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Matrix Structure and Physical Stability in Two Coupled Compartments

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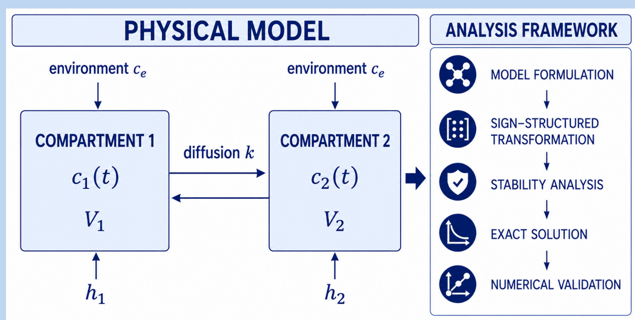
Abstract:
Reduced-order transport models are widely used in computational physics to describe diffusion, exchange, relaxation, and redistribution processes when full spatially resolved models are unnecessary, unavailable, or computationally expensive. This paper develops a sign-structured matrix framework for a two-compartment diffusion-relaxation system, the smallest nontrivial model combining environmental relaxation with bidirectional inter-compartment exchange. Starting from a physically interpretable mass-balance formulation, the system is written as a linear state-space model. A diagonal signature transformation converts the governing matrix into a matrix with nonpositive entries and strictly positive determinant for all positive physical parameters. In two dimensions, this is equivalent to the property that the determinant is positive while all proper minors are nonpositive. The same parameter regime is shown to imply nonsingularity, asymptotic stability of the original diffusion dynamics, and physically meaningful relaxation toward equilibrium. The exact matrix-exponential solution is used to interpret the spectral decay modes, while a forward Euler discretization is analyzed to connect continuous-time stability with time-step-dependent numerical stability and first-order global convergence. Computational experiments confirm decay toward equilibrium and slope-one convergence of the discretization error. The results suggest that sign-structured matrix analysis can serve as a compact framework for linking local dissipative coupling, global solvability, spectral stability, and numerical reliability in reduced-order diffusive transport models.



Keywords: Diffusive transport; Computational physics; Reduced-order modeling; Scientific computing; Stability analysis; Numerical stability; Environmental transport; Dynamical systems.

Research Highlights

- A two-compartment diffusion-relaxation model realizes a sign-structured matrix class exactly.
- A diagonal signature transformation gives positive determinant and nonpositive proper minors.
- The same physical parameter regime implies nonsingularity and asymptotic stability.
- Three figures connect physical structure, exact dynamics, and error convergence.
- The analysis links physical dissipativity with numerical reliability for reduced-order transport models.

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1 Introduction

Reduced-order transport models are central in computational physics because they provide interpretable and computationally efficient descriptions of diffusion, relaxation, exchange, and redistribution processes [1]. In many applications, full spatially resolved partial differential equation models contain more detail than is needed for the question being studied [2]. Compartmental models provide a useful alternative by replacing spatially distributed media with a finite number of well-mixed subsystems connected by transfer laws [3, 4]. Once such a model is written in state-space form, its qualitative behavior is controlled by the algebraic and spectral structure of the governing matrix. Thus, the same matrix simultaneously represents the physical transport operator and the generator of the computational dynamics [5, 6, 7]. Matrix structure is not a secondary detail in transport modeling. The signs of the entries, the determinant, the minors, and the eigenvalues determine whether the system is dissipative, whether perturbations decay, whether the equilibrium is unique, and whether numerical simulations preserve the intended physical behavior.

Compartmental models have long been used in diffusion, biological transport, pharmacokinetics, environmental exchange, and radiation dosimetry [3, 4, 8, 9]. Their analysis is closely related to nonnegative matrix theory, M-matrices, sign-regular matrices, and linear dynamical systems [10, 11, 12, 13, 14]. The theory of sign-regular matrices has a rich history, with foundational contributions from Schoenberg, Motzkin, and Gantmacher and Krein [15, 16, 17]. In recent decades, the theory has been significantly advanced by Peña [11], Garloff [18], Alseidi, Margaliot, and Garloff [12], and others.

In particular, sign-structured matrices provide qualitative information that is robust under parameter perturbation. A matrix whose determinant is positive while all proper minors are nonpositive forms a special sign-constrained class of sign-structured matrices. Such matrices are closely related to sign-regular matrices, whose theory has been extensively developed [11, 12, 18] and which have applications in economics, biology, and network analysis where qualitative sign information is sufficient to determine system behavior. Such algebraic conditions can encode physical features such as local opposition, global solvability, and stability.

1.1 Statement of Contribution and Aims

The present work investigates a two-compartment diffusion-relaxation model as a minimal computational physics system in which environmental relaxation and bidirectional exchange occur simultaneously. The contribution of this paper is methodological: we demonstrate, through a fully worked example, how four levels of analysis can be connected in a single transparent framework:

1. Physical mass-balance formulation leading to a linear state-space model;
2. Sign-structured matrix transformation that reveals local dissipativity and global nonsingularity;
3. Continuous-time stability analysis via trace-determinant conditions;
4. Numerical time-discretization analysis connecting physical stability to computational reliability.

