodelas et al. proposed a variational method using Wendland RBFs to solve linear Volterra integral equation systems of the second kind, establishing convergence theorems and demonstrating efficiency through numerical experiments [12].

However, a major limitation of most existing studies, including [12], is the assumption of precisely known input data and kernels. In practice, such parameters are often uncertain, vague, or imprecise, due to measurement errors, incomplete knowledge, or inherent variability. Traditional probability theory is not always suitable to model such vagueness. To address this, Zadeh (1965) introduced fuzzy set theory, which has since become a robust framework for handling imprecise data [13]. A fuzzy number, such as a triangular fuzzy number (TFN), captures uncertainty via a membership function, and a fuzzy function can be defined accordingly.

The solution of fuzzy integral equations typically employs the α -cut representation, reducing a fuzzy problem into a family of crisp problems parameterized by $\alpha \in [0, 1]$. This methodology has been widely adopted in fuzzy differential and integral equations, leading to numerical schemes tailored for uncertain systems [14–18]. These works emphasize the importance of combining numerical analysis with fuzzy modeling. Nevertheless, the integration of meshless RBF methods into the fuzzy framework remains almost unexplored.

Recent studies have also emphasized the importance of reliable numerical methods for fuzzy-type differential and integro-differential models. For example, Abu Arqub et al. developed a Lucas-polynomial Galerkin strategy for fuzzy-type differential models with an application to electrical circuit engineering, while Abu Arqub et al. proposed a Hilbert reproducing kernel algorithm for fuzzy M-fractional integro-differential models [19, 20]. These contributions further highlight the relevance of combining uncertainty modeling with robust numerical approximation techniques.

The motivation for combining RBFs with fuzzy set theory is manifold. First, fuzzy modeling increases realism by explicitly representing uncertainty in system parameters. Second, the flexibility of RBFs complements the variability of fuzzy inputs. Third, the convergence properties of RBF approximations for crisp Volterra systems can be extended to fuzzy systems by considering the Hausdorff metric between fuzzy solutions. Fourth, compactly supported Wendland RBFs ensure computational efficiency even when solving multiple crisp systems across different α -levels.

The contribution of this work is to extend the Wendland radial basis function framework to linear Volterra integral equation systems with fuzzy data [12]. The proposed approach formulates the uncertain problem through the α -cut representation and, under a suitable positivity assumption, reduces it to lower and upper deterministic endpoint systems. These endpoint systems are approximated by compactly supported Wendland RBFs, while the reconstructed fuzzy solution is analyzed in the Hausdorff metric. The numerical experiments illustrate the construction of fuzzy solution bands and show the decay of the lower and upper L^2 -errors as the number of Wendland centers increases.

This paper is structured as follows. The Preliminaries section re-

calls fuzzy numbers and RBFs. The Formulation section presents the fuzzy Volterra formulation and α -cut decomposition. The Numerical Method section introduces the FRBF discretization method. The Convergence Analysis section discusses the convergence properties of the proposed method. The Numerical Experiments section presents the numerical results, followed by the Conclusion and future directions.

Preliminaries

Before we introduce the proposed fuzzy radial basis function (FRBF) method for solving Volterra integral equations of the second kind, it is important to recall in detail some of the key theoretical concepts on which our approach is built. These concepts include fuzzy numbers and their α -cut representation, which allow us to model uncertainty and vagueness in mathematical systems, and radial basis functions (RBFs), particularly the compactly supported Wendland functions, which form the core of our numerical approximation framework.

The theory of fuzzy sets was introduced by Lotfi Zadeh in 1965. For a fuzzy set $\tilde{A}$ defined on a universe U , the membership function $\mu_{\tilde{A}} : U \rightarrow [0, 1]$ assigns a grade of membership to each element. Building upon this, fuzzy numbers are fuzzy sets on $\mathbb{R}$ that are normal, convex, and upper semi-continuous. Among the most widely used fuzzy numbers are triangular fuzzy numbers (TFNs) and trapezoidal fuzzy numbers (TrFNs). A TFN $\tilde{u} = (u_l, u_m, u_r)$ has the α -cut

$$\tilde{u}_\alpha = [u_l + \alpha(u_m - u_l), u_r - \alpha(u_r - u_m)], \quad \alpha \in [0, 1]. \quad (2.1)$$

This representation reduces a fuzzy problem to a parametric family of crisp problems at each α .

