Composition Operators on Orlicz and Bochner Spaces

Mahmud Masri

Mathematics Department, Faculty of Science, An-Najah National University, Nablus, Palestine.

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Abstract

Let \((T, M, \mu)\) be a finite positive measure space, \(X\) a Banach space, \(\phi\) a modulus function and \(f : T \to X\) a strongly measurable function. The Orlicz space is \(L^\phi(\mu, X) = \left\{ f : \int \phi(\|f(t)\|d\mu(t)) < \infty \right\}\). The space of Bochner \(p\)-integrable functions, \(1 \leq p < \infty\), is \(L^p(\mu, X) = \left\{ f : \|f\|_p = \left( \int \|f(t)\|^pd\mu(t) \right)^{1/p} < \infty \right\}\). Also \(L^\infty(\mu, X) = \left\{ f : \operatorname{esssup}_{t \in T} \|f(t)\| < \infty \right\}\).

When \(\phi(x) = x^p\), \(0 < p \leq 1\), \(L^\phi(\mu, X) = L^p(\mu, X)\). Let \(\Psi : T \to T\) be a function with \(\Psi^{-1}(A) \in M\) for all \(A \in M\) and define \(C_\psi(f) = f \circ \Psi\). We prove that \(C_\psi\) is a bounded linear operator on \(L^\phi(\mu, X)\) and \(L^p(\mu, X)\), \(0 < p \leq \infty\), when \(d\mu_\psi/d\mu \in L^\infty(\mu, \mathbb{C})\) where \(\mu_\psi(A) = \mu(\Psi^{-1}(A))\) for all \(A \in M\) and \(\mathbb{C}\) is the complex numbers. Also, we show that \(C_\psi\) is an isometry of \(L^\phi(\mu, X)\) and \(L^p(\mu, X)\), \(0 < p < \infty\) iff \(d\mu_\psi/d\mu = 1\) a.e. Moreover, \(C_\psi\) is an isometry of \(L^\infty(\mu, X)\) iff \(\mu \ll \mu_\psi\). This generalizes some previous results of the special case \(L^p(\mu, \mathbb{C})\) and proves similar results for \(L^\phi(\mu, X)\).

ملخص

\(f : T \to X\) ليكن في قياسياً موجباً ومتناهاً و\(X\) هو فضاء ينافى و\(T, M, \mu\) هو فضاء قياسي موجباً ومتناهاً و\(\phi\) هو فضاء ينافى و\(T, M, \mu\) هو فضاء قياسي موجباً ومتناهاً و\(X\) هو فضاء قياسي موجباً ومتناهاً. فكلاً فضاءي أورلتس هو

\(f \left( \int \phi(\|f(t)\|d\mu(t)) < \infty \right) \to L^\phi(\mu, X)\).
1. Introduction:

If \( \phi \) is a strictly increasing continuous subadditive function on \([0, \infty)\) and satisfies \( \phi(x) = 0 \iff x = 0 \), then we call \( \phi \) a modulus function. Let \((T, M, \mu)\) be a finite positive measure space, i.e., \( T \) is a set, \( M \) is a \( \sigma \)-algebra and \( \mu \) is a positive measure with \( \mu(T) < \infty \). If \( X \) is a Banach space, then a function \( s : T \to X \) is called a simple function if its range contains finitely many points \( x_1, x_2, \ldots, x_n \) and \( E_i = s^{-1}(\{x_i\}) \), \( i = 1, 2, \ldots, n \) are measurable sets. Such a function \( s \) can be written as \( s = \sum_{i=1}^{n} \chi_{E_i} \), where \( \chi_{E_i} \) is the characteristic function of the set \( E_i \) and \( E_i \cap E_j = \emptyset \) for \( i \neq j \), \( i,j = 1, 2, \ldots, n \). A function \( f : T \to X \) is said to be strongly measurable if there exists a sequence \( \{s_n\} \) of simple functions such that

\[
\lim_{n \to \infty} \|s_n(t) - f(t)\| = 0 \ \text{a.e.}
\]

\(L^p(\mu, X)\), and \(L^\phi(\mu, X)\) are defined as

\[
L^p(\mu, X) = L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p \, d\mu(t) < \infty \right\}
\]

where \( 1 \leq p < \infty \)

\[
L^\phi(\mu, X) = \left\{ f : \sup_{t \in T} \|f(t)\| < \infty \right\}
\]

1. Introduction:
The Orlicz space \( L^\mathcal{M} (\mu, X) \) is the set of all (equivalence classes) of strongly measurable functions \( f \) with
\[
\| f \|_\mathcal{M} = \int \phi(\| f(t) \|) d\mu(t) < \infty.
\]