The model is deliberately chosen to be low-dimensional because this allows every step of the analysis to be explicit and interpretable. The same physical parameter regime implies local dissipative structure in the transformed matrix, global nonsingularity through determinant positivity, asymptotic stability of the original diffusion dynamics, and first-order convergence of a stable forward Euler approximation. Our aim is to expose these connections clearly so that the paper can serve as a methodological template for researchers who wish to apply similar matrix-structure reasoning in their own reduced-order transport models. Figures illustrating the model architecture, exact relaxation, and global error convergence connect the mathematical structure to the computational behavior. The paper is organized as follows. Section 2 reviews related work. Section 3 formulates the two-compartment diffusion-relaxation model. Section 4 derives the sign-structured matrix transformation. Section 5 proves asymptotic stability. Section 6 discusses the exact dynamics. Section 7 analyzes forward Euler discretization and stability. Section 8 presents the computational results associated with the numerical figures. Sections 9 and 10 discuss interpretation, scope, and extensions. Section 11 concludes the paper.

2 Related Work

Diffusion and relaxation models are classical tools for describing mass transfer, heat transfer, concentration equilibration, and redistribution processes [1, 2]. When spatial details are not resolved, compartmental models approximate a distributed system by a finite set of well-mixed compartments. Each compartment has a storage capacity, and the exchange between compartments is represented by transfer coefficients. This reduction is particularly useful when the main objective is to understand global exchange rates, long-time relaxation, stability, or numerical behavior rather than detailed spatial gradients.

Classical compartmental analysis has been widely used in biological and medical modeling [3, 4, 8]. In radiation dosimetry and biokinetics, for example, compartmental models are used to describe redistribution and clearance of radionuclides [9]. The two-compartment model studied in this paper has the same structural form: diagonal terms represent loss or relaxation, while off-diagonal terms represent transfer from one compartment to another. Although the model is low-dimensional, it retains the essential competition between internal exchange and external relaxation that appears in larger transport networks.

The stability of linear compartmental systems is closely related to the theory of nonnegative matrices and M-matrices [10]. Matrices with nonpositive off-diagonal entries and positive principal minors often appear after sign transformations or after writing balance laws in loss-minus-gain form. These structures encode dissipativity, uniqueness, monotonicity, and stability. Sign-regular and sign-structured matrices provide a broader qualitative framework. Instead of focusing only on eigenvalues, these methods examine the signs of minors and determinants. Such properties can impose strong restrictions on spectral behavior and robustness under structured perturbations [11, 12, 14, 13].

Hassuneh, Adm, and Garloff [19] characterized matrices with positive determinant and nonpositive proper minors. This result is directly relevant to the determinant–minor structure considered in the present work.

Several authors have developed computational tests and algorithms for sign-regular matrices. Gasca and Peña [20] provided an algorithmic characterization of strict sign-regularity using Neville elimination. Cortés and Peña [21] later developed a stable test for strict sign regularity with optimal growth factor. Alonso, Peña, and Serrano [22] extended the analysis to almost strictly sign-regular matrices and Neville elimination with two-determinant pivoting, and later studied backward stability properties [23]. These computational methods are directly relevant to the present work, which connects sign-structured matrix theory to stability and numerical discretization.

The concept of matrices with positive determinant and nonpositive proper minors is closely related to the broader theory of sign-regular matrices. Peña [11] characterized nonsingular sign-regular matrices, establishing conditions under which such matrices have the sign variation diminution property. Garloff [18] developed criteria for sign regularity of sets of matrices. More recently, Alseidi, Margaliot, and Garloff [12] studied the spectral properties of sign-regular matrices, showing that their eigenvalues have specific sign and modulus structures. The

present work shows that such sign-structured matrices can arise naturally from a physical transport model.
The matrix class defined by the conditions

$$\det(A) > 0, \qquad \text{all proper minors of } A \text{ are nonpositive} \tag{1}$$

is closely related to sign-regular matrices and totally nonnegative matrices. For general $n \times n$ matrices, this class has several important algebraic properties, including nonsingularity and specific sign patterns in the inverse. These properties have been extensively studied in the context of sign-regular matrix theory [11, 18, 12, 19].
In the 2×2 case, which is the focus of the present work, the proper minors are simply the entries themselves. Thus the condition reduces to

$$a_{11} \leq 0, \quad a_{12} \leq 0, \quad a_{21} \leq 0, \quad a_{22} \leq 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0. \tag{2}$$

This is precisely the structure obtained in Theorem 1 after the signature transformation. The simplicity of the 2×2 case allows us to provide a complete physical interpretation and to connect the algebraic conditions to stability and numerical discretization.
Beyond the sign-structure analysis, structure-preserving discretizations have received increasing attention in computational physics and scientific computing. Methods that preserve physical properties such as positivity, dissipation, or conservation laws are known to improve numerical reliability over long-time simulations [6, 24]. Recent work has explored the preservation of matrix structure under discretization, showing that maintaining properties such as stability or monotonicity at the discrete level can be achieved through appropriate time-stepping schemes [25, 26]. Our analysis of forward Euler stability fits within this broader theme: we show explicitly that the continuous-time stability property of the physical system does not automatically guarantee numerical stability, and that the discrete amplification factors must be controlled through the time-step selection. For a stable continuous-time system, numerical stability is not automatic. If