RBFs approximate functions via isotropic kernels $\phi(\|x - \xi\|)$. Given centers $\{\xi_j\}_{j=1}^N$, an RBF approximant reads $s(x) = \sum_{j=1}^N \alpha_j \phi(\|x - \xi_j\|)$. Wendland functions are compactly supported, positive definite, and yield sparse, stable systems [10, 11]. For instance,

$$\phi_{1,1}(r) = (1-r)_+^3(3r+1), \quad (1-r)_+ = \max\{0, 1-r\}. \quad (2.2)$$

Wendland functions form a special class of compactly supported radial basis functions. They are constructed to be positive definite on Euclidean spaces and to possess a prescribed degree of smoothness with minimal polynomial degree. Their compact support means that the basis function vanishes outside a finite support radius; that is,

$$\phi(r) = 0 \quad \text{for } r \geq 1,$$

or, more generally, outside a prescribed compact support. This property leads to sparse interpolation or collocation matrices, which is computationally advantageous when the method must be applied repeatedly for several α -levels. In the present work, Wendland functions are used because they combine stability, locality, and computational efficiency in solving the deterministic endpoint Volterra systems generated by the fuzzy α -cut formulation.

Formulation of the Fuzzy Volterra Integral Equation System

The general form of a linear Volterra integral equation of the second kind is given by (1.1). For practical problems, one often deals with systems:

$$x_i(t) = f_i(t) + \sum_{j=1}^m \int_0^t k_{ij}(t, s) x_j(s) ds, \quad i = 1, \dots, m. \quad (3.1)$$

To model uncertainty, we treat the data as fuzzy functions $\tilde{f}_i$ and $\tilde{k}_{ij}$, yielding the fuzzy system

$$\tilde{x}_i(t) = \tilde{f}_i(t) + \sum_{j=1}^m \int_0^t \tilde{k}_{ij}(t, s) \tilde{x}_j(s) ds, \quad 0 \leq t \leq T. \quad (3.2)$$

Using α -cuts, each fuzzy function becomes an interval, e.g.,

$$\begin{aligned} \tilde{x}_i(t)_\alpha &= [\underline{x}_i^\alpha(t), \bar{x}_i^\alpha(t)], \\ \tilde{f}_i(t)_\alpha &= [\underline{f}_i^\alpha(t), \bar{f}_i^\alpha(t)], \\ \tilde{k}_{ij}(t, s)_\alpha &= [\underline{k}_{ij}^\alpha(t, s), \bar{k}_{ij}^\alpha(t, s)]. \end{aligned} \quad (3.3)$$

The endpoint-wise decoupling is not valid in full generality. In the α -cut framework, products of interval-valued quantities must be treated by interval arithmetic; that is, for intervals $[a_1, a_2]$ and $[b_1, b_2]$, one has

$$[a_1, a_2][b_1, b_2] = [\min\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}, \max\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}],$$

which in general couples the lower and upper bounds.

Assumption 1 (Positivity and endpoint decoupling). For every $\alpha \in [0, 1]$, every $(t, s) \in [0, T] \times [0, T]$, and all $i, j = 1, \dots, m$, assume that the lower and upper endpoint kernels satisfy

$$\underline{k}_{ij}^\alpha(t, s) \geq 0, \quad \bar{k}_{ij}^\alpha(t, s) \geq 0,$$

and that the lower and upper endpoint source functions satisfy

$$\underline{f}_i^\alpha(t) \geq 0, \quad \bar{f}_i^\alpha(t) \geq 0, \quad t \in [0, T].$$

Under this positivity assumption, interval multiplication preserves the ordering of the lower and upper endpoints inside the Volterra integral operator. Consequently, for each fixed α , the fuzzy Volterra integral equation system can be reduced to two deterministic endpoint systems: one for the lower solution $\underline{x}_i^\alpha(t)$ and one for the upper solution $\bar{x}_i^\alpha(t)$.

Consequently, under Assumption 1, the fuzzy system admits the following endpoint decomposition:

$$\underline{x}_i^\alpha(t) = \underline{f}_i^\alpha(t) + \sum_{j=1}^m \int_0^t \underline{k}_{ij}^\alpha(t, s) \underline{x}_j^\alpha(s) ds, \quad (3.4)$$

$$\bar{x}_i^\alpha(t) = \bar{f}_i^\alpha(t) + \sum_{j=1}^m \int_0^t \bar{k}_{ij}^\alpha(t, s) \bar{x}_j^\alpha(s) ds. \quad (3.5)$$

Equations (3.4) and (3.5) represent the lower and upper deterministic Volterra systems, respectively, which are solved independently for each fixed α -level.

