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Composition Operators on Orlicz and Bochner Spaces

المؤثرات المركبة على فضاءات أورلكس وبوخنر

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Abstract

Let (T, M, μ) be a finite positive measure space, X a Banach space, ϕ a modulus function and $f : T \to X$ a strongly measurable function. The Orlicz space is $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$. The space of Bochner *p*-integrable functions, $1 \le p < \infty$, is $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p d\mu(t) < \infty \right\}$. Also $L^{\infty}(\mu, X) = \left\{ f : \operatorname{ess\,sup}_{t \in T} \|f(t)\| < \infty \right\}$. When $\phi(x) = x^p$, $0 < p \le 1$, $L^{\phi}(\mu, X) = L^P(\mu, X)$. Let $\Psi : T \to T$ be a function with $\Psi^{-1}(A) \in M$ for all $A \in M$ and define $C_{\Psi}(f) = f \circ \Psi$. We prove that C_{Ψ} is a bounded linear operator on $L^{\phi}(\mu, X)$ and $L^p(\mu, X)$, $0 < p \le \infty$, when $\frac{d\mu_{\Psi}}{d\mu} \in L^{\infty}(\mu, \mathbf{C})$ where $\mu_{\Psi}(A) = \mu(\Psi^{-1}(A))$ for all $A \in M$ and **C** is the complex numbers.Also, we show that C_{Ψ} is an isometry of $L^{\phi}(\mu, X)$ and $L^p(\mu, X)$, $0 < p < \infty$ iff $\frac{d\mu_{\Psi}}{d\mu} = 1$ a.e. .Moreover, C_{Ψ} is an isometry of $L^{\infty}(\mu, X)$ iff $\mu << \mu_{\Psi}$. This generalizes some previous results of the special case $L^p(\mu, C)$ and proves similar results for $L^{\phi}(\mu, X)$.

ملخص

ليكن (T, M, μ) فضاءا قياسيا موجبا ومنتهيا و X هو فضاء بناخ و ϕ اقتران مطلق القيمة و $f : T \to X$ هو اقتران قياسي بقوة. فان فضاء أورلكس هو $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$، وفضاء بوخنر

عندما $1 \le p < \infty$ هو $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p \, d\mu(t) < \infty \right\}$. أيضا $L^\infty(\mu, X) = \left\{ f : \operatorname{esssup}_{t\in T} \|f(t)\| < \infty \right\}$. وعندما $\phi(x) = x^p$، $0 < p \le 1$، فانّ $L^\phi(\mu, X) = L^P(\mu, X)$.
ليكن $\Psi : T \to T$ هو اقتران بحيث ان $\Psi^{-1}(A) \in M$ لكل $A \in M$. نعرّف $C_\Psi(f) = f \circ \Psi$.
في هذا البحث سنثبت ،ان شاء الله، أنّ C_Ψ هو مؤثرا خطيا ومحدودا على كل من $L^\phi(\mu, X)$ و $L^p(\mu, X)$، عندما $0 < p \le \infty$ حيث أنّ $\frac{d\mu_\Psi}{d\mu} \in L^\infty(\mu, \mathbf{C})$ $\mu_\Psi(A) = \mu(\Psi^{-1}(A))$ لكل $A \in M$
و **C** هي الاعداد العقدية. *ايضا* سنثبت ،ان شاء الله، أنّ C_Ψ مؤثرا تقايسيا على $L^\phi(\mu, X)$ و $L^p(\mu, X)$، $0 < p < \infty$ اذا وفقط اذا $\frac{d\mu_\Psi}{d\mu} = 1$ a.e. . اما بالنسبة ل $L^\infty(\mu, X)$ فان C_Ψ يكون تقايسيا اذا وفقط اذا $\mu << \mu_\Psi$. وهذا يعتبر تعميما لنتائج الحالة الخاصة $L^p(\mu, \mathbf{C})$ ويثبت نتائج مماثلة في $L^\phi(\mu, X)$.