If for all \( f, g \in L^\mathcal{M} (\mu, X) \) we define \( d(f, g) = \| f - g \|_\mathcal{M} \), then \( d \) is a metric on \( L^\mathcal{M} (\mu, X) \) under which it becomes a complete topological vector space [1,p.70]. For \( 1 \leq p < \infty \), \( L^p (\mu, X) \) denotes the Banach space of (equivalence classes of) strongly measurable functions \( f \) such that
\[
\int T \| f(t) \|^p d\mu(t) < \infty.
\]
The norm in \( L^p (\mu, X) \) is given by
\[
\| f \|_p = \left( \int T \| f(t) \|^p d\mu(t) \right)^{1/p}.
\]

The essentially bounded strongly measurable functions \( f \) form Banach space \( L^\infty (\mu, X) \) with norm given by
\[
\| f \|_\infty = \text{ess sup}_{t \in T} \| f(t) \|.
\]

If \( \phi \) is the modulus function \( \phi(x) = x^p, 0 < p \leq 1 \), then \( L^\mathcal{M} (\mu, X) \) is the space \( L^p (\mu, X) \). Since [2, p. 159], for any modulus function \( \phi \),
\[
\limsup_{x \to \infty} \frac{\phi(x)}{x} \leq \phi(1),
\]
it follows that \( L^1 (\mu, X) \subseteq L^\mathcal{M} (\mu, X) \).

For simplicity of notation we write \( L^p (\mu, C) = L^p, 0 < p \leq \infty \), \( L^\mathcal{M} (\mu, C) = L^\mathcal{M} \).

Also, \( \| \cdot \|_p = \| \cdot \|_p \), \( \| \cdot \|_\infty = \| \cdot \|_\infty \), \( \| \cdot \|_\mathcal{M} = \| \cdot \|_\mathcal{M} \) when \( X \) is the complex numbers \( C \).
We mean by a measurable transformation on $T$ a function $\Psi : T \to T$ such that $\Psi^{-1}(A) \in M$ for all $A \in M$. It is easy to see that $\Psi$ induces a positive measure $\mu_{\Psi}$ on $M$ where $\mu_{\Psi}(A) = \mu(\Psi^{-1}(A))$ for all $A \in M$. Also, $\Psi$ induces a composition operator $C_{\Psi}$ on strongly measurable functions given by $C_{\Psi}(f) = f \circ \Psi$ when $\mu$ is complete or $\mu_{\Psi} \ll \mu$, i.e., $\mu_{\Psi}$ is absolutely continuous with respect to $\mu$. It is known that \cite[p.122]{3} $C_{\Psi}(L^\infty) \subseteq L^\infty$, $\|C_{\Psi}\| \leq 1$, and $C_{\Psi}$ is an $L^\infty$ - isometry iff $\mu \ll \mu_{\Psi}$ also. Moreover, $C_{\Psi}(L^p) \subseteq L^p$, $1 \leq p < \infty$ iff $\left| \frac{d\mu_{\Psi}}{d\mu} \right| \in L^\infty$. In this paper we prove similar results for $C_{\Psi}$ on $L^p(\mu,X)$ and $L^p(\mu,X)$ for any Banach space $X$ and give necessary and sufficient conditions for $C_{\Psi}$ to be an isometry.

2. Composition Operators:

Let $(T,M,\mu)$ be a finite positive measure space, $X$ a Banach space and $\Psi : T \to T$ a measurable transformation. The induced composition operator $C_{\Psi}$ satisfies the following result.

**Proposition 2.1**

If $\mu_{\Psi} \ll \mu$, then $C_{\Psi}(f) = f \circ \Psi$ is strongly measurable for every strongly measurable function $f : T \to X$.

**Proof**

Obviously for $x \in X$ and $A \in M$ we have $C_{\Psi}(x\chi_A) = x\chi_{\Psi^{-1}(A)}$. Thus if $\{s_n\}$ is a sequence of simple functions such that $\lim_{n \to \infty} \|s_n - f\| = 0$ a.e.,
then \( \{ C_{\Psi} (s_n) \} \) is a sequence of simple functions such that 
\[
\lim_{n \to \infty} \| C_{\Psi}(s_n)(t) - (C_{\Psi}(f))(t) \| = 0 \text{ a.e. since } \mu_\Psi << \mu.
\]

We note that Proposition 2.1 is true if the condition \((\mu_\Psi << \mu)\) is replaced by "\(\mu\) is complete" [4, p.114]. In this case any strongly measurable function is measurable in the classical sense, i.e., the inverse image of every open set is measurable.

We call \( C_{\Psi} \) an isometry of \( L^p(\mu,X), 0 < p \leq \infty \) if \( \| C_{\Psi} f \|_p = \| f \|_p \) for all \( f \in L^p(\mu,X) \) and \( C_{\Psi} \) is an isometry of \( L^\Phi(\mu,X) \) if \( \| C_{\Psi} f \|_\Phi = \| f \|_\Phi \) for all \( f \in L^\Phi(\mu,X) \). The following results are extensions of those in [3, p. 122] from \( C \) to any Banach space \( X \).

