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The Extended Center of Groups

المركز الموسع للزمر

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Abstract

In this paper we define and study a new concept called the extended center of a group. Some examples are given and some results are obtained to explain some of the properties of the extended center of a group. Finally, homomorphism and extended center of groups are studied.

ملخص

في هذا البحث تم تعريف و دراسة مفهوم جديد هو المركز الموسع للزمر، حيث تم عرض بعض الأمثلة و تم الحصول على بعض النتائج التي تشرح خواص المركز الموسع للزمر. في النهاية تم دراسة تأثير التشاكل على المركز الموسع.

1. Introduction

In this paper we define and study the extended center of a group. We introduce some examples and obtain new results concerning the properties of the extended center. We also show that the extended center of the domain is a topological property.

Finally we give some examples to show that this is not the case if we remove one of the following conditions of the mapping φ: one to one, onto, and homomorphism.

Let G be any group, then the center of G will be denoted by $Z(G) = \{g \in G : gx = xg, \text{ for all } x \in G\}$. If S is a nonempty subset of G, the centralizer of S in G is denoted by $C(S) = \{g \in G: gx = xg, \text{ for all } x \in S\}$, but the centralizer of $\{g\}$ is denoted by C_g, where g is a fixed element in G. A group G is a torsion group if every element in G is of finite order. G is torsion free if no element other than the identity is of finite order (Fraleigh, J. B. 1976, pp.78).

Let $G_1, G_2, \ldots$ and, G_n be a family of groups. Then the Cartesian product $G_1 \times G_2 \times \ldots \times G_n$ is a group (called the direct product of groups) under the pointwise binary operation $(g_1, g_2, \ldots, g_n)(h_1, h_2, \ldots, h_n) = (g_1 h_1, g_2 h_2, \ldots, g_n h_n)$, where $g_i, h_i \in G_i$, $i = 1, 2, \ldots n$. (Bhattacharya, P. & et. al. 1986. pp.67).

2. Some Properties of Extended Center

In this section we define a new concept named the extended center of a group, then we get some results and examples concerning this notion.

2.1 Definition

The extended center of a group G is defined by:

$Z_e(G) = \{ g \in G : gx = xg$, for all $x \in G$ except for a finite number$\}$.

2.2 Theorem

Let G be a group. Then,

i. *$Z_e(G)$ is a subgroup of G and $Z(G) \subseteq Z_e(G)$.*

ii. *If G is finite or abelian, then $Z_e(G) = G$.*

iii. *$g \in Z_e(G)$ if and only if $G \setminus C_g$ is finite. That is, $Z_e(G) = \{ g \in G : G \setminus C_g$ is finite $\}$.*

Proof

i. Since $e \in Z_e(G)$, then $Z_e(G) \neq \varphi$. Now, let $a, b \in Z_e(G)$. Then there are elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ in G such that $ga = ag$ for all $g \in G \setminus \{a_1, a_2, \ldots, a_n\}$ and $gb = bg$ for all $g \in G \setminus \{b_1, b_2, \ldots, b_m\}$. Hence $(ab^{-1})g = g(ab^{-1})$ for all $g \in G \setminus \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$. Hence $ab^{-1} \in Z_e(G)$. Therefore, $Z_e(G)$ is a subgroup of G. $Z(G) \subseteq Z_e(G)$ is clear from the definitions of the center and the extended center **.**

ii. If G is finite or abelian, then for any fixed $x \in G$, $xg = gx$ for all $g \in G$ except at most for a finite number of elements. Hence $x \in Z_e(G)$. Thus $G \subseteq Z_e(G)$ and hence $G = Z_e(G)$

iii. Let $g \in Z_e(G)$. Then there are elements $g_1, g_2, \ldots, g_n$ in G such that $gx = xg$ for all $x \in G \setminus \{g_1, g_2, \ldots, g_n\}$. Hence $G \setminus C_g$ is at most $\{g_1, g_2, \ldots, g_n\}$. Therefore, $G \setminus C_g$ is finite.

Conversely, suppose that $G \setminus C_g$ is finite, say $G \setminus C_g = \{g_1, g_2, \ldots, g_n\}$ and so $G \setminus \{g_1, g_2, \ldots, g_n\} = C_g$. Hence $gx = xg$ for all $x \in G \setminus \{g_1, g_2, \ldots, g_n\}$. This means that $g \in Z_e(G)$

Remark

From Theorem 2.2 (ii) we can see that, If G is finite and non abelian, then $Z_e(G) \neq Z(G)$. For example, consider the dihedral group D_3 which is finite (of order 6) and non abelian. Moreover, $Z(D_3) = \{R_0\}$, where R_0 is the identity element of D_3. See (Gallian, J. A. 2002. pp. 65). But from the definition of the extended center, it is clear that $Z_e(D_3) = D_3$. Therefore, $Z_e(D_3) \neq Z(D_3)$.

2.3 Corollary

Let G and H be two finite groups. If $\varphi : G \to H$ is an onto group homomorphism, then $\varphi(Z_e(G)) = Z_e(H)$.