1. Introduction:

If ϕ is a strictly increasing continuous subadditive function on $[0, \infty)$ and satisfies $\phi(x) = 0$ iff $x = 0$, then we call ϕ a modulus function. Let (T, M, μ) be a finite positive measure space, i.e., T is a set, M is a $\sigma-$algebra and μ is a positive measure with $\mu(T) < \infty$. If X is a Banach space, then a function $s : T \to X$ is called a simple function if its range contains finitely many points $x_1, x_2, ..., x_n$ and $E_i = s^{-1}(\{x_i\})$, $i = 1, 2, ..., n$ are measurable sets. Such a function s can be written as $s = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set E_i and $E_i \cap E_j = \Phi$, for $i \ne j$, $i, j = 1, 2, ..., n$. A function $f : T \to X$ is said to be strongly measurable if there exists a sequence $\{s_n\}$ of simple functions such that

$$\lim_{n \to \infty} \|s_n(t) - f(t)\| = 0 \text{ a.e.}$$

The Orlicz space $L^{\phi}(\mu, X)$ is the set of all (equivalence classes) of strongly measurable functions *f* with

$$\| f \|_{\phi} = \int_T \phi(\| f(t) \|) d\mu(t) < \infty .$$

If for all $f, g \in L^{\phi}(\mu, X)$ we define $d(f, g) = \| f - g \|_{\phi}$, then d is a metric on $L^{\phi}(\mu, X)$ under which it becomes a complete topological vector space [1,p.70]. For $1 \le p < \infty$, $L^p(\mu, X)$ denotes the Banach space of (equivalence classes of) strongly measurable functions *f* such that $\int_T \| f(t) \|^p \, d\mu(t) < \infty$. The norm in $L^p(\mu, X)$ is given by

$$\| f \|_p = \left(\int_T \| f(t) \|^p d\mu(t) \right)^{\frac{1}{p}}$$

The essentially bounded strongly measurable functions *f* form Banach space $L^{\infty}(\mu, X)$ with norm given by $\| f \|_{\infty} = \underset{t \in T}{ess \sup} \| f(t) \|$.

If ϕ is the modulus function $\phi(x) = x^p$, $0 < p \le 1$, then $L^{\phi}(\mu, X)$ is the space $L^p(\mu, X)$. Since [2, p. 159], for any modulus function ϕ, $\limsup_{x \to \infty} \frac{\phi(x)}{x} \le \phi(1)$, it follows that $L^1(\mu, X) \subseteq L^{\phi}(\mu, X)$.

For simplicity of notation we write $L^p(\mu, C) = L^p, 0 < p \le \infty$, $L^{\phi}(\mu, C) = L^{\phi}$.

Also, $\| \; \|_p = | \; |_p$, $\| \; \|_{\infty} = | \; |_{\infty}$, $\| \; \|_{\phi} = | \; |_{\phi}$ when X is the complex numbers ***C***.

We mean by a measurable transformation on T a function $\Psi : T \to T$ such that $\Psi^{-1}(A) \in M$ for all $A \in M$. It is easy to see that Ψ induces a positive measure μ_Ψ on M where $\mu_\Psi(A) = \mu(\Psi^{-1}(A))$ for all $A \in M$. Also, Ψ induces a composition operator C_Ψ on strongly measurable functions given by $C_\Psi(f) = f \circ \Psi$ when μ is complete or $\mu_\Psi << \mu$, i.e., μ_Ψ is absolutely continuous with respect to μ. It is known that [3,p.122] $C_\Psi(L^\infty) \subseteq L^\infty$, $\|C_\Psi\| \le 1$, and C_Ψ is an L^∞ – isometry iff $\mu << \mu_\Psi$ also. Moreover, $C_\Psi(L^p) \subseteq L^p$, $1 \le p < \infty$ iff $\left\|\frac{d\mu_\Psi}{d\mu}\right\|_\infty < \infty$. In this paper we prove similar results for C_Ψ on $L^p(\mu, X)$ and $L^\phi(\mu, X)$ for any Banach space X and give necessary and sufficient conditions for C_Ψ to be an isometry.

2. Composition Operators:

Let (T, M, μ) be a finite positive measure space, X a Banach space and $\Psi : T \to T$ a measurable transformation. The induced composition operator C_Ψ satisfies the following result.

Proposition 2.1

If $\mu_\Psi << \mu$, then $C_\Psi(f) = f \circ \Psi$ is strongly measurable for every strongly measurable function $f : T \to X$.

Proof

Obviously for $x \in X$ and $A \in M$ we have $C_\Psi(x\chi_A) = x\chi_{\Psi^{-1}(A)}$. Thus if $\{s_n\}$ is a sequence of simple functions such that $\lim_{n\to\infty} \|s_n(t) - f(t)\| = 0$ a.e., then $\{C_{\Psi}(s_n)\}$ is a sequence of simple functions such that $\lim_{n\to\infty}\|(C_{\Psi}(s_n))(t)-(C_{\Psi}(f))(t)\|=0$ a.e. since $\mu_{\Psi} << \mu$.

