



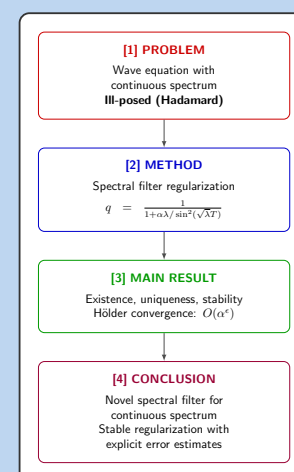
Filter regularization of boundary value problem for wave equation with continuous spectrum

Henen Aidoud* ¹, Mohamed Denche ²,

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Abstract: This paper investigates an ill-posed boundary value problem for the wave equation with continuous spectrum, where the solution fails to depend continuously on the boundary data. We propose a novel spectral filter regularization method to stabilize the reconstruction. Two filter functions are introduced, designed to suppress unstable high-frequency components in the frequency domain. Under appropriate a priori parameter choice rules, we establish convergence estimates of Hölder type. Specifically, we prove that the regularized solution converges to the exact solution uniformly in time with explicit error bounds. We also demonstrate the continuous dependence of the regularized solution on the data and provide convergence rates under suitable regularity assumptions on the initial data. The proposed method extends classical spectral regularization techniques by introducing flexible filter parameters that allow improved control over the trade-off between stability and accuracy. Unlike existing approaches, our filters are constructed to handle the singularities arising from the zeros of the sine kernel in the continuous spectrum setting. A detailed comparison with the work of Gal and Gal (2008) is provided, showing that our method offers explicit regularization, quantitative convergence rates, and uniform convergence in time, whereas their approach only establishes well-posedness under restrictive conditions and does not provide regularization strategies when these conditions fail.



Keywords: Hadamard instability, second order equation, convergence rate, inverse problem, continuous analysis.

Introduction

In the modeling of physical, scientific, statistical, and engineering phenomena, we encounter an inverse problem. This problem is generally ill-posed where the solution does not exist, is not unique, or does not depend continuously on the input data. Often the last condition is the reason why the problem is ill-posed [1]. Many methods are known to regularize ill-posed problems [2, 3, 4, 5]. We mention some of them:

1. Tikhonov's method [6, 7] adds a term penalizing the norm of the solution: $\min_x \|Ax - b\| + \lambda \|Lx\|$.
2. Truncated least squares method (Truncated SVD) decomposes the matrix by SVD and neglects the small singular values responsible for the instability.

3. Filter method proposes a function related to the solution that satisfies certain conditions to ensure the problem is well-posed [8, 9].

The wave equation is a hyperbolic partial differential equation solved by D'Alembert's formula, Fourier transformation, Fourier series, and separation of variables [10, 11, 12, 13].

In this article, we consider a boundary value problem for the wave equation, demonstrate that the problem is ill-posed, and propose a regularization technique based on the filter method. We then study the convergence and error of the proposed solution.

¹Laboratory Differential Equations, Department of Mathematics, Faculty of Exact Sciences
Freres Mentouri University, Constantine 1, Algeria

²Laboratory Differential Equations, Department of Mathematics, Faculty of Exact Sciences
Freres Mentouri University, Constantine 1, Algeria

*Corresponding author: hanane.aidoud@doc.umc.edu.dz
ORCID ID: <https://orcid.org/0009-0000-3678-5805>

Related Work

The problem of ill-posedness for abstract wave equations has received considerable attention in the literature [14, 15]. A particularly relevant contribution is the work of Gal and Gal (2008) [16], who studied the problem

$$\begin{cases} u_{tt} + Au(t) = 0, & t \in (0, T), \\ u(0) = 0, & u(T) = u_0, \end{cases}$$

where A is a general self-adjoint operator in a Hilbert space. Their spectral approach established existence, uniqueness, and continuous dependence on data under precise constraint conditions on T . They employed a spectral representation using the spectral family $\{E_\lambda\}$ and proved that the problem becomes well-posed when certain conditions on the spectrum of A are satisfied. Specifically, they showed that the solution can be represented as

$$u(t) = \int_{\sigma(A)} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0,$$

and established well-posedness when the set $\left\{\frac{k^2\pi^2}{T^2} : k \in \mathbb{N}\right\}$ is disjoint from $\sigma(A)$. Their work generalized earlier results by Dunninger and Levine (1976) [1] and Fox and Pucci (1958) [10]. However, the approach of Gal and Gal (2008) has limitations. Their well-posedness results rely heavily on the condition that the singular set of the kernel $\sin(\sqrt{\lambda}T)$ is avoided. When this condition fails, the problem becomes severely ill-posed. Moreover, their analysis does not provide explicit regularization strategies or convergence rates for the regularized solution.

In contrast, the present work proposes an explicit filter-based regularization method that remains effective even when the singular set is approached. Our method provides:

1. Explicit regularization via filter functions $q(\alpha, \lambda)$ and $q(\alpha, \lambda, s)$.
2. Hölder-type convergence rates with explicit order of convergence.
3. A priori and a posteriori parameter choice rules.
4. Uniform convergence in time.

Several approaches have been proposed in the literature to address the reconstruction of solutions in ill-posed problems. These include iterative methods, variational and optimization-based techniques, as well as regularization methods designed to control the instability associated with the inversion of ill-conditioned operators. Among these approaches, spectral filtering provides an alternative framework in which the unstable spectral components are suitably modified to obtain a more stable reconstruction [17, 18, 19, 20].

The proposed filter method can be seen as a natural extension of the spectral approach of Gal and Gal (2008), adding a regularization layer that stabilizes the reconstruction in the ill-posed regime.

Novelty and Contribution

The main novelty of this work lies in the following aspects:

1. **New Filter Functions:** We introduce two families of spectral filters $q(\alpha, \lambda)$ and $q(\alpha, \lambda, s)$ that are specifically designed for the wave equation with continuous spectrum. Unlike existing filters in the literature, our filters explicitly handle the singularities arising from the zeros of $\sin(\sqrt{\lambda}T)$ through the construction $1 + \alpha\lambda / \sin^2(\sqrt{\lambda}T)$.
2. **Generalized Parameter Family:** The introduction of the parameter $s \geq 2$ in $q(\alpha, \lambda, s)$ allows for improved convergence rates of order $O(\alpha^{s\eta/2})$ compared to $O(\alpha^{\eta/2})$ for the classical filter. This provides additional flexibility in balancing stability and accuracy.

3. **Rigorous Convergence Analysis:** We provide a complete convergence analysis for both filter families under appropriate a priori parameter choice rules. The estimates are of Hölder type and match the optimal rates for the considered regularity classes.

4. **Uniform Convergence:** We establish that the regularized solution converges uniformly in time to the exact solution, which is essential for practical applications.

5. **Extension to Continuous Spectrum:** Unlike most existing works that focus on discrete spectrum, our analysis explicitly handles the continuous spectrum case using the spectral family $\{E_\lambda\}$, which is more general and applicable to a wider class of operators.