Based on Assumption 1, the fuzzy Volterra system is transformed, for each fixed α -level, into the lower and upper deterministic endpoint systems (3.4) and (3.5). These endpoint systems are then solved by the FRBF procedure described in the Numerical Method section, and the fuzzy solution is reconstructed from the resulting family of α -cut intervals.

Numerical Method: Fuzzy Radial Basis Function Approximation

For each fixed α , (3.4)–(3.5) are deterministic VIE systems. On a grid of centers $\{t_j\}_{j=1}^N \subset [0, T]$, we approximate each component by

$$\underline{x}_i^\alpha(t) \approx \sum_{j=1}^N c_{ij}^\alpha \phi(\|t - t_j\|), \quad \bar{x}_i^\alpha(t) \approx \sum_{j=1}^N d_{ij}^\alpha \phi(\|t - t_j\|), \quad (4.1)$$

with ϕ of Wendland type; cf. (2.2). The coefficients are obtained by minimizing the residual functional. For the lower system,

$$R_i^\alpha(t) := \underline{x}_i^\alpha(t) - \underline{f}_i^\alpha(t) - \sum_{j=1}^m \int_0^t \underline{k}_{ij}^\alpha(t, s) \underline{x}_j^\alpha(s) ds, \quad (4.2)$$

and the least-squares functional

$$J^\alpha := \sum_{i=1}^m \int_0^T |R_i^\alpha(t)|^2 dt \quad (4.3)$$

is minimized (analogously for the upper system). Compact support of ϕ yields sparse matrices and stable solves.

Algorithm 1 (FRBF procedure for fuzzy Volterra systems).

1. Select a finite set of α -levels, $0 = \alpha_0 < \alpha_1 < \dots < \alpha_L = 1$.
2. For each α_ℓ , compute the lower and upper endpoints of the fuzzy data $\tilde{f}_i(t)^{\alpha_\ell}$ and $\tilde{k}_{ij}(t, s)^{\alpha_\ell}$.
3. Construct the lower and upper deterministic Volterra systems using (3.4) and (3.5).
4. Choose N Wendland RBF centers $t_1, \dots, t_N$ on $[0, T]$.
5. Approximate $\underline{x}_i^{\alpha_\ell}(t)$ and $\bar{x}_i^{\alpha_\ell}(t)$ using Wendland basis expansions.
6. Substitute the approximations into the endpoint systems and determine the unknown coefficients by minimizing the corresponding least-squares residuals.
7. Reconstruct the fuzzy numerical solution as $\tilde{x}_{i,N}(t)^{\alpha_\ell} = [\underline{x}_{i,N}^{\alpha_\ell}(t), \bar{x}_{i,N}^{\alpha_\ell}(t)]$.
8. Repeat the process for all selected α -levels and use the endpoint intervals to plot the fuzzy bands and compute the L^2 -errors.

After computing (4.1) across a discrete set of α -levels, the fuzzy solution $\tilde{x}_i(t)$ is reconstructed from the family $\{[\underline{x}_i^\alpha(t), \bar{x}_i^\alpha(t)] : \alpha \in [0, 1]\}$.

Convergence Analysis

Let h denote the fill distance associated with the set of Wendland RBF centers $\{t_j\}_{j=1}^N \subset [0, T]$, defined by

$$h = \sup_{t \in [0, T]} \min_{1 \leq j \leq N} |t - t_j|.$$

The fill distance measures the largest distance from any point in the computational domain to the nearest RBF center. Thus, smaller values of h correspond to denser distributions of centers and are expected to improve the accuracy of the RBF approximation. In the fuzzy setting, the convergence analysis is applied to the lower and upper endpoint systems generated by the α -cut representation.

To make the convergence argument precise, we state the assumptions under which the endpoint estimates are transferred to the fuzzy setting.

Assumption 2 (Well-posedness and uniform stability). For every $\alpha \in [0, 1]$, the lower and upper endpoint Volterra systems admit unique solutions $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in $L^2(0, T)$. Moreover, the endpoint data and kernels are uniformly bounded with respect to α ; that is, there exist constants $M_f > 0$ and $M_k > 0$, independent of α , such that

$$\sup_{\alpha \in [0, 1]} \left(\|\underline{f}^\alpha\|_{L^2(0, T)} + \|\bar{f}^\alpha\|_{L^2(0, T)} \right) \leq M_f,$$

$$\sup_{\alpha \in [0, 1]} \left(\|\underline{k}^\alpha\| + \|\bar{k}^\alpha\| \right) \leq M_k.$$

We also assume that the stability constants of the deterministic endpoint Volterra systems and the approximation constants of the Wendland-RBF scheme are bounded independently of α . This uniformity condition ensures that a single constant C can be used in the lower and upper endpoint error estimates for all $\alpha \in [0, 1]$.