$$\dot{x} = Bx, \tag{3}$$

then the exact solution is

$$x(t) = \mathrm{e}^{Bt}x(0). \tag{4}$$

A time-stepping method replaces this exact semigroup by a discrete update matrix. Explicit methods such as forward Euler are conditionally stable: even if all eigenvalues of B have negative real part, the discrete solution can grow if the time step is too large [6, 24, 25]. Thus, a complete computational analysis must connect physical stability with numerical stability. This is one reason the present work combines sign-structured matrix analysis, spectral stability, exact matrix-exponential dynamics, and explicit time-discretization behavior in a single framework. In the context of compartmental systems, several authors have examined the relationship between matrix structure and stability. Linear compartmental models with nonnegative off-diagonal entries and negative diagonal entries (often called compartmental matrices or Metzler matrices with negative diagonal) have been studied extensively [5, 27]. When such matrices are nonsingular and have all principal minors positive, they are M-matrices, which guarantee stability and positivity of the inverse. The determinant-minor condition studied in this paper is closely related but does not require the matrix itself to have nonpositive off-diagonal entries; rather, the sign structure emerges after a signature transformation that reverses the sign of one compartment.
The present work complements the algebraic characterizations of Peña [11] and the computational tests of Cortés and Peña [21] by demonstrating that sign-structured matrices arise naturally from physical transport models and that the determinant-minor condition has direct implications for both stability and numerical reliability. The value of worked examples as methodological templates in computational physics has been recognized in the literature. Well-structured examples that connect physical modeling, mathematical analysis, and numerical computation can serve as effective references for researchers entering the field [28, 29]. The present paper aims to contribute to this tradition by providing a complete, self-contained demonstration of how sign-structured matrix analysis can be applied to a physically meaningful transport problem, with potential applications to larger compartmental networks and other dissipative systems.

3 Model Formulation

Consider two well-mixed compartments that exchange material with each other and with an external environment at equilibrium concentration c_e. Let $c_1(t)$ and $c_2(t)$ denote the concentrations in compartments 1 and 2. Let $V_1 > 0$ and $V_2 > 0$ denote effective compartment volumes, $h_1 > 0$ and $h_2 > 0$ environmental relaxation coefficients, and $k > 0$ the diffusive exchange coefficient between compartments.
Each compartment relaxes toward the environmental concentration, while the two compartments exchange material according to their concentration difference. The physical configuration is shown in Figure 1.

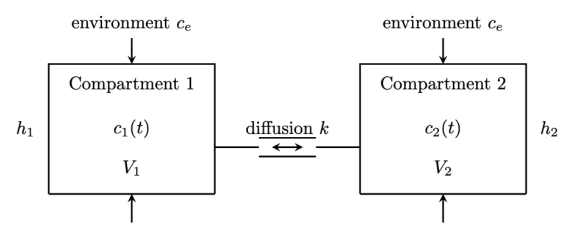

Figure 1: Schematic of the two-compartment diffusion-relaxation model. Here, $c_1(t)$ and $c_2(t)$ are compartment concentrations, V_1 and V_2 are effective volumes, h_1 and h_2 are environmental relaxation coefficients, k is the bidirectional diffusive coupling coefficient, and c_e is the environmental equilibrium concentration.

The mass-balance equations are

$$V_1\dot{c}_1 = -h_1(c_1 - c_e) - k(c_1 - c_2), \tag{5}$$
$$V_2\dot{c}_2 = -h_2(c_2 - c_e) - k(c_2 - c_1). \tag{6}$$

The first term on the right-hand side of each equation represents relaxation toward the environment. The second term represents diffusive exchange between compartments.
Define concentration deviations from environmental equilibrium:

$$x_1(t) = c_1(t) - c_e, \qquad x_2(t) = c_2(t) - c_e. \tag{7}$$

Then Eqs. (5)–(6) become

$$V_1\dot{x}_1 = -(h_1 + k)x_1 + kx_2, \tag{8}$$
$$V_2\dot{x}_2 = kx_1 - (h_2 + k)x_2. \tag{9}$$

Dividing by the volumes gives

$$\dot{x}_1 = -a_1x_1 + b_1x_2, \tag{10}$$
$$\dot{x}_2 = b_2x_1 - a_2x_2, \tag{11}$$

where

$$a_1 = \frac{h_1 + k}{V_1}, \qquad a_2 = \frac{h_2 + k}{V_2}, \qquad b_1 = \frac{k}{V_1}, \qquad b_2 = \frac{k}{V_2}. \tag{12}$$

All four coefficients are positive.
In state-space form,

$$\dot{x} = Bx, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad B = \begin{bmatrix} -a_1 & b_1 \\ b_2 & -a_2 \end{bmatrix}. \tag{13}$$

The diagonal entries of B are negative because each compartment loses deviation through relaxation and exchange. The off-diagonal entries are positive because each compartment receives material from the other compartment.