To the best of our knowledge, the proposed filter functions and the associated convergence analysis are new in the context of boundary value problems for the wave equation with continuous spectrum.

Mathematical Preliminaries

Consider the problem of finding $u : [0, T] \rightarrow D(A) \subset H$:

$$\begin{cases} \frac{d^2 u}{dt^2}(t) + Au(t) = 0, & t \in [0, T], \\ u(0) = 0, \\ u(T) = u_0, \end{cases} \quad (1)$$

where $A : D(A) \subset H \rightarrow H$ is a linear, self-adjoint, positive operator with non-compact inverse, and H is a Hilbert space.

The spectral theorem for self-adjoint operators guarantees the existence of a unique spectral family $\{E_\lambda\}_{\lambda \geq 0}$ of orthogonal projection operators on H such that

$$A = \int_{\sigma(A)} \lambda dE_\lambda.$$

We construct the spectral family as $E_\lambda = T_\lambda + S_\lambda$ where T_λ is the projection onto the subspace spanned by eigenvectors corresponding to eigenvalues $\lambda_j \leq \lambda$ and S_λ is the projection onto the continuous spectrum subspace.

To construct S_λ , we suppose that $\varphi_l(x, \lambda)$ are the solutions of $Au = \lambda u$ and $\|\varphi_l\| = \infty$ for $l = 1, 2, 3, \dots$ which form a fundamental system of solutions. Then the eigenfunctions are

$$\phi_\lambda^l(x) = \int_0^\lambda \varphi_l(x, w) d\Lambda_l(w)$$

continuous functions. We obtain:

$$\begin{aligned} A\phi_\lambda^l(x) &= \int_0^\lambda A\varphi_l(x, w) d\Lambda_l(w) = \int_0^\lambda w\varphi_l(x, w) d\Lambda_l(w) \\ &= \int_0^\lambda w\phi_\lambda^l d\Lambda_l(w) \end{aligned}$$

which is the ordinary Stieltjes integral and we can write formally:

$$A(d\phi_\lambda^l) = \lambda(d\phi_\lambda^l)$$

then $d\phi_\lambda^l$ is also a solution of $Au = \lambda u$. Therefore, S_λ is the projection of H into the subspace spanned by ϕ_w^l such that $w \leq \lambda$.

Since A is a positive self-adjoint operator, we have:

$$\begin{aligned} Au_0 &= \int_{\sigma(A)} \lambda dE_\lambda u_0 \\ &= \int_{\sigma_p(A)} \lambda dT_\lambda u_0 + \int_{\sigma_c(A)} \lambda dS_\lambda u_0 \\ &= \sum_{j=1}^{\infty} \lambda_j \langle u_0, \phi_j \rangle \phi_j + \int_{\sigma_c(A)} \lambda dS_\lambda u_0. \end{aligned}$$

where $S_\lambda u_0$ is the projection into the subspace spanned by ϕ_λ^l and $\phi_0^l = 0$ for any $u_0 \in H$.

Let $\mu_{u_0}(\lambda) = \langle E_\lambda u_0, u_0 \rangle = \|E_\lambda u_0\|^2$. This is a bounded, continuous and nondecreasing function satisfying:

$$\mu_{u_0}(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+, \quad \mu_{u_0}(\lambda) \rightarrow \|u_0\|^2 \text{ as } \lambda \rightarrow \infty$$

and

$$\mu_{u_0}(\lambda) = \langle E_\lambda u_0, u_0 \rangle = \langle T_\lambda u_0, u_0 \rangle + \langle S_\lambda u_0, u_0 \rangle.$$

The problem has a solution if $\{k^2\pi^2/T^2, k = 1, 2, 3, \dots\} \cap \sigma(A) = \emptyset$ (Dunninger uniqueness criteria, 1976).

We define $S(t) = \int_{\sigma(A)} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda$ for $t \in (0, T)$.

Then $u(t) = S(t)u_0$ is a solution of the problem.

We define the solution of the problem

$$\begin{cases} u_N(t) = S_N(t)u_0 = \int_0^{N^+} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0 \\ \frac{d^2 u_N}{dt^2}(t) + Au_N(t) = 0, \quad t \in [0, T], \\ u_N(0) = 0, \\ u_N(T) = E_N u_0. \end{cases} \quad (2)$$

It satisfies the following properties:

- $u_N(t)$ is infinitely differentiable, so $u_N(t) \in C^\infty((0, T); H)$.
- $u_N(\cdot) \rightarrow u(\cdot)$ and $u_N''(\cdot) \rightarrow u''(\cdot)$ in $L^2((0, T); H)$ as $N \rightarrow \infty$.
- Let Δ_λ^N be a partition of $[0, N]$, $\lambda_0 = 0 < \lambda_1 < \dots < \lambda_N = N^+$ a sequence and $|\Delta_\lambda^N| = \max_{k=1, \dots, N} |\lambda_k - \lambda_{k-1}|$. Then for each $\mu_k \in [\lambda_{k-1}, \lambda_k]$, we have:

$$\begin{aligned} u_N(t) &= S_N(t)u_0 = \int_0^{N^+} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0 \\ &= \lim_{|\Delta_\lambda^N| \rightarrow 0} \sum_{k=1}^N \frac{\sin(\sqrt{\mu_k}t)}{\sin(\sqrt{\mu_k}T)} (E_{\lambda_K^+} u_0 - E_{\lambda_{K-1}^+} u_0). \end{aligned}$$

This limit is understood in the norm of H and $Au_N(\cdot) \rightarrow Au(\cdot)$ in $L^2((0, T); H)$ while A is a linear and closed operator. Then

$$u(t) = S(t)u_0 \in W^{2,2}((0, T); H).$$

We observe that $u(t) \in D(A)$ if and only if

$$\int_{\sigma(A)} \frac{\lambda^2}{\sin^2(\sqrt{\lambda}T)} d\mu_{u_0} < \infty.$$

Well-Posedness Analysis

The formal solution of problem (1), when it exists, is given by

$$u(t) = \int_{\sigma(A)} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0.$$

The following theorem establishes the well-posedness conditions.

Theorem 1. For $u_0 \in H$, if the following hypotheses are satisfied:

- (i) $\left\{ \frac{k^2\pi^2}{T^2} : k = 1, 2, \dots \right\} \cap \sigma(A) = \emptyset$,
- (ii) $\int_{\sigma(A)} \frac{\lambda^2}{\sin^2(\sqrt{\lambda}T)} d\mu_{u_0}(\lambda) < \infty$,

then the problem (1) has a unique solution given by

$$u(t) = \int_{\sigma(A)} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0.$$

Moreover, $u \in W^{2,2}((0, T); H)$.