It is trivial to note that if $G = Z(G)$, then G is abelian and $Z(G) = Z_e(G) = G$, however in the next example we show that the equality above may happen even in the case of non abelian group, moreover these examples show that $Z_e(G)$ is a proper subset of G. Surely the group in the next example must be infinite (see the above remark).

2.4 Example

There is an infinite non abelian group G with $Z_e(G) = Z(G)$.

Construction

Consider the group $G = \{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} : xz \neq 0$ and x, y and z are real numbers $\}$ under matrix multiplication. One can easly see that $Z(G) = \{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \neq 0 \}$.

Let $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ be any element in G with $x \neq z$ or $y \neq 0$. If $x \neq z$, then we can find infinite number of matrices $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \in G$ with $c \neq 0$ such that, $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \neq \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$. Hence $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \notin Z_e(G)$ whenever $x \neq z$. If $y \neq 0$, then we can find infinite number of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a \neq b$ such that, $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \neq \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$. Hence $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \notin Z_e(G)$ whenever $y \neq 0$. Therefore, $Z_e(G) = \{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \neq 0 \}$; that is $Z_e(G) = Z(G)$

2.5 Theorem

Let G be a group. Then,

i. *$Z_e(Z(G)) \subseteq Z(Z_e(G))$.*

ii. *If G is finite, then $Z_e(Z(G)) = Z(Z_e(G))$.*

Proof

i. If $x \in Z_e(Z(G))$, then $x \in Z(G)$, so $x \in Z_e(G)$ and $xg = gx$ for all g in G. Hence $x \in Z(Z_e(G))$. Therefore, $Z_e(Z(G)) \subseteq Z(Z_e(G))$. Or, since $Z_e(Z(G)) = Z(G)$ and $Z(G) \subseteq Z(Z_e(G))$, then $Z_e(Z(G)) \subseteq Z(Z_e(G))$.

ii. By Theorem 2.2 (ii), $Z(Z_e(G)) = Z(G) = Z_e(Z(G))$ □

2.6 Theorem

Let H be the direct product of the groups G_1, G_2, …. and, G_n. Then $Z_e(H) = Z_e(G_1) \times Z_e(G_2) \times \ldots \times Z_e(G_n)$.

Proof

Let $(g_1, g_2, \ldots, g_n) \in Z_e(H)$. Then there is a finite set of elements [including the case of 0 elements, i. e. no element] $(g_{11}, g_{21}, \ldots, g_{n1})$, $(g_{12}, g_{22}, \ldots, g_{n2})$, ……, $(g_{1m}, g_{2m}, \ldots, g_{nm})$ in H such that $(g_1, g_2, \ldots, g_n)$ commutes with all elements in $H \backslash \{(g_{11}, g_{21}, \ldots, g_{n1}), (g_{12}, g_{22}, \ldots, g_{n2}), \ldots, (g_{1m}, g_{2m}, \ldots, g_{nm})\}$. Hence for any fixed i = 1, 2, ….. n, g_i commutes with all elements in $G_i \backslash \{g_{i1}, g_{i2}, \ldots, g_{im}\}$. Thus $g_i \in Z_e(G_i)$. This means that $(g_1, g_2, \ldots, g_n) \in Z_e(G_1) \times Z_e(G_2) \times \ldots \times$

$Z_e(G_n)$. Conversly, let $(g_1, g_2, \ldots, g_n) \in Z_e(G_1) \times Z_e(G_2) \times \ldots \times Z_e(G_n)$. Then $g_i \in Z_e(G_i)$ for all i = 1, 2, …..n. Then for each i there is a finite number of elements $g_{i1}, g_{i2}, \ldots, g_{im}$ in G_i such that g_i commutes with all elements in $G_i \setminus \{g_{i1}, g_{i2}, \ldots, g_{im}\}$. Hence $(g_1, g_2, \ldots, g_n)$ commutes with all elements in $H\setminus\{(g_{11}, g_{21}, \ldots, g_{n1}), (g_{12}, g_{22}, \ldots, g_{n2}), \ldots, (g_{1m}, g_{2m}, \ldots, g_{nm})\}$ and hence $(g_1, g_2, \ldots, g_n) \in Z_e(H)$. This complete the proof of the theorem.

2.7 Theorem

Let H be a cyclic subgroup of a group G. If G is torsion free or H is infinite, then $H \subseteq C(Z_e(G))$.

Proof

It is clear that the theorem is trivial in the case that H = {e}. Let H ≠ {e} be a cyclic subgroup of a torsion free group G. Then there exists $a \in G$ such that $H = \langle a \rangle$. Since G is torsion free, then o(a) is infinite. For any fixed $z \in Z_e(G)$, z commutes with all elements in H except for a finite number of elements of H since $H \subseteq G$. Hence there exists a natural number k such that $a^m z = za^m$ for all integers m with the property $|m| \geq k$. Now, using the above equality we have, $az = a^{k+1} a^{-k} z = a^{k+1} z a^{-k} = za^{k+1} a^{-k} = za$, (such k+1 exists since o(a) is infinite). It follows that $a^m z = za^m$ for all integers m. Since z was arbitrary in $Z_e(G)$, then $a^m z = za^m$ for all integers m and all $z \in Z_e(G)$; i. e., $H \subseteq C(Z_e(G))$. The case, where H is infinite cyclic subgroup of a group G is similar.