4 Sign-Structured Matrix Transformation

The matrix B has a physically natural sign pattern: negative diagonal entries and positive off-diagonal entries. The negative diagonal entries represent relaxation and loss of deviation within each compartment, while the positive off-diagonal entries represent the gain received from the other compartment. To connect this dynamical matrix to a sign-structured matrix class with nonpositive proper minors, introduce the diagonal signature matrix

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad S^{-1} = S. \tag{14}$$

Define

$$A = SBS. \tag{15}$$

Using Eq. (13), this gives

$$A = \begin{bmatrix} -a_1 & -b_1 \\ -b_2 & -a_2 \end{bmatrix}. \tag{16}$$

Thus every entry of A is nonpositive. This transformation does not change the eigenvalues because A and B are similar. It only changes the sign representation of the coupling so that the physical exchange process can be interpreted through the lens of sign-structured matrix theory.
The determinant of A is

$$\begin{aligned} \det(A) &= a_1a_2 - b_1b_2 \\ &= \frac{(h_1+k)(h_2+k)-k^2}{V_1V_2} \\ &= \frac{h_1h_2 + k(h_1+h_2)}{V_1V_2}. \end{aligned} \tag{17}$$

Since $V_1, V_2, h_1, h_2, k > 0$, it follows that

$$\det(A) > 0. \tag{18}$$

Because A is 2×2, its proper minors are precisely its entries. Hence A has positive determinant and all proper minors nonpositive.
The following result summarizes the structural property generated by the physical model.
Theorem 1. For the two-compartment diffusion-relaxation model with

$$V_1, V_2, h_1, h_2, k > 0,$$

the transformed matrix

$$A = \begin{bmatrix} -\dfrac{h_1+k}{V_1} & -\dfrac{k}{V_1} \\ -\dfrac{k}{V_2} & -\dfrac{h_2+k}{V_2} \end{bmatrix}$$

has positive determinant and all proper minors nonpositive.
Proof. Each entry of A is nonpositive because all physical parameters are positive. Moreover,

$$\det(A) = \left(-\frac{h_1+k}{V_1}\right)\left(-\frac{h_2+k}{V_2}\right) - \left(-\frac{k}{V_1}\right)\left(-\frac{k}{V_2}\right),$$

so

$$\det(A) = \frac{(h_1+k)(h_2+k)-k^2}{V_1V_2} = \frac{h_1h_2+k(h_1+h_2)}{V_1V_2} > 0.$$

Since A is two-dimensional, its proper minors are exactly its entries. Therefore all proper minors are nonpositive and the determinant is positive. □
This theorem shows that the sign-structured matrix property is not a numerical accident. It follows directly from the physical assumptions of positive volumes, positive environmental relaxation, and positive diffusive coupling. In this sense, the sign structure is inherited from the transport mechanism itself. The negative entries of the transformed matrix represent local dissipative opposition to deviations, while the positive determinant represents global nonsingularity of the coupled system.
The determinant

$$D = \det(A) = \frac{h_1h_2 + k(h_1+h_2)}{V_1V_2} \tag{19}$$

has a clear physical interpretation. Increasing either environmental relaxation coefficient increases the determinant:

$$\frac{\partial D}{\partial h_1} = \frac{h_2+k}{V_1V_2} > 0, \tag{20}$$

$$\frac{\partial D}{\partial h_2} = \frac{h_1+k}{V_1V_2} > 0. \tag{21}$$

Increasing the inter-compartment coupling also increases the determinant:

$$\frac{\partial D}{\partial k} = \frac{h_1+h_2}{V_1V_2} > 0. \tag{22}$$

Thus stronger relaxation and stronger exchange increase global nonsingularity. Larger volumes decrease the determinant because larger storage capacity slows the response.
A limiting case clarifies the role of environmental relaxation. If $h_1 = h_2 = 0$, then

$$D = 0.$$

The system becomes a closed two-compartment diffusion system with no environmental relaxation. In that case, diffusion redistributes material internally but does not force convergence to a prescribed external equilibrium. Determinant positivity therefore distinguishes open dissipative relaxation from internal redistribution alone. This distinction is important because it shows that the positive determinant is not caused by coupling alone; it results from the combination of internal exchange and external relaxation.

5 Asymptotic Stability

The transformed matrix A encodes the sign structure, while the original matrix B governs the physical dynamics in the deviation variables x_1 and x_2. Since

$$A = SBS,$$

the matrices A and B are similar and therefore have the same eigenvalues.
For a real 2×2 system

$$\dot{x} = Bx,$$

the equilibrium $x = 0$ is asymptotically stable if

$$\operatorname{tr}(B) < 0, \qquad \det(B) > 0. \tag{23}$$

For the present model,

$$\operatorname{tr}(B) = -(a_1+a_2) < 0, \tag{24}$$

and

$$\det(B) = a_1a_2 - b_1b_2 = \frac{h_1h_2+k(h_1+h_2)}{V_1V_2} > 0. \tag{25}$$

Therefore, all eigenvalues of B have negative real part.
Corollary 1. For every physically admissible parameter set $V_1, V_2, h_1, h_2, k > 0$, the equilibrium $x = 0$ of the two-compartment diffusion-relaxation system is asymptotically stable.
The eigenvalues can be written explicitly as

$$\lambda_\pm = -\frac{a_1+a_2}{2} \pm \frac{1}{2}\sqrt{(a_1-a_2)^2 + 4b_1b_2}. \tag{26}$$

The determinant condition $a_1a_2 - b_1b_2 > 0$ ensures that the larger eigenvalue remains negative. Thus every perturbation from environmental equilibrium decays exponentially.