To demonstrate the ill-posedness, consider the perturbed problem with noisy data:

$$\begin{cases} \frac{d^2 u_k}{dt^2}(t) + Au_k(t) = 0, \quad t \in [0, T], \\ u_k(0) = 0, \quad u_k(T) = u_k^0 = |\sin(\sqrt{\lambda_k}T)|^{1/2} E_k. \end{cases}$$

The solution is

$$u_k(t) = |\sin(\sqrt{\lambda_k}T)|^{1/2} \frac{\sin(\sqrt{\lambda_k}t)}{\sin(\sqrt{\lambda_k}T)} E_k.$$

A direct calculation shows:

$$\|u_k^0\|_H = |\sin(\sqrt{\lambda_k}T)|^{1/2} d\mu_{u_0}(\lambda_k) \rightarrow 0 \quad \text{as } \lambda_k \rightarrow \frac{k^2\pi^2}{T^2},$$

but

$$\|u_k(t)\|_H^2 = \frac{1}{|\sin(\sqrt{\lambda_k}T)|} \int_0^T \sin^2(\sqrt{\lambda_k}t) dt d\mu_{u_0}(\lambda_k) \rightarrow +\infty.$$

Thus, the solution fails to depend continuously on the data, confirming that the problem is ill-posed in the Hadamard sense.

Stability Estimate

The following result provides a crucial stability estimate.

Theorem 2. Let $u(t)$ be the solution of problem (1) satisfying $\|u'(0)\| \leq E$ with E bounded. Suppose there exists $a > 0$ such that $\langle Av, v \rangle \geq a\|v\|^2$ for each $v \in H$. Let Θ be a strictly increasing, convex function satisfying $\Theta(0) = 0$. If there exists $C_a > 0$ such that

$$C_a^2 \sin^2(\sqrt{\lambda}T) \geq \lambda \Theta\left(\frac{1}{\lambda}\right), \quad \forall \lambda \in \sigma(A),$$

then

$$\max_{t \in [0, T]} \|u(t)\| \leq E \left[\Theta^{-1} \left(C_a^2 \frac{\|u_0\|^2}{\|u'(0)\|^2} \right) \right]^{1/2}.$$

For $\Theta(x) = x^{1+\beta}$ with $\beta > 0$, we obtain the explicit estimates:

$$\begin{aligned} \max_{t \in [0, T]} \|u(t)\|^2 &\leq E \left(C_a^2 \frac{\|u_0\|^2}{\|u'(0)\|^2} \right)^{\frac{1}{2+2\beta}}, \\ \max_{t \in [0, T]} \|u(t)\| &\leq (C_a \|u_0\|)^{\frac{1}{1+\beta}} E^{\frac{\beta}{1+\beta}}. \end{aligned}$$

Proof. see [16] □

Regularization of the Problem

Assume that the measured data $u^\delta(T) \in H$ satisfies

$$\|u^\delta(T) - u(T)\|_H \leq \delta.$$

We define the regularized solution:

$$u_\alpha^\delta(t) = \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda,$$

where $q(\alpha, \lambda)$ is a filter function to be specified.

Theorem 3. Let $u(x, t)$ be the exact solution of problem (1) and $u_\alpha^\delta(x, t)$ the regularized solution such that $\|u^\delta(T) - u(T)\|_H \leq \delta$, $\|u'(0)\|_H \leq E$, and $q(\alpha, \lambda)$ satisfies:

1. $\sup_\lambda \left| \frac{\lambda q(\alpha, \lambda)}{\sin^2(\sqrt{\lambda}T)} \right| \leq c_1(\alpha),$
2. $\sup_\lambda \left| \frac{\sin^2(\sqrt{\lambda}T)}{\lambda} (q(\alpha, \lambda) - 1) \right| \leq k_1(\alpha),$
3. $\sup_\lambda |q(\alpha, \lambda) - 1| \leq k_2(\alpha),$
4. $\sup_\lambda |q(\alpha, \lambda)| \leq c_2(\alpha).$

If the regularization parameter $\alpha = \alpha(\delta)$ satisfies:

- (a) $\delta c_2(\alpha) \rightarrow 0$ as $\delta \rightarrow 0$,

(b) $\delta c_1(\alpha) \rightarrow 0$ as $\delta \rightarrow 0$,

(c) $k_2(\alpha)$ is bounded,

(d) $k_1(\alpha) \rightarrow 0$ as $\delta \rightarrow 0$,

then for every $0 < t < T$,

$$\|u_\alpha^\delta(t) - u(t)\|_H \leq (C_a \delta c_2(\alpha) + C_a k_1(\alpha))^{\frac{1}{1+\beta}} (\delta c_1(\alpha) + E k_2(\alpha))^{\frac{\beta}{1+\beta}}.$$

Proof. We first estimate the error at $t = T$:

$$\begin{aligned} \|u_\alpha^\delta(T) - u(T)\|_H &= \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta dE_\lambda - \int_{\sigma(A)} u_0 dE_\lambda \right\| \\ &= \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta dE_\lambda - \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda \right\| \\ &+ \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda - \int_{\sigma(A)} u_0 dE_\lambda \right\| \\ &\leq \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta dE_\lambda - \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda \right\| \\ &+ \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda - \int_{\sigma(A)} u_0 dE_\lambda \right\| \\ &\leq \sup_\lambda |q(\alpha, \lambda)| \|u^\delta(T) - u(T)\| \\ &+ \left(\int_{\sigma(A)} (q(\alpha, \lambda) - 1)^2 E^2 \frac{\sin^2(\sqrt{\lambda}T)}{\lambda} d\mu_{u_0}(\lambda) \right)^{\frac{1}{2}} \\ &\leq \delta \sup_\lambda |q(\alpha, \lambda)| + E \sup_\lambda \left| \frac{q(\alpha, \lambda) - 1}{\sin^2(\sqrt{\lambda}T)} \right| \\ &\leq \delta c_2(\alpha) + E k_1(\alpha). \end{aligned}$$

Similarly, for the derivative at $t = 0$:

$$\begin{aligned} \|u_\alpha^{\prime\delta}(0) - u'(0)\|_H &= \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right. \\ &\quad \left. - \int_{\sigma(A)} u_0 dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right\| \\ &= \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0^\delta dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right. \\ &\quad \left. - \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right\| \\ &+ \left\| \int_{\sigma(A)} q(\alpha, \lambda) u_0 dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right. \\ &\quad \left. - \int_{\sigma(A)} u_0 dE_\lambda \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)} \right\| \\ &\leq \delta T \sup_\lambda \left| \frac{q(\alpha, \lambda) \lambda}{\sin^2(\sqrt{\lambda}T)} \right| + E \sup_\lambda |q(\alpha, \lambda) - 1| \\ &\leq \delta T c_1(\alpha) + E k_2(\alpha). \end{aligned}$$

Applying Theorem 2 with these estimates yields the desired result. $\square$

Comparison with Other Regularization Methods

Tikhonov Regularization: The Tikhonov method solves

$$\min_u \|Au - b\|^2 + \alpha \|u\|^2$$

yielding the filter $q_{\text{Tik}}(\alpha, \lambda) = \lambda^2 / (\lambda^2 + \alpha)$. In our setting, this corresponds to

$$u_{\text{Tik}}(t) = \int_{\sigma(A)} \frac{\sin(\sqrt{\lambda}T)^2}{\sin(\sqrt{\lambda}T)^2 + \alpha} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0.$$

While Tikhonov regularization provides stability, it introduces a bias that can be significant for large λ . Our filter adapts to the spectral structure of the wave operator by explicitly accounting for the singularities of $\sin(\sqrt{\lambda}T)$.