Deterministically, the Wendland variational method converges with order $O(h^k)$ (fill distance h , smoothness-driven k) [12], i.e.,

$$\|x - x_N\|_{L^2(0, T)} \leq C h^k. \quad (5.1)$$

The following result summarizes the convergence behavior of the proposed FRBF method in the Hausdorff metric.

Theorem 1 (Hausdorff convergence of the FRBF method).

Under Assumption 2, suppose that the Wendland-RBF approximation of each deterministic endpoint problem satisfies

$$\|\underline{x}^\alpha - \underline{x}_N^\alpha\|_{L^2(0, T)} \leq C h^k, \quad \|\bar{x}^\alpha - \bar{x}_N^\alpha\|_{L^2(0, T)} \leq C h^k. \quad (5.2)$$

If the constant C is independent of both α and the number of centers N , then the reconstructed fuzzy numerical solution satisfies

$$d_H(\tilde{x}, \tilde{x}_N) \leq C h^k. \quad (5.3)$$

Proof. For each fixed α , Assumption 1 transforms the fuzzy Volterra system into two deterministic Volterra endpoint systems. By the convergence of the Wendland-RBF variational approximation, the lower and upper endpoint errors are bounded by $C h^k$. Since the stability and approximation constants are assumed to be uniform with respect to α , taking the supremum over all $\alpha \in [0, 1]$ gives

$$\sup_{\alpha \in [0, 1]} \max \left\{ \|\underline{x}^\alpha - \underline{x}_N^\alpha\|_{L^2(0, T)}, \|\bar{x}^\alpha - \bar{x}_N^\alpha\|_{L^2(0, T)} \right\} \leq C h^k.$$

By the definition of the Hausdorff metric for fuzzy functions, this yields

$$d_H(\tilde{x}, \tilde{x}_N) \leq C h^k. \quad \square$$

To pass from the pointwise-in- α endpoint estimates to a fuzzy convergence result, the error bounds must hold uniformly with respect to α . This is precisely the role of Assumption 2, which guarantees that the endpoint data, kernels, stability constants, and approximation

constants are bounded independently of α . Hence, the same constant C can be used for the lower and upper endpoint estimates for all $\alpha \in [0, 1]$.

The Hausdorff metric is used here because a fuzzy-valued function is represented by a family of α -cut intervals rather than by a single crisp function. Therefore, convergence of the fuzzy numerical solution requires controlling the errors of both the lower and upper endpoint functions uniformly over all $\alpha \in [0, 1]$. The Hausdorff metric provides a natural way to measure the distance between two fuzzy functions by taking the largest endpoint discrepancy over all α -levels.

$$d_H(\tilde{u}, \tilde{v}) := \sup_{\alpha \in [0, 1]} \max \left\{ \|\underline{u}^\alpha - \underline{v}^\alpha\|_{L^2(0, T)}, \|\bar{u}^\alpha - \bar{v}^\alpha\|_{L^2(0, T)} \right\}. \quad (5.4)$$

Using the endpoint estimates in (5.2), we obtain

$$d_H(\tilde{x}, \tilde{x}_N) \leq C h^k. \quad (5.5)$$

Consequently, the proposed FRBF method converges in the Hausdorff metric with the same order as the deterministic Wendland-RBF approximation, provided that the uniformity assumptions stated in Assumption 2 are satisfied.

A further requirement for the reconstructed family of α -cuts to define a valid fuzzy-valued solution is the nesting property. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, the α -cut intervals must satisfy

$$[\underline{x}^{\alpha_2}(t), \bar{x}^{\alpha_2}(t)] \subseteq [\underline{x}^{\alpha_1}(t), \bar{x}^{\alpha_1}(t)], \quad t \in [0, T].$$

Equivalently, this means that

$$\underline{x}^{\alpha_1}(t) \leq \underline{x}^{\alpha_2}(t) \leq \bar{x}^{\alpha_2}(t) \leq \bar{x}^{\alpha_1}(t), \quad t \in [0, T].$$

In the continuous fuzzy formulation, this property follows from the nesting of the input fuzzy data and kernels together with the monotonicity of the endpoint Volterra operators under the positivity assumption. In the numerical implementation, the reconstructed α -cut intervals are checked to preserve this ordering across the selected α -levels. If small violations occur due to numerical round-off errors, they can be corrected by a standard monotone post-processing step that enforces the endpoint ordering without changing the overall convergence behavior.