6 Exact Continuous-Time Dynamics

Since the system is linear with constant coefficients, the exact solution is

$$x(t) = e^{Bt}x(0). \tag{27}$$

If B is diagonalizable, the solution may be written as

$$x(t) = C_+u_+e^{\lambda_+ t} + C_-u_-e^{\lambda_- t}, \tag{28}$$

where $\lambda_\pm$ are given by Eq. (26), $u_\pm$ are the corresponding eigenvectors, and $C_\pm$ depend on the initial condition.

This representation shows that relaxation is not governed by the individual coefficients a_1 and a_2 alone. The coupled decay modes depend on the full matrix, including the inter-compartment transfer coefficients b_1 and b_2. The slow eigenvalue controls the long-time approach to equilibrium, while the fast eigenvalue controls the initial transient.

The representative parameter set used for the computational results is

$$V_1 = 1, \qquad V_2 = 2, \qquad h_1 = 2, \qquad h_2 = 3, \qquad k = 1. \tag{29}$$

Then

$$a_1 = 3, \qquad a_2 = 2, \qquad b_1 = 1, \qquad b_2 = \frac{1}{2}. \tag{30}$$

Therefore,

$$B = \begin{bmatrix} -3 & 1 \\ 1/2 & -2 \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & -1 \\ -1/2 & -2 \end{bmatrix}. \tag{31}$$

The determinant is

$$\det(A) = 6 - \frac{1}{2} = \frac{11}{2} > 0. \tag{32}$$

The eigenvalues are

$$\lambda_\pm = \frac{-5 \pm \sqrt{3}}{2}, \tag{33}$$

which are both strictly negative.

7 Forward Euler Discretization

The exact solution in Eq. (27) serves as a reference for the numerical approximation. Applying the forward Euler method to

$$\dot{x} = Bx$$

yields

$$x^{n+1} = x^n + \Delta t B x^n = (I + \Delta t B)x^n. \tag{34}$$

The local truncation error is $O(\Delta t^2)$, which accumulates to a global error of $O(\Delta t)$ over a stable finite time interval. Consequently, the forward Euler method is first-order accurate in time.

Accuracy alone, however, is not sufficient; numerical stability must also be ensured. Stability requires that all eigenvalues of the update matrix $I + \Delta t B$ lie within the unit disk. For $\lambda_j \in \mathrm{spec}(B)$, the corresponding amplification factor is

$$g_j = 1 + \Delta t \lambda_j. \tag{35}$$

The forward Euler stability condition is therefore

$$|1 + \Delta t \lambda_j| < 1, \qquad j = 1, 2, \tag{36}$$

which, for real negative eigenvalues, reduces to

$$0 < \Delta t < \frac{2}{\max_j |\lambda_j|}. \tag{37}$$

For the representative example, the eigenvalue with the largest magnitude is

$$\lambda_- = \frac{-5 - \sqrt{3}}{2}.$$

For the representative parameter set, the stability threshold is given by Eq. (37). This condition demonstrates that even a physically stable continuous system can produce nonphysical numerical growth if the explicit time step is chosen outside the stability region.

8 Computational Results

The computational results are designed to connect the analytical structure of the model with its physical and numerical behavior. All simulations use the representative parameter set introduced in Section 6, with B and eigenvalues as previously defined. The initial condition used for the exact relaxation and forward Euler experiments is

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

8.1 Numerical Implementation

All computations were performed in Python using the NumPy and SciPy libraries in double precision. The matrix exponential was evaluated using the `scipy.linalg.expm` function, which implements the scaling and squaring algorithm with Padé approximation. The exact solution $x(t) = e^{Bt}x(0)$ served as the reference for all error measurements.

For the convergence study, the final time was fixed at $T = 2$, and the time step was chosen as

$$\Delta t = \frac{T}{N}, \qquad N = 20, 40, 80, \ldots, 5120. \tag{38}$$

The global error at final time was measured as

$$E(\Delta t) = \|x(T) - x^N\|_2, \tag{39}$$

where $x(T)$ is the exact matrix-exponential solution, x^N is the forward Euler approximation, and $\|\cdot\|_2$ denotes the Euclidean vector norm.

Convergence was verified by ensuring that the error at the smallest time step was at least four orders of magnitude smaller than the error at the largest time step. No convergence failures were observed within the stable regime.

Table 1: Representative parameter values used in the computational experiments.

V_1	V_2	h_1	h_2	k	$\Delta t_{\max}$
1	2	2	3	1	$4/(5+\sqrt{3}) \approx 0.59417$

Figure 2 shows the exact matrix-exponential solution of the coupled diffusion-relaxation system. The concentration deviations converge to zero, indicating that the compartment concentrations approach the environmental equilibrium concentration c_e, which confirms the asymptotic stability predicted by the trace–determinant analysis. The response is governed by the two exponential modes associated with $\lambda_\pm = (-5 \pm \sqrt{3})/2$. The slow mode controls the long-time return to equilibrium, while the fast mode controls the initial transient. The initial rise of $x_2(t)$ occurs because compartment 2 first receives material from compartment 1, which starts above equilibrium; as the exchange input weakens, environmental relaxation dominates and $x_2(t)$ decays to zero.