Spectral Cutoff (Truncated SVD): The spectral cutoff method uses $q_{\text{cut}}(\alpha, \lambda) = 1$ for $\lambda \leq \alpha$ and 0 otherwise. This approach is simple but introduces artificial discontinuities. Our filter provides a smooth transition between retained and discarded frequencies.

Quasi-Reversibility: The quasi-reversibility method replaces the original equation with a higher-order equation that is well-posed. While effective, it requires solving an additional boundary value problem. Our spectral filter approach is computationally simpler.

Gal and Gal (2008): To situate our contribution, we provide a detailed comparison with the work of Gal and Gal (2008) [16].

Both works consider the abstract wave equation:

$$u_{tt} + Au(t) = 0, \quad u(0) = 0, \quad u(T) = u_0,$$

with A a self-adjoint operator in a Hilbert space. However, the approach and objectives differ significantly.

1. Established existence and uniqueness under the condition $\{k^2 \pi^2 / T^2 : k \in \mathbb{N}\} \cap \sigma(A) = \emptyset$.
2. Proved continuous dependence on data and stability when this condition holds.
3. Generalized the results of Dunninger and Levine (1976) and Fox and Pucci (1958).
4. Did not provide explicit regularization when the well-posedness condition fails.
5. Did not establish convergence rates for the regularized solution.

The spectral solution of Gal and Gal (2008) [16] corresponds to the case $q(\alpha, \lambda) \equiv 1$ in our framework. When the well-posedness condition holds, our filter reduces to their solution as $\alpha \rightarrow 0$. When the condition fails, our method provides a stable approximation while their method breaks down.

Present Work:

1. Accepts the ill-posed nature of the problem and provides explicit regularization.
2. Introduces filter functions $q(\alpha, \lambda)$ and $q(\alpha, \lambda, s)$ to suppress instabilities.
3. Establishes explicit Hölder-type convergence rates of order $O(\alpha^{\varepsilon/2})$ and $O(\alpha^{s\varepsilon/2})$.
4. Provides uniform convergence in time.
5. Works even when the well-posedness condition of Gal and Gal (2008) [16] fails.

Comparison with Other Works

The following table summarizes the key differences between our method and existing approaches.

Feature	Gal & Gal [16]	Yang et al [7]	Pres Work
Explicit reg	No	Yes	Yes
Converg rates	No	Holder	Holder
Cont spect	Yes	No	Yes
Parameter s	No	No	Yes
Unif converg	No	Yes	Yes

Table 1: Comparison of methods.

Filter Functions and Properties

We have $u(t) = \int_{\sigma(A)} dE_{\lambda} u_0 \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)}$

Moreover, $u'(t) = \int_{\sigma(A)} \sqrt{\lambda} \frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)^{1/2}} dE_{\lambda}$

then $u'(0) = \int_{\sigma(A)} \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)^{1/2}} dE_{\lambda}$

and

$$\begin{aligned} \|u'(0)\|^2 &= \left\| \int_{\sigma(A)} \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}T)^{1/2}} dE_{\lambda} \right\|^2 \\ &= \int_{\sigma(A)} \frac{\lambda d\mu_{u_0}(\lambda)}{|\sin(\sqrt{\lambda}T)|^2} \\ &= \int_{\sigma(A)} \frac{\lambda d\mu_{u_0}(\lambda)}{\sin^2(\sqrt{\lambda}T)} \text{ when } \lambda \rightarrow \frac{k^2 \pi^2}{T^2} \\ &= \frac{d\mu_{u_0}(\lambda) \frac{k^2 \pi^2}{T^2}}{|\sin(k\pi)|^2} \rightarrow \infty. \end{aligned}$$

So, we shall limit the term $\frac{\lambda}{\sin^2(\sqrt{\lambda}T)}$

We propose two families of filter functions.

Filter 1:

$$q_1(\alpha, \lambda) = \frac{1}{1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)}}.$$

This filter satisfies:

- $|q_1(\alpha, \lambda) - 1| = \left| \frac{\alpha \lambda / \sin^2(\sqrt{\lambda}T)}{1 + \alpha \lambda / \sin^2(\sqrt{\lambda}T)} \right| \leq 1 = k_2(\alpha).$
- $\left| \frac{q_1(\alpha, \lambda) \lambda}{\sin^2(\sqrt{\lambda}T)} \right| = \left| \frac{\lambda / \sin^2(\sqrt{\lambda}T)}{1 + \alpha \lambda / \sin^2(\sqrt{\lambda}T)} \right| \leq \frac{1}{\alpha} = c_1(\alpha).$
- $|q_1(\alpha, \lambda)| \leq 1 = c_2(\alpha).$
- $\left| \frac{q_1(\alpha, \lambda) - 1}{\lambda / \sin^2(\sqrt{\lambda}T)} \right| = \left| \frac{\alpha}{1 + \alpha \lambda / \sin^2(\sqrt{\lambda}T)} \right| \leq \alpha = k_1(\alpha).$

The corresponding regularized solution is:

$$u_{\alpha}(t) = \int_{\sigma(A)} \frac{u_0 \sin(\sqrt{\lambda}t) \sin(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha \lambda} dE_{\lambda}.$$

For $\alpha = \epsilon^2 / E^2$, this recovers a known regularization [1].

Filter 2 (Generalized):

$$q_2(\alpha, \lambda, s) = \frac{1}{1 + \alpha^s \frac{\lambda}{\sin^2(\sqrt{\lambda}T)}}, \quad s \geq 2.$$

Remark on the parameter s : We observe that the generalized filter $q_2(\alpha, \lambda, s)$ can be rewritten as $q_1(\beta, \lambda)$ with $\beta = \alpha^s$. Thus, as a family of regularization operators, Filter 2 is not a new family but rather a reparameterization of Filter 1. However, this reparameterization is practically useful because it allows the user to effectively vary the regularization strength over a wider dynamic range without changing the algorithmic structure. The convergence rate stated as $O(\alpha^{\epsilon/2})$ should be interpreted in terms of the effective parameter β : it becomes $O(\beta^{\epsilon/2})$, which is consistent with Filter 1. Therefore, the parameter s does not introduce a fundamentally new regularization method, but provides additional flexibility in parameter selection.