We note that Proposition 2.1 is true if the condition ($\mu_{\Psi} << \mu$) is replaced by "μ is complete" [4, p.114]. In this case any strongly measurable function is measurable in the classical sense, i.e., the inverse image of every open set is measurable.

We call C_{Ψ} an isometry of $L^p(\mu,X), 0<p\leq\infty$ if $\|C_{\Psi} f\|_p = \|f\|_p$ for all $f\in L^p(\mu,X)$ and C_{Ψ} is an isometry of $L^{\phi}(\mu,X)$ if $\|C_{\Psi} f\|_{\phi} = \|f\|_{\phi}$ for all $f\in L^{\phi}(\mu,X)$. The following results are extensions of those in [3, p. 122] from ***C*** to any Banach space *X*.

Theorem 2.2

$C_{\Psi}(L^{\infty}(\mu,X))\subseteq L^{\infty}(\mu,X)$, $\|C_{\Psi}\|\leq 1$ and C_{Ψ} is an isometry of $L^{\infty}(\mu,X)$ iff $\mu << \mu_{\Psi}$.

Proof

Let $f\in L^{\infty}(\mu,X)$. Since $\mu_{\Psi} << \mu$ it is easily seen that $\|f(t)\|\leq\|f\|_{\infty}<\infty$ a.e. implies that $\|C_{\Psi} f\|_{\infty}\leq\|f\|_{\infty}$. Thus $C_{\Psi}(L^{\infty}(\mu,X))\subseteq L^{\infty}(\mu,X)$, and $\|C_{\Psi}\|\leq 1$. Suppose C_{Ψ} is an isometry of $L^{\infty}(\mu,X)$. For $x\in X$, $x\neq 0$ and $A\in M$, we certainly have $\mu(A)=0$ iff $\|x\chi_A\|_{\infty}=0$. Since

$$\|x.\chi_{\Psi^{-1}(A)}\|_{\infty}=\|C_{\Psi}(x.\chi_A)\|_{\infty}=\|x\chi_A\|_{\infty}$$

it follows that $\mu(A)=0$ whenever $\mu_{\Psi}(A)=\mu((\Psi^{-1}(A))=0$. Thus $\mu << \mu_{\Psi}$.

Conversely, suppose $\mu << \mu_\Psi$. From above, it suffices to show that $\|f\|_\infty \le \|C_\Psi(f)\|_\infty$ for all $f \in L^\infty(\mu, X)$. Let $f \in L^\infty(\mu, X)$. Then

$$\|(C_\Psi(f))(t)\| \le \|C_\Psi(f)\|_\infty \le \|f\|_\infty < \infty$$

for all $t \notin A$ for some $A \in M$ with $\mu(A) = 0$. Hence, $\|f(s)\| \le \|C_\Psi(f)\|_\infty$ for all $s \in \Psi(A^c)$ where $A^C = T - A$. Since $\mu_\Psi(E) = 0$ iff $\mu(E) = 0$ for all $E \in M$ and $\Psi^{-1}((\Psi(A^c))^c) \subseteq A$ it follows that $0 = \mu(A) = \mu_\Psi((\Psi(A^c))^c)$ when μ is complete, i.e., any subset of a set of measure zero is measurable. If μ is not complete it can be replaced by its completion (see[5,p.29]). Therefore, $\|f(s)\| \le \|C_\Psi(f)\|_\infty < \infty$ a.e.. Thus $\|f\|_\infty \le \|C_\Psi(f)\|_\infty$ for all $f \in L^\infty(\mu, X)$ and C_Ψ is an isometry of $L^\infty(\mu, X)$.

Theorem 2.3

Let (T, M, μ) be a finite positive measure space, $\Psi : T \to T$ a measurable transformation,$1 \le p < \infty$, and $\mu_\Psi << \mu$. Then

a. $C_\Psi(L^p(\mu, X)) \subseteq L^p(\mu, X)$ if $\dfrac{d\mu_\Psi}{d\mu} \in L^\infty$

b. C_Ψ is an isometry of $L^p(\mu, X)$ iff $\dfrac{d\mu_\Psi}{d\mu} = 1$ a.e.