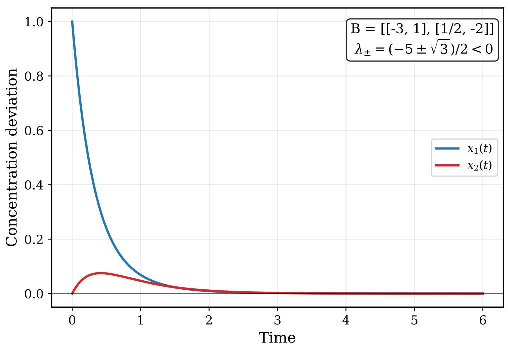

Figure 2: Exact matrix-exponential solution of the two-compartment diffusion-relaxation model. The concentration deviations $x_1(t)$ and $x_2(t)$ converge to zero, indicating that the compartment concentrations approach the environmental equilibrium concentration c_e, consistent with the negative eigenvalues $\lambda_{\pm} = (-5 \pm \sqrt{3})/2$.

The convergence behavior of the forward Euler method is shown in Figure 3. The log-log error curve follows a reference line of slope one, confirming the expected first-order global convergence of the method in the stable time-step regime.

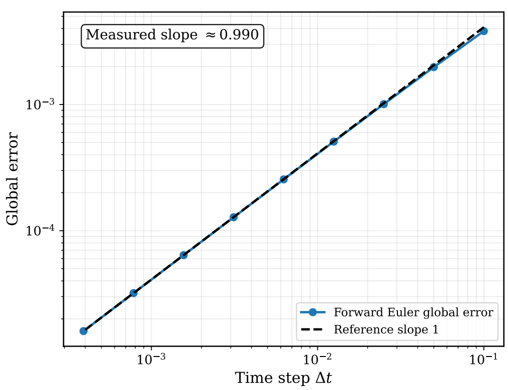

Figure 3: Forward Euler global error at fixed final time. The error follows a reference slope of one on the log-log scale, confirming first-order global convergence for time steps inside the stability regime.

Overall, the computational figures provide complementary evidence for the theoretical claims. Figure 1 defines the physical architecture introduced in the model formulation. Figure 2 confirms exact exponential relaxation toward equilibrium. Figure 3 verifies first-order global convergence. Together, these computational results show that the model is analytically tractable, physically interpretable, and numerically transparent, making it a useful testbed for studying the interaction between matrix sign structure, physical stability, and numerical reliability in reduced-order transport systems.

9 Discussion

The principal result of this study is that a physically meaningful two-compartment diffusion-relaxation system generates, after a diagonal signature transformation, a matrix with positive determinant and nonpositive proper minors. This result provides a direct bridge between physical balance laws and sign-structured matrix theory. Rather than presenting this as a novel mathematical discovery, we emphasize the methodological value of the demonstration: the model serves as a fully worked example that exposes the connections between physical modeling, algebraic sign structure, spectral stability, and numerical discretization in a transparent and interpretable way.

The connection to the broader theory of sign-regular matrices [11, 12, 18, 19] is worth highlighting. While much of the existing literature provides purely algebraic characterizations of sign-structured matrices, the present paper demonstrates that this class is not merely an abstract mathematical curiosity—it emerges naturally from a physically meaningful transport process. In the 2×2 case, the determinant-minor condition is equivalent to the transformed matrix having all entries nonpositive and positive determinant, which follows directly from the physical assumptions of positive volumes, positive relaxation coefficients, and positive diffusive coupling. Thus, the algebraic structure studied in the sign-regular matrix literature has a concrete physical origin in the competition between internal exchange and external relaxation.

The nonpositive proper minors of the transformed matrix encode local suppressive interactions. Physically, diffusion and environmental exchange oppose concentration deviations and reduce imbalance. The positive determinant encodes global nonsingularity, meaning that the coupled system has a unique equilibrium response. Thus, the transformed matrix combines local dissipativity and global solvability in a single algebraic condition. This interpretation is valuable because it shows how abstract matrix conditions can be given concrete physical meaning.

The stability analysis strengthens this interpretation. The same parameter regime that gives the determinant-minor property also gives negative trace and positive determinant for the original dynamical matrix. Hence the system is asymptotically stable. The physical, algebraic, and spectral interpretations are therefore mutually reinforcing. This coherence is one of the strengths of the model as a methodological template: researchers can see the same property expressed in different mathematical languages (determinant minors, eigenvalues, trace-determinant conditions) and understand how they relate to the same physical phenomenon of dissipation.

The numerical analysis adds an important computational dimension. Even when the continuous model is stable, explicit time stepping can fail if the time step does not respect the stability region of the method. The forward Euler method is useful here because its stability mechanism is transparent. The convergence figure shows why numerical reliability requires both a stable physical model and an appropriate discretization. This is a lesson that generalizes well beyond the specific model studied here: physical stability and numerical stability are distinct properties, and both must be considered in computational simulations.

The sensitivity results further clarify the role of environmental relaxation. Increasing h_1 or h_2 strengthens determinant positivity and contributes to faster relaxation. In contrast, if $h_1 = h_2 = 0$, the determinant vanishes, corresponding to a closed system with internal redistribution but no forced relaxation to the environment. Thus determinant positivity separates genuine environmental relaxation from purely internal diffusion. This distinction is physically important: it shows that the sign-structured property is not merely an algebraic curiosity but encodes the presence of dissipative coupling to an external reservoir.