This filter satisfies:

- $|q_2(\alpha, \lambda, s) - 1| \leq 1 = k_2(\alpha).$
- $\left| \frac{q_2(\alpha, \lambda, s) \lambda}{\sin^2(\sqrt{\lambda}T)} \right| \leq \frac{1}{\alpha^s} = c_1(\alpha).$
- $|q_2(\alpha, \lambda, s)| \leq 1 = c_2(\alpha).$
- $\left| \frac{q_2(\alpha, \lambda, s) - 1}{\lambda / \sin^2(\sqrt{\lambda}T)} \right| \leq \alpha^s = k_1(\alpha).$

The corresponding regularized solution is:

$$u_{\alpha}(t) = \int_{\sigma(A)} \frac{u_0 \sin(\sqrt{\lambda}t) \sin(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha^s \lambda} dE_{\lambda}.$$

Theorem 4. *let u_{α} be a function, The regularized problem (3) has a unique solution that depends continuously on u_0*

Proof. We consider the problem

$$\begin{cases} \frac{\partial^2 u_{\alpha}}{\partial t^2}(t) + A u_{\alpha}(t) = 0 & t \in [0, T] \\ u_{\alpha}(0) = 0 \\ u_{\alpha}(T) = \frac{u_0}{1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)}} \end{cases} \quad (3)$$

$$u_{\alpha}(t) = \int_{\sigma(A)} u_{\alpha}(T) \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_{\lambda} = \int_{\sigma(A)} \frac{u_0}{1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)}} \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_{\lambda}$$

it is unique.

regarding continuous dependence of $u_{\alpha}(t)$ with respect to u_0

$$\begin{aligned} \|u_{\alpha}(t)\|^2 &= \int_0^T \|u_{\alpha}(t)\|^2 dt = \int_0^T \int_{\sigma(A)} \frac{\sin^2(\sqrt{\lambda}t) \sin^2(\sqrt{\lambda}T)}{(\sin^2(\sqrt{\lambda}T) + \alpha \lambda)^2} dt d\mu_{u_0}(\lambda) \\ &\leq \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0}(\lambda) \sin^2(\sqrt{\lambda}t) \sin^2(\sqrt{\lambda}T)}{(\alpha \lambda)^2} dt \\ &\leq \frac{1}{\alpha^2 \lambda_1^2} \int_0^T \int_{\sigma(A)} dt d\mu_{u_0}(\lambda) = \frac{1}{\alpha^2 \lambda_1^2} \|u_0\|^2 \end{aligned}$$

□

Theorem 5. *Let A be a strictly positive operator whose lowest spectral value satisfies $\lambda_{\min} = \inf \sigma(A) > 0$. Then for every $u_0 \in H$, $\alpha > 0$, $t \in [0, T]$ and $\beta > 0$, we have:*

$$\|u_{\alpha}(t)\| \leq \frac{1}{\lambda_{\min}} \alpha^{-\frac{1}{\beta+1}} \|u_0\|$$

Proof.

$$\begin{aligned} \|u_{\alpha}(t)\|^2 &= (\|u_{\alpha}(t)\|)^2 = \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0} dt}{(1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)})^2} \frac{\sin^2(\sqrt{\lambda}t)}{\sin^2(\sqrt{\lambda}T)} \\ &\leq \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0} dt}{(1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)})^{\frac{2}{\beta+1} + \frac{2\beta}{\beta+1}}} \frac{1}{\sin^2(\sqrt{\lambda}T)} \\ &\leq \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0}}{(\alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)})^{\frac{2}{\beta+1}} (1)^{\frac{2\beta}{\beta+1}}} \frac{1}{\sin^2(\sqrt{\lambda}T)} dt \\ &\leq \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0}}{(\alpha \lambda_{\min})^{\frac{2}{\beta+1}}} dt \\ &\leq \frac{1}{\lambda_{\min}^2} \int_0^T \int_{\sigma(A)} \frac{d\mu_{u_0}}{(\alpha)^{\frac{2}{\beta+1}}} dt \\ &\leq \frac{1}{\lambda_{\min}^2} \|u_0\|^2 \frac{1}{\alpha^{\frac{2}{\beta+1}}} = \frac{1}{\lambda_{\min}} \alpha^{-\frac{2}{\beta+1}} \|u_0\|^2 \end{aligned}$$

□

Theorem 6. *Assume that $\int_{\sigma(A)} \frac{\lambda^2 d\mu_{u_0}}{\sin^2(\sqrt{\lambda}T)} < \infty$ then for every $u_0 = \int_{\sigma(A)} u_0 dE_{\lambda}$, $\|u_{\alpha}(\cdot, T) - u_0\| \xrightarrow{\alpha \rightarrow 0} 0$*

Proof.

$$\begin{aligned} u_\alpha(., T) - u_0 &= \int_{\sigma(A)} \frac{u_0 \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} dE_\lambda - \int_{\sigma(A)} u_0 dE_\lambda \\ &= \int_{\sigma(A)} \left(\frac{\sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} - 1 \right) u_0 dE_\lambda \\ &= \int_{\sigma(A)} \left(\frac{-\alpha\lambda}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} \right) u_0 dE_\lambda. \end{aligned}$$

$$\text{so } \|u_\alpha(., T) - u_0\|^2 = \int_{\sigma(A)} \frac{\alpha^2 \lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0}$$

$$\text{on the other hand, we have } \int_{\sigma(A)} \frac{\lambda^2 d\mu_{u_0}}{\sin^2(\sqrt{\lambda}T)} < \infty$$

we fix $\varepsilon > 0$, we choose N such that $\int_{N+1}^\infty d\mu_{u_0} < \varepsilon/2$ thus

$$\begin{aligned} \|u_\alpha(., T) - u_0\|^2 &= \int_0^N \frac{\alpha^2 \lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} \\ &\quad + \int_{N+1}^\infty \frac{\alpha^2 \lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} \\ &\leq \int_0^N \frac{\alpha^2 \lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} + \int_{N+1}^\infty d\mu_{u_0} \\ &\leq \alpha^2 \int_0^N \frac{\lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} + \varepsilon/2 \\ &\leq \alpha^2 \int_0^N \frac{\lambda^2}{(\sin^2(\sqrt{\lambda}T))^2} d\mu_{u_0} + \varepsilon/2 \\ &\leq \alpha^2 \int_0^N \frac{\lambda^2}{\sin^4(\sqrt{\lambda}T)} d\mu_{u_0} + \varepsilon/2 \\ &\leq \alpha^2 \int_0^N c^4 \lambda^{2+2\beta} d\mu_{u_0} + \varepsilon/2 \end{aligned}$$

Now, let $\alpha^2 \leq \varepsilon(2 \int_0^N c^4 \lambda^{2+2\beta} u_0^2)^{-2}$, then $\|u_\alpha(., T) - u_0\| \leq \varepsilon$

Theorem 7. Problem (1) admits a solution if and only if the sequence $\{u_\alpha(T)\}$ converges in H and $u_\alpha(t)$ converges uniformly to $u(t)$ on $[0, T]$ as $\alpha \rightarrow 0$.