Numerical Experiments

We report two test problems that clarify the FRBF workflow from fuzzy formulations in equation form to α -cut crisp subproblems, together with figures and tables that quantify errors versus the number of Wendland centers N . Throughout, triangular fuzzy numbers (TFNs) are used. For an α -level, a TFN (a_l, a_m, a_r) reduces to the interval $[a_l + \alpha(a_m - a_l), a_r - \alpha(a_r - a_m)]$. Figures (1)–(2) show the fuzzy bands at $\alpha = 0.5$, while Tables (1)–(2) summarize L^2 -errors.

For a fixed α -level, the fuzzy band represents the interval-valued numerical solution obtained from the lower and upper endpoint problems. More precisely, for each $t \in [0, T]$, the fuzzy band is given by

$$[\underline{x}^\alpha(t), \bar{x}^\alpha(t)].$$

The lower curve $\underline{x}^\alpha(t)$ and the upper curve $\bar{x}^\alpha(t)$ bound the uncertainty region associated with the fuzzy solution. Thus, the width of the band reflects the propagated uncertainty from the fuzzy input functions and fuzzy kernels to the numerical solution. In the present experiments, the fuzzy bands are displayed at $\alpha = 0.5$ to illustrate an intermediate level of uncertainty.

The level $\alpha = 0.5$ is selected as a representative intermediate α -level, since it provides a clear visualization of the uncertainty band between the widest support level and the core of the fuzzy solution. The same FRBF procedure can be applied to any other α -level in $[0, 1]$, and the nesting discussion in the Convergence Analysis section describes the expected ordering of the reconstructed intervals as α varies. A more extensive numerical comparison over several α -levels will be considered in future work.

For systems, this interpretation is applied componentwise; that is, each component $\tilde{x}_i(t)$ has its own α -level band $[\underline{x}_i^\alpha(t), \bar{x}_i^\alpha(t)]$.

The L^2 -errors reported in Tables (1) and (2) are computed separately for the lower and upper endpoint approximations. For a generic endpoint approximation $x_N(t)$ and a reference solution $x_{\text{ref}}(t)$, the L^2 -error is defined by

$$E_{L^2} = \left(\int_0^T |x_{\text{ref}}(t) - x_N(t)|^2 dt \right)^{1/2}.$$

This definition is applied to both the lower endpoint solution $\underline{x}^\alpha(t)$ and the upper endpoint solution $\bar{x}^\alpha(t)$ in order to quantify the accuracy of the fuzzy numerical approximation.

In the present numerical experiments, the reference solution $x_{\text{ref}}(t)$ is obtained by solving the corresponding endpoint Volterra systems using a finer Wendland-RBF discretization than the values of N reported in the error tables. This reference solution is then used consistently to compute the lower and upper endpoint L^2 -errors for each reported value of N . Thus, the tabulated errors measure the difference between the FRBF approximation with N centers and a highly resolved endpoint reference solution.

Example 1. We consider a scalar fuzzy VIE on $[0, T]$:

$$\tilde{x}(t) = \tilde{f}(t) + \int_0^t \tilde{k}(t, s) \tilde{x}(s) ds, \quad (6.1)$$

with TFN data

$$\begin{aligned} \tilde{f}(t) &= (0.9 + 0.1t, 1 + 0.1t, 1.1 + 0.1t), \\ \tilde{k}(t, s) &= (0.45, 0.50, 0.55)(t - s). \end{aligned} \quad (6.2)$$

For each α , (6.1) induces the lower/upper crisp equations as in (3.4)–(3.5). The fuzzy band at $\alpha = 0.5$ is in Figure (1). Table (1) shows monotone L^2 -error decay as N doubles.

The results in Table (1) show that both the lower and upper L^2 -errors decrease monotonically as the number of Wendland centers increases. In particular, when N is doubled from 10 to 20 and then from 20 to 40, the errors are approximately reduced by a factor close to two. This behavior indicates a nearly first-order empirical convergence rate for the scalar fuzzy Volterra problem.

Example 2. A coupled fuzzy system on $[0, T]$:

$$\begin{aligned} \tilde{x}_1(t) &= \tilde{f}_1(t) + \int_0^t \tilde{k}_{11}(t, s) \tilde{x}_1(s) ds, \\ \tilde{x}_2(t) &= \tilde{f}_2(t) + \int_0^t \tilde{k}_{22}(t, s) \tilde{x}_2(s) ds. \end{aligned} \quad (6.3)$$

with TFN data

$$\begin{aligned} \tilde{f}_1(t) &= (0.95, 1, 1.05), & \tilde{k}_{11}(t, s) &= (0.90, 1.00, 1.10)(t - s), \\ \tilde{f}_2(t) &= (1.90, 2, 2.10), & \tilde{k}_{22}(t, s) &= (0.95, 1.00, 1.05)(t - s). \end{aligned} \quad (6.4)$$

The fuzzy bands at $\alpha = 0.5$ are shown in Figure (2); the errors appear in Table (2).