Although the analysis is two-dimensional, the conceptual framework can guide future extensions to larger transport networks. In higher dimensions, proper minors are no longer just entries, and the sign structure becomes more complex. The two-compartment model is therefore valuable as a minimal exact realization: it exposes the mechanism without obscuring it through high-dimensional notation. This is precisely the value of methodological case studies in computational physics: they provide a foundation upon which more complex analyses can be built.

A key insight from this work is the connection between the determinant condition and the qualitative behavior of the system. The determinant in Eq. (19) is positive for all positive parameter values, and this positivity ensures both invertibility of the transformed matrix and asymptotic stability of the original dynamics. This robustness under parameter variation is important: as long as the physical parameters remain positive (which they must, by definition), the qualitative behavior does not change. This is a form of structural stability that is characteristic of dissipative compartmental systems.

The forward Euler analysis also reveals a practical lesson for computational modeling: the stability limit $\Delta t < 2/\max|\lambda_j|$ depends on the fast eigenvalue, which may be much larger in magnitude than the slow eigenvalue if the compartmental parameters vary widely. This means that the time step may need to be chosen based on the fastest dynamics, even if the slower dynamics are of primary interest. This is a common challenge in multiscale modeling, and the present example illustrates it in a simple setting.

It is also worth noting that the sign-structured transformation is not unique. Other signature matrices could be chosen, leading to different sign patterns. The transformation used here ($S = \mathrm{diag}(1, -1)$) is natural because it makes both off-diagonal entries negative, creating a matrix with all entries nonpositive. This choice emphasizes the dissipative nature of the interactions. Other transformations might reveal different structural properties, depending on the questions being asked.

The present work complements the algebraic characterizations of Peña [11] and the computational tests of Cortés and Peña [21] by demonstrating that sign-structured matrices arise naturally from physical transport models. In summary, the two-compartment diffusion-relaxation model provides a unified framework in which physical assumptions, matrix structure, spectral properties, and numerical behavior are connected through explicit analytical expressions. This makes it a useful case study for researchers interested in the intersection of transport modeling, matrix analysis, and computational methods.

Possible Applications

The sign-structured matrix framework developed in this paper can be applied in several computational and physical contexts beyond the specific two-compartment example.

First, it provides a certification criterion for reduced-order models. In transport simulations, one often replaces a detailed spatially resolved model by a coarser compartmental approximation. The determinant-minor condition offers a quick algebraic test to verify that the reduced matrix retains the essential dissipative coupling and global nonsingularity of the original system. If the transformed matrix fails the condition, this may indicate that the reduction has introduced an unphysical coupling or that the parameter regime is non-physical.

Second, the framework supports parameter estimation from measured relaxation data. Given concentration histories, one can estimate the entries of the dynamical matrix B using standard least-squares or observer-based methods. The sign-structured condition then serves as a constraint that can be imposed during estimation, ensuring that the identified matrix remains physically plausible (nonpositive off-diagonals in the transformed representation) and globally solvable (positive determinant). This is particularly relevant in biokinetic models where the transfer coefficients are not directly measurable.

Third, the analysis guides the design of structure-preserving numerical methods. Although forward Euler is conditionally stable, other discretizations can be designed to preserve the sign-structured property exactly at the discrete level. For example, implicit Euler and certain exponential integrators generally provide improved stability properties compared with explicit methods and may better preserve the qualitative behavior of the continuous system. An analysis of their structure-preserving properties is left for future work. The framework can also be used to derive step-size restrictions for explicit methods that are sharper than general spectral bounds.

Fourth, the determinant-minor condition can be used as a screening tool in model-order reduction and network aggregation. When coupling larger networks, one can reduce a group of compartments to a single effective compartment if the aggregated matrix satisfies the sign-structured criterion. This reduces computational cost while maintaining the qualitative stability properties of the full network.

Fifth, the framework has potential applications in control and inverse problems. The transformed matrix $A = SBS$ is closely related to M-matrices; its positive determinant and nonpositive off-diagonals imply that the inverse A^{-1} is a nonnegative matrix (up to a scaling). This property can be exploited to construct Lyapunov functions, to design observers for state estimation, or to solve inverse problems where the positivity of the inverse is required.

Sixth, and of particular relevance to radiation protection and internal dosimetry, the two-compartment model is a foundational building block for interpreting bioassay data. The intake of a radionuclide is estimated from measurements of its activity $A(t)$ in urine, feces, or whole-body content using retention fractions $F(t)$ derived from the matrix solution. The retention fraction $F(t)$ is defined as the fraction of the initial intake that remains in a given compartment at time t. For a single measurement, the intake is estimated as $I = A(t)/F(t)$. When multiple measurements are available, weighted least-squares fitting (WLSF), unweighted least-squares fitting (ULSF), or slope averaging can be used to obtain a best estimate [9, 30, 31]. The interpretation of bioassay measurements has been further examined through case studies involving ingestion of phosphorus-32 [32] and international intercomparison exercises on internal dose assessment [33].