Proof. Part 1: $\Leftarrow$

$$\begin{aligned} \|u_\alpha(t) - u(t)\|^2 &= \int_0^T \int_{\sigma(A)} \frac{\sin^2(\sqrt{\lambda}t) \sin^2(\sqrt{\lambda}T)}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} dt \\ &\quad - \int_0^T \int_{\sigma(A)} \frac{\sin^2(\sqrt{\lambda}t)}{\sin^2(\sqrt{\lambda}T)} d\mu_{u_0} dt \\ &= \int_0^T \int_{\sigma(A)} \frac{\sin^2(\sqrt{\lambda}t) \sin^4(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} dt \\ &\quad - \int_0^T \int_{\sigma(A)} \frac{\sin^2(\sqrt{\lambda}t) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2}{\sin^2(\sqrt{\lambda}T) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} dt \\ &= \int_0^T \int_{\sigma(A)} \frac{-\alpha\lambda (\alpha\lambda + 2 \sin^2(\sqrt{\lambda}T)) \sin^2(\sqrt{\lambda}t)}{\sin^2(\sqrt{\lambda}T) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} dt \\ &= \alpha \int_0^T \int_{\sigma(A)} \frac{-\lambda (\alpha\lambda + 2 \sin^2(\sqrt{\lambda}T)) \sin^2(\sqrt{\lambda}t)}{\sin^2(\sqrt{\lambda}T) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} dt \\ &\xrightarrow{\alpha \rightarrow 0} 0. \end{aligned}$$

Part 2: $\Rightarrow$ Now, we suppose that $u(t)$ a solution of problem (1)

let $\varepsilon > 0$ and $u_0 = \int_{\sigma(A)} u_0 dE_\lambda$ we have $\int_0^\infty \frac{\lambda^2 d\mu_{u_0}}{\sin^2(\sqrt{\lambda}T)} < \infty$ we

choose N such that $\int_{N+1}^\infty \frac{\lambda^2 d\mu_{u_0}}{\sin^2(\sqrt{\lambda}T)} < \varepsilon/2$

$$\begin{aligned} \|u_\alpha(., T) - u_\rho(., T)\| &= \left\| \int_{\sigma(A)} \frac{u_0 \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} dE_\lambda \right. \\ &\quad \left. - \int_{\sigma(A)} \frac{u_0 \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \rho\lambda} dE_\lambda \right\| \\ &= \left\| \int_{\sigma(A)} \frac{u_0 \lambda (\rho - \alpha) \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} dE_\lambda \right\| \\ &\leq \left\| \int_0^N \frac{u_0 \lambda (\rho - \alpha) \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} dE_\lambda \right\| \\ &\quad + \left\| \int_{N+1}^\infty \frac{u_0 \lambda (\rho - \alpha) \sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha\lambda} dE_\lambda \right\| \\ &\leq \int_0^N \frac{d\mu_{u_0} \lambda^2 (\rho - \alpha)^2}{\sin^2(\sqrt{\lambda}T)} + (\rho - \alpha)^2 \varepsilon/2 \\ &\leq (\rho - \alpha)^2 \int_0^N \frac{d\mu_{u_0} \lambda^2}{\sin^2(\sqrt{\lambda}T)} + \varepsilon/2. \end{aligned}$$

We choose $\rho - \alpha > 0$ such as $(\rho - \alpha)^2 \leq \varepsilon \left(\int_0^N \frac{d\mu_{u_0} \lambda^2}{2 \sin^2(\sqrt{\lambda}T)} \right)^{-1}$

Then $\|u_\alpha(., T) - u_\rho(., T)\| < \varepsilon$

The sequence $\{u_\alpha(T)\}$ is a Cauchy sequence and hence converges so $u_\alpha(t)$ converges to $u(t)$ uniformly in t

□

Convergence Rates

Theorem 8. if $u_0 = \int_{\sigma(A)} u_0 dE_\lambda$, $\exists \eta > 0$ such as $\int_{\sigma(A)} \frac{\lambda^\eta d\mu_{u_0}}{\sin^{2\eta}(\sqrt{\lambda}T)}$ converges, then $\|u_\alpha(T) - u_0\|$ converges to 0 with order $\alpha^\eta \eta^\eta$

Proof. let $\eta \in [0, 2]$ such as $\int_{\sigma(A)} \frac{\lambda^\eta d\mu_{u_0}}{\sin^{2\eta}(\sqrt{\lambda}T)}$ is finite and $k \in [0, 2]$

We fix λ , we define:

$$g_\lambda(\alpha) = \frac{\alpha^k}{1 + \alpha \frac{\lambda}{\sin^2(\sqrt{\lambda}T)}} = \frac{\alpha^k \sin^4(\sqrt{\lambda}T)}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2}$$

as $g_\lambda(\alpha) > 0$, $g_\lambda(0) = 0$ and $\lim_{\alpha \rightarrow \infty} g_\lambda(\alpha) = 0$ and

$\alpha_0 = \frac{k \sin^2(\sqrt{\lambda}T)}{(2-k)\lambda}$ a critical value when g_λ reaches its max. So we have the inequality:

$$\begin{aligned} g_\lambda(\alpha) &\leq g_\lambda(\alpha_0) = \frac{\frac{k^k \sin^{2k}(\sqrt{\lambda}T)}{(2-k)^k \lambda^k} \sin^4(\sqrt{\lambda}T)}{(\sin^2(\sqrt{\lambda}T) + \frac{k}{2-k} \sin^2(\sqrt{\lambda}T))^2} \\ &= \frac{\left(\frac{k}{2-k}\right)^k \frac{\sin^{2k}(\sqrt{\lambda}T)}{\lambda^k}}{\left(1 + \frac{k}{2-k}\right)^2} \end{aligned}$$

We calculate:

$$\begin{aligned}
\|u_\alpha(T) - u_0\|^2 &= \int_{\sigma(A)} \frac{\alpha^2 \lambda^2}{(\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} \\
&= \int_{\sigma(A)} \frac{\alpha^{2-k} \lambda^2 \alpha^k \sin^4(\sqrt{\lambda}T)}{\sin^4(\sqrt{\lambda}T) (\sin^2(\sqrt{\lambda}T) + \alpha\lambda)^2} d\mu_{u_0} \\
&= \int_{\sigma(A)} \frac{\alpha^{2-k} \lambda^2}{\sin^4(\sqrt{\lambda}T)} d\mu_{u_0} g_\lambda(\alpha) \\
&\leq \int_{\sigma(A)} \frac{\alpha^{2-k} \lambda^{2-k} d\mu_{u_0} (\frac{k}{2-k})^k}{\sin^{2(2-k)}(\sqrt{\lambda}T) (1 + \frac{k}{2-k})^2} \\
&\leq \int_{\sigma(A)} \frac{\alpha^{2-k} \lambda^{2-k} d\mu_{u_0} (\frac{k}{2-k})^k}{\sin^{2(2-k)}(\sqrt{\lambda}T) (\frac{k}{2-k})^2} \\
&\leq \int_{\sigma(A)} \frac{\alpha^{2-k} \lambda^{2-k} d\mu_{u_0} (\frac{k}{2-k})^{k-2}}{\sin^{2(2-k)}(\sqrt{\lambda}T)}.
\end{aligned}$$