A similar monotone decay is observed in Table (2) for the coupled fuzzy Volterra system. As the number of Wendland centers increases from 15 to 30 and then from 30 to 60, the lower and upper endpoint errors of both components decrease consistently. This confirms that the proposed FRBF method remains stable and accurate for systems of fuzzy Volterra integral equations and supports the Hausdorff convergence behavior discussed in the Convergence Analysis section.

The numerical results in Figures (1)–(2) and Tables (1)–(2) consistently report the endpoint behavior of the FRBF approximation at the selected α -level. Together, the figures illustrate the fuzzy solution bands, while the tables quantify the decay of the lower and upper L^2 -errors as the number of Wendland centers increases.

The two test problems collectively validate the theoretical claim (5.5): in smooth cases, nearly first-order empirical rates are observed as N doubles (Figure (1), Table (1)); and in the coupled system, coupling does not degrade accuracy (Figure (2), Table (2)).

Conclusion

In this paper, we developed a fuzzy radial basis function methodology for solving linear Volterra integral equation systems of the second kind under uncertainty. The proposed approach represents fuzzy data and kernels through the α -cut formulation and reduces the fuzzy system, under a suitable positivity assumption, into lower and upper deterministic Volterra endpoint systems. Each deterministic endpoint problem is then approximated using compactly supported Wendland radial basis functions.

The convergence of the proposed FRBF method was established in the Hausdorff metric by combining the deterministic convergence of the Wendland-RBF variational scheme with uniform boundedness and stability assumptions on the α -cut endpoint systems. The numerical experiments confirmed the theoretical findings: the fuzzy bands at $\alpha = 0.5$ illustrate the propagation of uncertainty, while the tabulated L^2 -errors show monotone decay as the number of Wendland centers increases.

The main advantage of the proposed method lies in its ability to combine meshless approximation, sparse computation, and fuzzy uncertainty modeling within a unified framework. Future work will focus on adaptive center selection, nonlinear fuzzy Volterra systems, Volterra–Fredholm extensions, and applications to engineering and biological models involving uncertain data.

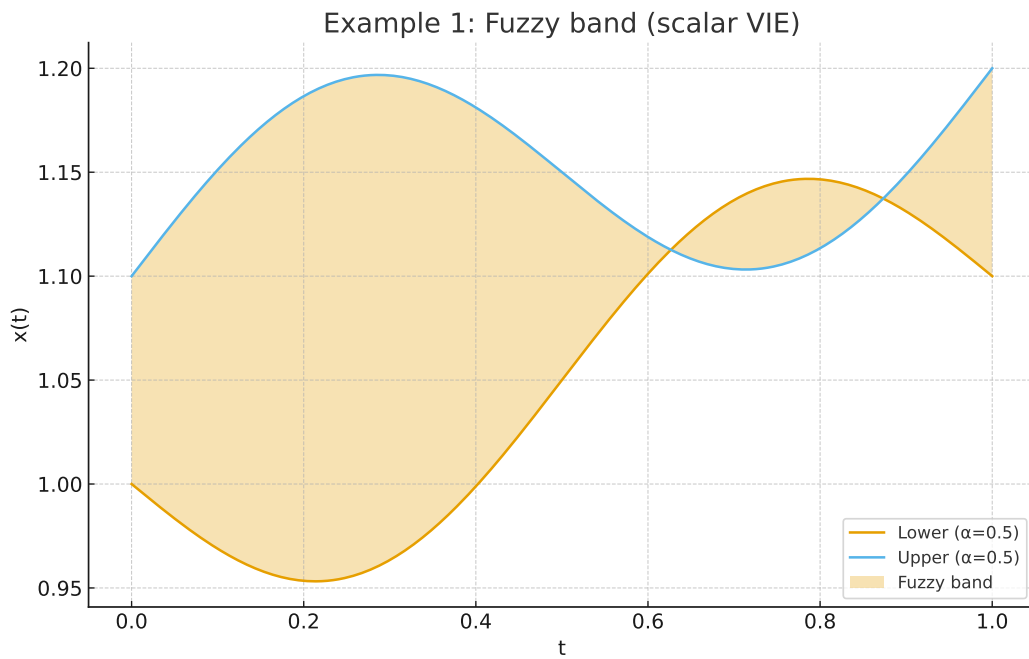


Figure (1): Fuzzy band at $\alpha = 0.5$ for Example 1. The shaded region represents the interval-valued numerical solution bounded by the lower and upper endpoint approximations.