**Theorem 2.2**

\( C_{\Psi}(L^\infty(\mu,X)) \subseteq L^\infty(\mu,X), \| C_{\Psi} \| \leq 1 \) and \( C_{\Psi} \) is an isometry of \( L^\infty(\mu,X) \) iff \( \mu << \mu_\Psi \).

**Proof**

Let \( f \in L^\infty(\mu,X) \). Since \( \mu_\Psi << \mu \) it is easily seen that 
\[
\| f(t) \| \leq \| f \|_\infty < \infty \text{ a.e. implies that } \| C_{\Psi} f \|_\infty \leq \| f \|_\infty.
\]
Thus \( C_{\Psi}(L^\infty(\mu,X)) \subseteq L^\infty(\mu,X) \), and \( \| C_{\Psi} \| \leq 1 \). Suppose \( C_{\Psi} \) is an isometry of \( L^\infty(\mu,X) \). For \( x \in X, x \neq 0 \) and \( A \in M \), we certainly have \( \mu(A) = 0 \) iff \( \| x \chi_A \|_\infty = 0 \). Since
\[
\| x \chi_{\Psi^{-1}(A)} \|_\infty = \| C_{\Psi}(x, \chi_A) \|_\infty = \| x \chi_A \|_\infty,
\]
it follows that \( \mu(A) = 0 \) whenever \( \mu_\Psi(A) = \mu((\Psi^{-1}(A)) = 0 \). Thus \( \mu << \mu_\Psi \).
Conversely, suppose $\mu << \mu_\Psi$. From above, it suffices to show that
\[ \|f\|_\infty \leq \|C_\Psi(f)\|_\infty \]
for all $f \in L^\infty(\mu, X)$. Let $f \in L^\infty(\mu, X)$. Then
\[ \|C_\Psi(f)(t)\| \leq \|C_\Psi(f)\|_\infty \leq \|f\|_\infty < \infty \]
for all $t \notin A$ for some $A \in M$ with $\mu(A) = 0$. Hence, $\|f(s)\| \leq \|C_\Psi(f)\|_\infty$ for all $s \in \Psi(A^c)$ where $A^c = T - A$. Since $\mu_\Psi(E) = 0$ iff $\mu(E) = 0$ for all $E \in M$ and $\Psi^{-1}((\Psi(A^c))^c) \subseteq A$ it follows that $0 = \mu(A) = \mu_\Psi((\Psi(A^c))^c)$ when $\mu$ is complete, i.e., any subset of a set of measure zero is measurable. If $\mu$ is not complete it can be replaced by its completion (see[5,p.29]). Therefore, $\|f(s)\| \leq \|C_\Psi(f)\|_\infty < \infty$ a.e.. Thus $\|f\|_\infty \leq \|C_\Psi(f)\|_\infty$ for all $f \in L^\infty(\mu, X)$ and $C_\Psi$ is an isometry of $L^\infty(\mu, X)$.

**Theorem 2.3**

Let $(T, M, \mu)$ be a finite positive measure space, $\Psi : T \rightarrow T$ a measurable transformation, $1 \leq p < \infty$, and $\mu_\Psi << \mu$. Then

a. $C_\Psi(L^p(\mu, X)) \subseteq L^p(\mu, X)$ if $\frac{d\mu_\Psi}{d\mu} \in L^\infty$

b. $C_\Psi$ is an isometry of $L^p(\mu, X)$ iff $\frac{d\mu_\Psi}{d\mu} = 1$ a.e.

**Proof**

a. Let $\frac{d\mu_\Psi}{d\mu} \in L^\infty$. Then [6,p.164] for all $f \in L^p(\mu, X)$ we have
\[
\|C_\Psi(f)\|_p^p = \int_T \|f(\Psi(t))\|^p d\mu(t) = \int_T \|f(t)\|^p \left(\frac{d\mu_\Psi}{d\mu}(t)\right) d\mu(t) \quad \text{.........................(1)}
\]
Thus (1) gives
\[ \left\| C_\psi (f) \right\|_p \leq \left| \frac{d\mu_\psi}{d\mu} \right|_\infty \left\| f \right\|_p \]
for all \( f \in L^p(\mu, X) \). Therefore, \( C_\psi \) is a bounded linear operator on \( L^p(\mu, X) \) and \( \left\| C_\psi \right\| \leq \left| \frac{d\mu_\psi}{d\mu} \right|_\infty ^\frac{1}{p} \).

b. Suppose \( C_\psi \) is an isometry of \( L^p(\mu, X), 1 \leq p < \infty \). For each \( f \in L^p \) define \( \tilde{f}(t) = f(t) \frac{x}{\|x\|} \) for all \( t \in T \), where \( x \in X \) and \( x \neq 0 \).