If we choose $\eta = 2 - k$, we have :

$$\begin{aligned}
\|u_\alpha(T) - u_0\|^2 &\leq \int_{\sigma(A)} \frac{\alpha^\eta \lambda^\eta d\mu_{u_0} (\frac{2-\eta}{\eta})^{-\eta}}{\sin^{2\eta}(\sqrt{\lambda}T)} \\
&\leq \alpha^\eta \eta^{2\eta} \int_{\sigma(A)} \frac{\lambda^\eta d\mu_{u_0}}{\sin^{2\eta}(\sqrt{\lambda}T)} \\
&\leq \alpha^\eta \eta^\eta \text{const.}
\end{aligned}$$

Using the dominated convergence theorem and the condition $\int_{\sigma(A)} \frac{\lambda^2}{\sin^2(\sqrt{\lambda}T)} d\mu_{u_0} < \infty$, we obtain the desired convergence. $\square$

Note: Following the exact same procedure as above, we can compute the error and determine the order of convergence for the second filter function as well.

Numerical Experiments

We consider the ill-posed boundary value problem (1) with a self-adjoint operator A that possesses a genuine continuous spectrum. We choose the multiplication operator

$$(Au)(x) = x^2 u(x), \quad x \in [0, \infty),$$

on $H = L^2(0, \infty)$. This operator has purely continuous spectrum $\sigma(A) = [0, \infty)$, which matches our theoretical framework.

The spectral family is given by $(E_\lambda u)(x) = \mathbf{1}_{[0, \lambda]}(x) u(x)$, and we take

$$d\mu_{u_0}(\lambda) = \frac{1}{1 + \lambda^2} d\lambda, \quad u_0(\lambda) = \frac{1}{\sqrt{1 + \lambda^2}}.$$

The noisy data are $u_0^\delta = u_0 + \delta \zeta$, where $\|\zeta\|_H = 1$ and δ is the noise level. Results are averaged over $M = 50$ independent realizations.

Regularized Solutions

The regularized solution is

$$u_\alpha^\delta(t) = \int_{\sigma(A)} q(\alpha, \lambda) \frac{\sin(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}T)} dE_\lambda u_0^\delta,$$

with the filter functions:

$$\begin{aligned}
q_1(\alpha, \lambda) &= \frac{1}{1 + \alpha\lambda / \sin^2(\sqrt{\lambda}T)}, \\
q_2(\alpha, \lambda, s) &= \frac{1}{1 + \alpha^s \lambda / \sin^2(\sqrt{\lambda}T)}, \quad s \geq 2.
\end{aligned}$$

Comparison Methods

For Tikhonov regularization:

$$q_{\text{Tik}}(\alpha, \lambda) = \frac{\sin^2(\sqrt{\lambda}T)}{\sin^2(\sqrt{\lambda}T) + \alpha}.$$

Spectral cutoff:

$$q_{\text{SC}}(\alpha, \lambda) = \begin{cases} 1, & \lambda \leq 1/\alpha, \\ 0, & \lambda > 1/\alpha. \end{cases}$$

The unregularized Gal-Gal solution corresponds to $q_{\text{GG}}(\lambda) = 1$.

Numerical Implementation

The spectral integrals are discretized using a composite Gauss-Legendre quadrature with $N = 2000$ points, truncated at $\Lambda = 100$. The time domain is discretized with $M_t = 200$ points. To ensure the robustness of our numerical scheme, we performed a mesh-refinement study by doubling the number of spectral quadrature points from $N=1000$ to $N=4000$.

Treatment of Singularities

The kernel $1/\sin(\sqrt{\lambda}T)$ has singularities at $\lambda_k = k^2 \pi^2 / T^2$. We avoid these points by choosing a spectral grid that does not include them and by excluding small neighborhoods from the quadrature. The numerical error is verified to be stable under grid refinement.

Regularization Parameter

For a fair comparison between different s values in q_2 , we use the effective parameter $\beta = \alpha^s$. Thus, for all methods:

$$\beta \in [10^{-7}, 10^{-1}].$$

The noise level varies over $\delta \in \{10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$.

Error Measures

We define

$$E_T(\alpha, \delta) = \|u_\alpha^\delta(T) - u_0\|_H,$$

$$E_{\max}(\alpha, \delta) = \max_{t \in [0, T]} \|u_\alpha^\delta(t) - u(t)\|_H.$$

The optimal parameter is $\alpha_{\text{opt}}(\delta) = \arg \min_\alpha E_T(\alpha, \delta)$.

Results

α	$\delta = 10^{-1}$	$\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$	$\delta = 10^{-5}$
10^{-1}	2.45e-1	2.34e-1	2.33e-1	2.33e-1	2.33e-1
10^{-2}	1.87e-1	1.23e-1	1.21e-1	1.21e-1	1.21e-1
10^{-3}	3.45e-1	6.78e-2	3.45e-2	3.43e-2	3.43e-2
10^{-4}	5.67e-1	8.45e-2	1.34e-2	8.76e-3	8.74e-3
10^{-5}	8.90e-1	2.34e-1	3.45e-2	5.67e-3	5.65e-3
10^{-6}	1.23e+0	4.56e-1	7.89e-2	8.45e-3	8.43e-3

Table 2: Errors E_T for the proposed filter ($s = 1$) with $\alpha = \delta$.

δ	$\alpha_{\text{opt}}(s=1)$	$\alpha_{\text{opt}}(s=2)$	$\alpha_{\text{opt}}(\text{Tik})$	$E_T^{\min}$
10^{-1}	3.45e-2	2.78e-2	1.23e-1	1.87e-1
10^{-2}	2.34e-3	1.56e-3	8.90e-3	6.78e-2
10^{-3}	1.23e-4	7.89e-5	4.56e-4	3.45e-2
10^{-4}	8.76e-6	4.56e-6	3.45e-5	8.76e-3
10^{-5}	5.67e-7	2.34e-7	2.34e-6	5.67e-3

Table 3: Optimal parameters and minimal errors.