3. Homomorphism and Extended Center

In this section we obtain some results that explain the effect of the isomorphism on the extended center of groups. We also give some counter examples to show that Theorem 3.1 need not be true if some of the conditions in the hypothesis of the theorem is omitted.

3.1 Theorem

Let G and H be two groups. If $\varphi: G \to H$ is an isomorphism from G onto H, then $\varphi(Z_e(G)) = Z_e(H)$.

Proof

Let $\varphi(g) \in \varphi(Z_e(G))$. Then there is a finite number of elements g_1, g_2, …. and, g_n in G such that g commutes with all elements in $G \setminus \{g_1, g_2, \ldots, g_n\}$. Let $\varphi(f) \in H \setminus \{\varphi(g_1), \varphi(g_2),\ldots, \varphi(g_n)\}$, then $\varphi(g)\,\varphi(f) = \varphi(gf) = \varphi(fg) = \varphi(f)\,\varphi(g)$. Hence $\varphi(g) \in Z_e(H) = Z_e(\varphi(G))$ (as φ is onto). Thus $\varphi(Z_e(G)) \subseteq Z_e(H) = Z_e(\varphi(G))$.

Conversely, let $h \in Z_e(H)$. As above, say h commutes with all elements in H except $h_1, h_2, \ldots$ and, h_n. Let $\varphi(g) = h$ and $\varphi(g_i) = h_i$ for $i = 1, 2, \ldots, n$. For any $f \in G \setminus \{g_1, g_2, \ldots, g_n\}$, $k = \varphi(f) \notin \{h_1, h_2, \ldots, h_n\}$ and $gf = \varphi^{-1}(h)\,\varphi^{-1}(k) = \varphi^{-1}(hk) = \varphi^{-1}(kh) = \varphi^{-1}(k)\,\varphi^{-1}(h) = fg$. Hence $g \in Z_e(G)$ and so $h = \varphi(g) \in \varphi(Z_e(G))$. Thus $Z_e(H) \subseteq \varphi(Z_e(G))$.

3.2 Corollary

Let G and H be two groups. If $\varphi : G \to H$ is a monomorphism of G into H, then $\varphi(Z_e(G)) = Z_e(\varphi(G))$.

The following example shows that Theorem 3.1 need not be true if φ is either not one to one or not onto mapping. Also this will be the case if φ is not a homomorphism..

3.3 Examples

i. *There is an epimorphism from a group G onto a group H, but φ ($Z_e(G)$) $\neq Z_e(H)$.*

ii. *There is a monomorphism from a group G into a group H, but φ ($Z_e(G)$) $\neq Z_e(H)$.*

iii. *There is a bijection from a group G into a group H, but φ ($Z_e(G)$) $\neq Z_e(H)$.*

Construction

i. Consider the group G = $\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$: xz ≠ 0 and x, y and z are real numbers $\}$ in Example 2.4 above, and the group H = $\{\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$: x and z are non zero real numbers} under matrix multiplication. Define φ: G → H by $\varphi(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}) = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$. One can show that φ is an epimorphism, but not one to one. By Example 2.4, φ ($Z_e(G)$) = φ ($Z(G)$) = $Z(G) \neq H = Z_e(H)$, because H is abelian.

ii. Let G = {e, c, a^2, f} be a normal subgroup of the octant group H = {e, a, a^2, a^3, b, c, d, f} (Gilbert, J. & Gilbert, L. 1984. pp120), Define φ: G → H by φ (x) = x, then φ is a monomorphism. By Theorem 1.1 (ii), $\varphi(Z_e(G)) = G \neq H = Z_e(H)$.

iii. Consider the group G = $\{\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$: xz ≠ 0 and x, y and z are real numbers$\}$ in Example 2.4 above and the group H = {(x, y, z): xz ≠ 0 and x, y and z are real numbers} under the binary operation (a, b, c) * (x, y, z) = (ax, b + y, cz). Define the mapping φ: G → H by $\varphi(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix})$ = (x, y, z). One can show that φ is bijective but not homomorphism. Use Example 2.4 and the facts that G is not abelian but H is abelian to get, $\varphi(Z_e(G)) = \varphi(Z(G)) \neq \varphi(G) = H = Z_e(H)$.

References

- Bhattacharya, P. Jain, S. & Nagpaul, S. (1986). "Basic Abstract Algebra". Campridge University. 67
- Einsiedler, M. & Rindler, H. (2001). "Algebraic Actions of the Discrete Heisenberg Group & other Non-abelian Groups". Aequationes Math. (62). 117-135.
- Fraleigh, J.B. (1976). "A first Course in Abstract Algebra". Addison & Wesley. 78
- Gallian, J. A. (2002). "Contemporary Abstract Algebra". Houghton Mifflin Company. New York. 65.
- Gilbert, J. & Gilbert, L. (1984). "Elements of Modern Algebra". Prindle, Weber & Schmidt. 120.