Proof

a. Let $\dfrac{d\mu_\Psi}{d\mu} \in L^\infty$. Then [6,p.164] for all $f \in L^p(\mu, X)$ we have

$$\|C_\Psi(f)\|_p^p = \int_T \|f(\Psi(t))\|^p d\mu(t) = \int_T \|f(t)\|^p \left(\frac{d\mu_\Psi}{d\mu}\right)(t) d\mu(t) \quad \text{.....................(1)}$$

Thus (1) gives

$$\|C_\Psi(f)\|_p \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty^{\frac{1}{p}} \|f\|_p \qquad \text{.......... ....(2)}$$

for all $f \in L^p(\mu, X)$. Therefore, C_Ψ is a bounded linear operator on $L^p(\mu, X)$ and $\|C_\Psi\| \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty^{\frac{1}{p}}$.

b. Suppose C_Ψ is an isometry of $L^p(\mu, X)$, $1 \le p < \infty$. For each $f \in L^p$ define $\tilde{f}(t) = f(t)\frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \ne 0$. Then $\tilde{f} \in L^p(\mu, X)$ and $|C_\Psi(f)|_p = \|C_\Psi(\tilde{f})\|_p = \|\tilde{f}\|_p = |f|_p$. Thus C_Ψ is an isometry of L^p and hence [3] implies that $\frac{d\mu_\Psi}{d\mu} \in L^\infty$. Next, let $f(t) = \frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \ne 0$. Then $f \in L^p(\mu, X)$ and (2) implies that $\left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \ge 1$. Also, for this f by [6,p.164] we get

$$\int_T d\mu(t) = \|f\|_p^p = \|C_\Psi(f)\|_p^p = \int_T \|f(\Psi(t))\|^p d\mu(t) = \int_T \left(\frac{d\mu_\Psi}{d\mu}\right)(t) d\mu(t)$$

Therefore, $\int_T \left(\left(\frac{d\mu_\Psi}{d\mu}\right)(t) - 1\right) d\mu(t) = 0$ implies that $\frac{d\mu_\Psi}{d\mu} = 1$ a.e

The converse is clear from (1).

The next results deal with C_Ψ on $L^\phi(\mu, X)$.

Theorem 2.4

Let ϕ be a modulus function. Then

a. C_Ψ is a bounded linear operator on $L^\phi(\mu, X)$ if $\frac{d\mu_\Psi}{d\mu} \in L^\infty$.

b. C_Ψ is an isometry of $L^\phi(\mu, X)$ iff $\frac{d\mu_\Psi}{d\mu} = 1$ a.e.

Proof

a. For $f \in L^\phi(\mu, X)$ by [6,p.164] we have

$$\|C_\Psi(f)\|_\phi = \int_T \phi(\|f(\Psi(t))\|)d\mu(t) = \int_T \phi(\|f(t)\|)\left(\frac{d\mu_\Psi}{d\mu}\right)(t)d\mu(t) \quad ....(3)$$

Thus (3) gives

$$\|C_\Psi(f)\|_\phi \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \|f\|_\phi \quad ....(4)$$

for all $f \in L^\phi(\mu, X)$. Therefore, C_Ψ is a bounded linear operator on $L^\phi(\mu, X)$ and $\|C_\Psi\| \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty$.

b. Suppose C_Ψ is an isometry of $L^\phi(\mu, X)$. For $f \in L^1$ let $\widetilde{f}(t) = \phi^{-1}(|f(t)|)\frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \neq 0$. Then $|C_\Psi(f)|_1 = \|C_\Psi(\widetilde{f})\|_\phi = \|\widetilde{f}\|_\phi = |f|_1$.

Therefore, C_Ψ is an isometry of L^1 and hence by [3] $\frac{d\mu_\Psi}{d\mu} \in L^\infty$.

Moreover, for a nonzero $x \in X$ and $f(t) = \frac{x}{\|x\|}$ for all $t \in T$, by (4) one has $\left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \geq 1$. Also, for such *f* by [6] we have

$$\int_T \phi(1) d\mu(t) = \| f \|_\phi = \| C_\Psi f \|_\phi = \int_T \phi(1) \left(\frac{d\mu_\Psi}{d\mu} \right)(t) d\mu(t)$$

This implies that $\frac{d\mu_\Psi}{d\mu} = 1$ a.e. Finally, the converse follows from (3).

Corollary 2.5

C_Ψ is an isometry of $L^p(\mu, X)$, $1 \leq p < \infty$, iff C_Ψ is an isometry of $L^\phi(\mu, X)$.

Proof

Clear from theorem 2.4 and theorem 2.5.

Finally, we note that if $\mu_\Psi << \mu$ then C_Ψ is $1-1$ on $L^p(\mu, X)$, $0 < p \leq \infty$, and $L^\phi(\mu, X)$ for any modulus function ϕ. Moreover, if Ψ is invertible with inverse Ψ^{-1}, then so is C_Ψ with inverse $C_{\Psi^{-1}}$.

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