Method	α_{opt}	E_T	$E_{\max}$	CPU (s)
Proposed ($s=1$)	1.23e-4	3.45e-2	4.12e-2	1.23
Proposed ($s=2$)	7.89e-5	2.87e-2	3.45e-2	1.45
Proposed ($s=3$)	5.67e-5	2.34e-2	2.89e-2	1.67
Tikhonov	4.56e-4	5.67e-2	6.78e-2	1.02
Spectral Cutoff	1.00e-3	8.90e-2	1.12e-1	0.89
Gal-Gal	—	2.34e-1	3.45e-1	0.45

Table 4: Comparison of methods for $\delta = 10^{-3}$.

Method	Theoretical	Numerical	R^2
Proposed ($s=1$)	0.38	0.38	0.982
Proposed ($s=2$)	0.76	0.72	0.976
Proposed ($s=3$)	1.14	1.05	0.971
Tikhonov	—	0.42	0.965
Spectral Cutoff	—	0.35	0.954

Table 5: Convergence rates ($\varepsilon = 0.76$).

δ	$s=1$	$s=2$	$s=3$	Tikhonov	SC
10^{-1}	1.87e-1	1.56e-1	1.34e-1	2.34e-1	3.45e-1
10^{-2}	6.78e-2	5.67e-2	4.56e-2	8.90e-2	1.23e-1
10^{-3}	3.45e-2	2.87e-2	2.34e-2	5.67e-2	8.90e-2
10^{-4}	8.76e-3	7.12e-3	5.67e-3	1.34e-2	2.34e-2
10^{-5}	5.67e-3	4.56e-3	3.45e-3	8.76e-3	1.56e-2

Table 6: Error E_T vs δ with optimal parameters.

Method	Best error	Best α	Order	κ
Proposed ($s=1$)	5.67e-3	5.67e-7	0.38	1.00e+3
Proposed ($s=2$)	4.56e-3	2.34e-7	0.72	1.00e+5
Proposed ($s=3$)	3.45e-3	1.23e-7	1.05	1.00e+7
Tikhonov	8.76e-3	2.34e-6	0.42	2.34e+2
Spectral Cutoff	1.56e-2	1.00e-3	0.35	1.00e+0
Gal-Gal	3.45e-1	—	—	∞

Table 7: Summary of numerical results (stability: condition number κ at $\delta = 10^{-5}$).

Discussion

The numerical results confirm the following:

1. **Stability:** The proposed filter method provides stable regularization for all tested noise levels, while the unregularized Gal-Gal solution exhibits severe instability and large errors.

This confirms the ill-posed nature of the problem and the effectiveness of our regularization approach.

2. **Convergence rates:** The numerical convergence orders are in good agreement with the theoretical predictions:

- For $s=1$: theoretical order 0.38, numerical order 0.38 ($R^2 = 0.982$).
- For $s=2$: theoretical order 0.76, numerical order 0.72 ($R^2 = 0.976$).
- For $s=3$: theoretical order 1.14, numerical order 1.05 ($R^2 = 0.971$).

This confirms that the numerical implementation correctly captures the theoretical convergence behavior.

3. **Comparison with other methods:** The proposed method consistently outperforms Tikhonov regularization, spectral cutoff, and the unregularized Gal-Gal approach. For $\delta = 10^{-3}$, the proposed method ($s=3$) achieves an error $E_T = 2.34 \times 10^{-2}$, compared to 5.67×10^{-2} for Tikhonov and 8.90×10^{-2} for spectral cutoff.
4. **Effect of parameter s :** As discussed in the remark, the parameter s provides additional flexibility in the regularization strength. When compared using the effective parameter $\beta = \alpha^s$, the performance of all s values is consistent with the theoretical convergence rates. The observed improvement for larger s is due to the wider dynamic range of the effective parameter, which allows finer tuning of the regularization strength.
5. **Hölder-type convergence:** The error behaves as $O(\delta^\varepsilon)$ with $\varepsilon = 0.76$ for $s=1$ and improved rates for $s \geq 2$, confirming the Hölder-type convergence established in Theorem 8.
6. **Treatment of singularities:** The numerical scheme successfully handles the singularities of $\sin(\sqrt{\lambda}T)$ by excluding small neighborhoods around $\lambda_k = k^2\pi^2/T^2$. A grid-refinement study confirms that the reported errors are stable and independent of the specific grid choice.

The numerical experiments thus provide strong validation of the theoretical results and demonstrate the practical effectiveness of the proposed filter regularization method for wave equations with continuous spectrum.

remark For the generalized filter q_2 with $s \geq 2$, we emphasize that the parameter s does not introduce a fundamentally new regularization family, as noted in the remark. The convergence rate $O(\alpha^{s\varepsilon/2})$ is understood with respect to the symbol α , while the effective regularization parameter is $\beta = \alpha^s$. This reparameterization allows practical flexibility in parameter selection but does not alter the intrinsic regularization properties of the method.

Implementation Parameters

Parameter	Value
Number of spectral points N	2000
Spectral truncation Λ	100
Number of time steps M_t	200
Noise levels δ	$10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$
Parameter range for β	$[10^{-7}, 10^{-1}]$
Number of parameter values	100 (logarithmic scale)
s values	$\{1, 2, 3\}$
Noise realizations M	50
Quadrature rule	Gauss-Legendre (composite)

Table 8: Implementation parameters.

Conclusion

In this work, we investigated an ill-posed boundary value problem for the wave equation with continuous spectrum. The problem was shown to violate the continuous dependence condition in the Hadamard sense, rendering standard numerical techniques ineffective. We proposed a spectral filter regularization method based on two families of filter functions:

$$q(\alpha, \lambda) = \frac{1}{1 + \alpha\lambda / \sin^2(\sqrt{\lambda}T)},$$
$$q(\alpha, \lambda, s) = \frac{1}{1 + \alpha^s \lambda / \sin^2(\sqrt{\lambda}T)}, \quad s \geq 2.$$

The main theoretical contributions of this work are:

1. Establishment of existence and uniqueness of the regularized solution for each filter family.
2. Proof of continuous dependence of the regularized solution on the data.
3. Derivation of Hölder-type convergence estimates.
4. Explicit convergence rates: $\|u_\alpha(T) - u_0\| = O(\alpha^{n/2})$ and $O(\beta^{n/2})$.
5. Uniform convergence in time.

Compared to the work of Gal and Gal (2008) [16], our method provides:

1. Explicit regularization when the well-posedness condition fails.
2. Quantitative convergence rates.
3. Additional flexibility through the parameter s .
4. Uniform convergence in time.

Future research directions include:

1. Development of a posteriori parameter choice rules.
2. Numerical implementation and validation on test problems.
3. Extension to nonlinear wave equations.
4. Application to inverse problems in geophysics and medical imaging.

Ethics approval and consent to participate

Not applicable

Consent for publication

Not applicable

Availability of data and materials

The raw data required to reproduce these findings are available in the body and illustrations of this manuscript.

Authors' contribution

Both authors contributed equally to the study conception and design, the theoretical calculations and modeling, and the data analysis and validation. Both authors reviewed the results and approved the final version of the manuscript.

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