Finally, although the present analysis is two-dimensional, the same reasoning extends to block-structured systems where each compartment represents a cluster of states. The sign-structured condition can be verified blockwise, providing a hierarchical certification framework for large-scale transport networks.

These applications highlight the practical value of connecting algebraic sign structure with physical and numerical properties. The two-compartment model serves as a transparent testbed for developing and testing such methods before they are applied to more complex systems.

10 Limitations and Extensions

Several limitations should be noted, and these also suggest directions for future work.

First, the model is low-dimensional and assumes that each compartment is perfectly mixed. It does not resolve spatial gradients inside the compartments. This is a standard limitation of compartmental modeling, and it is appropriate when the spatial structure is not the focus of the analysis. However, for problems where spatial gradients are important, a full partial differential equation model or a multi-compartment discretization would be more appropriate.

Second, the exchange laws are linear and are most appropriate near equilibrium. Nonlinear diffusion, concentration-dependent transfer, saturation, and nonlinear boundary exchange are outside the present scope. Extending the analysis to nonlinear compartmental systems would require different mathematical tools, such as Lyapunov methods or contraction theory.

Third, the determinant-minor characterization is especially simple because the system is 2×2. In higher-dimensional systems, the analysis of all proper minors is more difficult and may require graph-theoretic or combinatorial tools. The conditions for a matrix to have positive determinant and nonpositive proper minors become much more complex in higher dimensions, and not all matrices with this sign pattern are stable or nonsingular. Future work should explore whether higher-dimensional compartmental systems can be analyzed using similar sign-structured methods, perhaps by exploiting the graph structure of the compartmental network.

Fourth, the numerical analysis focuses on forward Euler because its stability and convergence behavior are easy to interpret. Other methods, including implicit Euler, Crank–Nicolson schemes, Runge–Kutta methods, exponential integrators, and structure-preserving discretizations, may offer improved stability or accuracy. A natural extension would be to analyze the stability and convergence of these methods in the same framework, showing how different numerical schemes preserve or modify the sign-structured properties of the continuous model.

Fifth, the sensitivity analysis is limited to the determinant and the slow eigenvalue. A more comprehensive sensitivity analysis could examine how all eigenvalues, the matrix norm, or the conditioning of the system depend on the physical parameters. This would provide further insight into the robustness of the model.

Sixth, the connection to data and inverse problems is not explored. In practice, compartmental models are often used to infer parameters from measured data. The sign-structured properties of the matrix might provide constraints for parameter estimation or help identify identifiable parameter combinations. Future work could explore the implications of determinant positivity for parameter identifiability in compartmental systems.

Seventh, the model assumes constant parameters. In many applications, volumes, relaxation coefficients, and exchange coefficients may vary with time or with the state of the system. Time-varying compartmental models introduce additional complexity, and the stability analysis would need to be reconsidered using tools from time-varying systems theory.

Despite these limitations, the two-compartment model remains valuable as a methodological case study because it is simple enough to be fully analyzable while still capturing the essential competition between internal exchange and external relaxation that appears in larger transport networks. The extensions listed above suggest numerous directions for future research that could build on the foundation established here.

11 Conclusion

This paper developed a sign-structured matrix framework for a two-compartment diffusion-relaxation model. Starting from a physically interpretable mass-balance formulation, the system was written as a linear state-space model. A diagonal signature transformation converted the dynamical matrix into a matrix with nonpositive entries and strictly positive determinant for all positive physical parameters. Since the system is two-dimensional, this is equivalent to the condition that the determinant is positive while all proper minors are nonpositive.

The same parameter regime was shown to imply asymptotic stability of the original diffusion dynamics. The exact matrix-exponential solution clarified the spectral decay modes, while forward Euler analysis connected continuous-time stability to time-step-dependent numerical stability and first-order global convergence. The computational results confirmed relaxation to equilibrium and slope-one global-error convergence.

The significance of the model lies in its unified interpretation. Local dissipative interactions, global nonsingularity, asymptotic stability, and numerical reliability all follow from the same physically admissible parameter structure. The two-compartment model therefore serves as a testbed for studying how sign-structured matrices can arise naturally in computational transport models.

The methodological value of this work is twofold. First, it demonstrates how abstract algebraic conditions (determinant positivity, nonpositive proper minors) can be interpreted with concrete physical meaning in the context of a transport model. Second, it illustrates how physical stability and numerical stability are distinct properties that must both be considered in computational simulations. The framework also has potential applications in parameter estimation, model reduction, and bioassay interpretation for compartmental systems.

Future work could extend the framework to multi-compartment networks, uncertain transfer coefficients, inverse parameter estimation, nonlinear exchange laws, and numerical schemes designed to preserve sign-structured stability properties. Another promising direction is to connect determinant-minor conditions with graph topology in larger transport networks. The two-compartment model presented here provides a useful building block for understanding the relationship between physical modeling, matrix structure, and computational methods in reduced-order transport models.

Ethics approval and consent to participate

Not applicable

Consent for publication

Not applicable

Availability of data and materials

All data generated or analysed during this study are included in this published article.

Author Contributions

Othman H. Y. Zalloum: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization, Project administration, Corresponding author.

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The author declares no conflict of interest.

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