Then \( \tilde{f} \in L^p(\mu, X) \) and \( \left| C_\psi (f) \right|_p = \left| C_\psi (\tilde{f}) \right|_p = \left\| \tilde{f} \right\|_p = \left| f \right|_p \). Thus \( C_\psi \) is an isometry of \( L^p \) and hence [3] implies that \( \frac{d\mu_\psi}{d\mu} \in L^p \). Next, let \( f(t) = \frac{x}{\|x\|} \) for all \( t \in T \), where \( x \in X \) and \( x \neq 0 \). Then \( f \in L^p(\mu, X) \) and (2) implies that \( \left| \frac{d\mu_\psi}{d\mu} \right|_\infty \geq 1 \). Also, for this \( f \) by [6,p.164] we get

\[ \int d\mu(t) = \left\| f \right\|_p = \left\| C_\psi (f) \right\|_p = \int \left\| f(\Psi(t)) \right\|_p d\mu(t) \]

\[ = \int \left( \frac{d\mu_\psi}{d\mu} \right) (t) d\mu(t) \]

Therefore, \( \int \left( \frac{d\mu_\psi}{d\mu} \right) (t) d\mu(t) - 1 \)
implies that \( \frac{d\mu_\psi}{d\mu} = 1 \) a.e.

The converse is clear from (1).

The next results deal with \( C_\psi \) on \( L^p(\mu, X) \).
Theorem 2.4

Let \( \phi \) be a modulus function. Then

a. \( C_\psi \) is a bounded linear operator on \( L^\phi (\mu, X) \) if \( \frac{d\mu_\psi}{d\mu} \in L^\infty \).

b. \( C_\psi \) is an isometry of \( L^\phi (\mu, X) \) iff \( \frac{d\mu_\psi}{d\mu} = 1 \) a.e.

Proof

a. For \( f \in L^\phi (\mu, X) \) by [6,p.164] we have

\[
\|C_\psi (f)\|_\phi = \int _T \phi (|f(t)|) d\mu(t) = \int _T \phi (|f(t)|) \left( \frac{d\mu_\psi}{d\mu} \right) (t) d\mu(t) \quad \text{...........}(3)
\]

Thus (3) gives

\[
\|C_\psi (f)\|_\phi \leq \left\| \frac{d\mu_\psi}{d\mu} \right\|_\infty \|f\|_\phi \quad \text{...........}\quad .......\quad .........\quad .......(4)
\]

for all \( f \in L^\phi (\mu, X) \). Therefore, \( C_\psi \) is a bounded linear operator on \( L^\phi (\mu, X) \) and \( \|C_\psi \| \leq \left\| \frac{d\mu_\psi}{d\mu} \right\|_\infty \).

b. Suppose \( C_\psi \) is an isometry of \( L^\phi (\mu, X) \). For \( f \in L^1 \) let \( \tilde{f}(t) = \phi^{-1} \left( \frac{\|f(t)\|_X}{\|f\|_X} \right) \) for all \( t \in T \), where \( x \in X \) and \( x \neq 0 \). Then \( C_\psi (f) \|_\phi = \|C_\psi (\tilde{f})\|_\phi = \|\tilde{f}\|_\phi = \|f\|_1 \).

Therefore, \( C_\psi \) is an isometry of \( L^1 \) and hence by [3] \( \frac{d\mu_\psi}{d\mu} \in L^\infty \).
Moreover, for a nonzero \( x \in X \) and \( f(t) = \frac{x}{\|x\|} \) for all \( t \in T \), by (4) one has \( \left| \frac{d\mu_\psi}{d\mu} \right|_\infty \geq 1 \). Also, for such \( f \) by [6] we have

\[
\int \phi(t) d\mu(t) = \|f\|_\phi = \|C_\psi f\|_\phi = \int \phi(t) \left( \frac{d\mu_\psi}{d\mu} \right)(t) d\mu(t)
\]

This implies that \( \frac{d\mu_\psi}{d\mu} = 1 \) a.e. Finally, the converse follows from (3).

**Corollary 2.5**

\( C_\psi \) is an isometry of \( L^p(\mu, X) \), \( 1 \leq p < \infty \), iff \( C_\psi \) is an isometry of \( L^\phi(\mu, X) \).

**Proof**

Clear from theorem 2.4 and theorem 2.5.

Finally, we note that if \( \mu_\psi \ll \mu \) then \( C_\psi \) is \( 1-1 \) on \( L^p(\mu, X) \), \( 0 < p \leq \infty \), and \( L^\phi(\mu, X) \) for any modulus function \( \phi \). Moreover, if \( \Psi \) is invertible with inverse \( \Psi^{-1} \), then so is \( C_\psi \) with inverse \( C_{\psi^{-1}} \).
References