Table (1): Convergence of FRBF in Example 1: L^2 -errors for lower/upper solutions vs. number of centers N .

N	E_{lower}	E_{upper}	Rate (emp.)
10	2.1×10^{-2}	2.5×10^{-2}	—
20	1.1×10^{-2}	1.4×10^{-2}	≈ 0.93
40	5.3×10^{-3}	6.0×10^{-3}	≈ 0.97

Table (2): Convergence of FRBF in Example 2: L^2 -errors for both components (lower/upper) vs. N .

N	$E_{x_1, \text{lower}}$	$E_{x_1, \text{upper}}$	$E_{x_2, \text{lower}}$	$E_{x_2, \text{upper}}$
15	3.0×10^{-2}	3.5×10^{-2}	2.5×10^{-2}	2.9×10^{-2}
30	1.6×10^{-2}	1.9×10^{-2}	1.4×10^{-2}	1.7×10^{-2}
60	7.8×10^{-3}	9.0×10^{-3}	6.7×10^{-3}	7.5×10^{-3}

Ethics approval and consent to participate

Not applicable.

Availability of data and materials

The data required to reproduce the findings of this study are available within the manuscript and its accompanying figures and tables.

Consent for publication

Not applicable.

Author Contributions

Mohammed Yasin: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing—original draft, Writing—review & editing, Visualization, and Project administration.

The author has read and approved the final version of the manuscript.

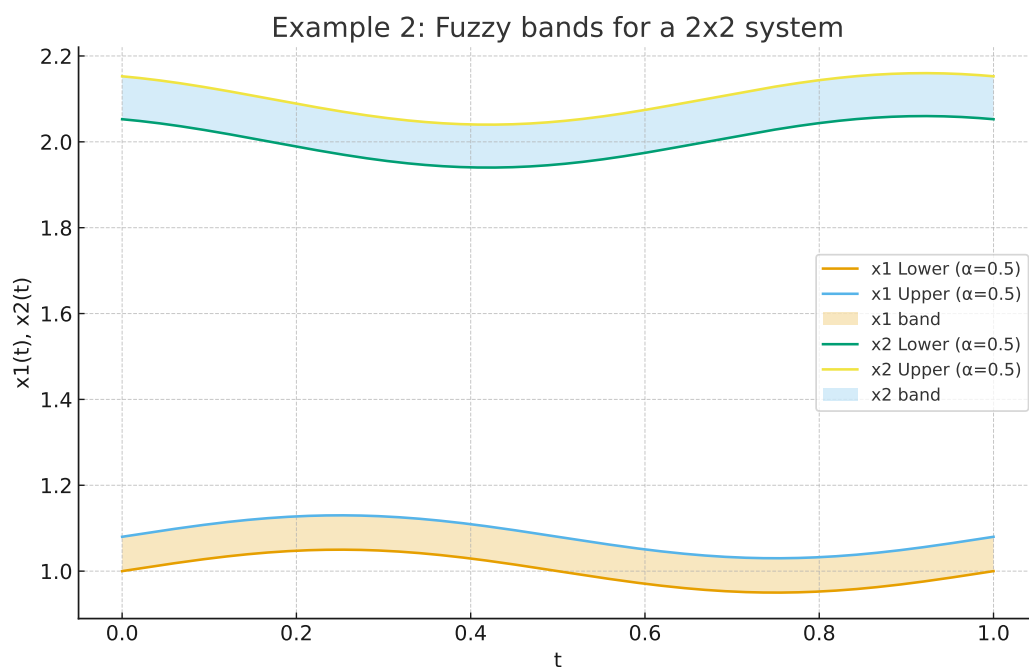


Figure (2): Fuzzy band at $\alpha = 0.5$ for Example 2. The shaded region represents the interval-valued numerical solution bounded by the lower and upper endpoint approximations.

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Conflicts of interest

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Source of Research

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References

- [1] Brunner H. Volterra Integral Equations: An Introduction to Theory and Applications. Cambridge University Press; 2017. Available from: <https://doi.org/10.1017/9781316162491>.
- [2] Corduneanu C, Sandberg I. Volterra Equations and Applications. Gordon and Breach; 2000. Available from: <https://doi.org/10.1201/9781482287424>.
- [3] Elnagar GN, Kazemi M. Chebyshev spectral solution of nonlinear Volterra–Hammerstein integral equations. J Comput Appl Math. 1996;76:147-58. Available from: [https://doi.org/10.1016/S0377-0427\(96\)00098-2](https://doi.org/10.1016/S0377-0427(96)00098-2).
- [4] Maleknejad K, Almasieh H, Roodaki M. Triangular functions method for nonlinear Volterra–Fredholm equations. Commun Nonlinear Sci Numer Simul. 2010;15:3293-8. Available from: <https://doi.org/10.1016/j.cnsns.2009.12.015>.
- [5] Deb A, Sarkar G, Sengupta A. Triangular Orthogonal Functions for the Analysis of Continuous Time Systems. Anthem Press; 2011. Available from: <https://doi.org/10.7135/UP09781843318118>.
- [6] Daubechies I. Ten Lectures on Wavelets. SIAM; 1992. Available from: <https://doi.org/10.1137/1.9781611970104>.
- [7] Aziz I, Siraj-ul-Islam. New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets. J Comput Appl Math. 2013;239:333-45. Available from: <https://doi.org/10.1016/j.cam.2012.08.031>.
- [8] Wazwaz AM. The combined Laplace transform–Adomian decomposition method for handling nonlinear Volterra integro–differential equations. Appl Math Comput. 2010;216(4):1304-9. Available from: <https://doi.org/10.1016/j.amc.2010.02.023>.

- [9] Ghasemi M, Kajani MT, Babolian E. Numerical solutions of the nonlinear Volterra–Fredholm integral equations by using homotopy perturbation method. *Appl Math Comput.* 2007;188:446-9. Available from: <https://doi.org/10.1016/j.amc.2006.10.015>.
- [10] Fasshauer GE. *Meshfree Approximation Methods with MATLAB*. World Scientific; 2007. Available from: <https://doi.org/10.1142/6437>.
- [11] Wendland H. Error estimates for interpolation by compactly supported radial basis functions of minimal degree. *J Approx Theory.* 1998;93:258-72. Available from: <https://doi.org/10.1006/jath.1997.3137>.
- [12] González-Rodelas P, Pasadas M, Kouibia A, Mustafa B. Numerical solution of linear Volterra integral equation systems by radial basis functions. *Mathematics (Basel).* 2022;10(2):223. Available from: <https://doi.org/10.3390/math10020223>.
- [13] Zadeh LA. Fuzzy sets. *Inf Control.* 1965;8:338-53. Available from: [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X).
- [14] Sadatrasoul SM, Ezzati R. Iterative method for numerical solution of two-dimensional nonlinear fuzzy integral equations. *Fuzzy Sets Syst.* 2015;280:91-106. Available from: <https://doi.org/10.1016/j.fss.2014.12.008>.
- [15] Sadatrasoul SM, Ezzati R. Numerical solution of two-dimensional nonlinear Hammerstein fuzzy integral equations based on optimal fuzzy quadrature formula. *J Comput Appl Math.* 2016;292:430-46. Available from: <https://doi.org/10.1016/j.cam.2015.07.023>.
- [16] Zeinali M, Shahmorad S, Mirnia K. Fuzzy integro-differential equations: discrete solution and error estimation. *Iran J Fuzzy Syst.* 2013;10(1):107-22. Available from: <https://doi.org/10.22111/IJFS.2013.169>.
- [17] Mosayebi Z. Numerical solution of fuzzy linear Volterra–Fredholm–Hammerstein integral equations via collocation method based on radial basis functions. *J Fuzzy Set Valued Anal.* 2014;2014:1-10. Available from: <https://doi.org/10.5899/2014/jfsva-00201>.
- [18] Behzadi SS. Solving fuzzy nonlinear Volterra–Fredholm integral equations by using homotopy analysis and Adomian decomposition methods. *J Fuzzy Set Valued Anal.* 2011;2011:1-13. Available from: <https://doi.org/10.5899/2011/jfsva-00067>.
- [19] Abu Arqub O, Abukhaled M, Sweis H, Shawagfeh N. Numerical analysis of Lucas polynomials innovative technique within the Galerkin strategy for solving fuzzy-type differential models with an application in the electrical circuit engineering field. *Results Eng.* 2025;27:106298. Available from: <https://doi.org/10.1016/j.rineng.2025.106298>.
- [20] Abu Arqub OA, Mezghiche R, Maayah BM. Fuzzy M-fractional integrodifferential models: theoretical existence and uniqueness results, and approximate solutions utilizing the Hilbert reproducing kernel algorithm. *Front Phys.* 2023;11:1252919. Available from: <https://doi.org/10.3389/FPHY.2023.1252919>.