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The Hamilton-Jacobi Treatment of Complex Fields as Constrained Systems

معالجة نظم المجال المركبة كأنظمة مقيدة باستخدام طريقة هاميلتيون جاكوبي

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Abstract

The complex scalar field is treated as a constrained system using the Hamilton-Jacobi approach. The reduced phase space Hamiltonian density is obtained without introducing Lagrange multipliers and without any additional gauge fixing condition. The quantization of this system is also discussed.

Keywords: Hamiltonian and Lagrangian approach, Hamilton-Jacobi approach, path integral quantization, reparametrization invariant theories, quantization of complex field systems.

ملخص

تتم معالجة نظم المجال المركبة كنظم مقيدة باستخدام طريقة هاميلتيون جاكوبي. يتم الحصول على الطور الفراغي المصغر بدون تضمين مضاعفات لاجرانج وبدون تثبيت أية شروط خارجية. وأيضا تتم دراسة تكميم هذه الأنظمة باستخدام طريقة التكميم المساري.

1. Introduction

There are some famous theories of parameterization which can be described as invariant. Einstein's theory of gravitation, relativistic point particle and relativistic string theories are good examples. The first systematic study of mechanical systems including field theories with constraints was done by Dirac (Dirac, 1964), (Dirac, 1950, p.129). He showed that the algebra of poisson bracket determines a division of constraints into two classes: first- class constraints that have vanishing poisson's brackets with all other constraints and second-class constraints that have non-vanishing poisson's brackets. The presence of constraints in such theories makes one be careful when applying Dirac's method, especially when first-class constraints arise since the first class constraints are generators of gauge transformations which lead to the gauge freedom. In other words the equations of motion are still degenerate and depend on the functional arbitrariness. Recently the Hamilton-Jacobi (Guler, 1992, p.1389), (Guler, 1992, p.1143), (Rabi, & Guler,. 1992, p.3513), (Muslih, 1998, p.277) method has been developed to investigate constrained systems. In this method, the distinguish between the first and second class constraints is not necessary. The equations of motion are written as a total differential equation in many variables, which require the investigation of the integrability conditions. In other words the integrability conditions may lead to new constraints Moreover, it is shown that the gauge fixing, which is an essential procedure to study singular systems by Dirac's method, is not necessary if the Hamilton-Jacobi method is used. The aim of this paper is to analyze field systems as singular systems by using the Hamilton-Jacobi method and by considering that the reparametrization invariant theories have vanishing Hamiltonians. Besides, we discuss the quantization of this system using the canonical path integral quantization (Muslih, 2000, p7), (Muslih, 2000, p. 203), (Muslih, 2000, p. 2495), (Muslih, 2002, p.1), (Muslih, 2002, p. 919), (Muslih, & *et.al.* 2004, p. 119).

The plan of this paper is the following: A brief information of the Hamilton- Jacobi method is given in section 2. In section 3 the parameterization invariant field theory is treated as a constrained system using the Hamilton-Jacobi method. In section 4 the Hamilton-Jacobi analysis for the time "t" as an evolution parameter is given. In section 5 we obtain the path integral quantization by using the canonical path integral formulation. In section 6 conclusions are presented.

2. The Hamilton-Jacobi method

In this section we will briefly review the Hamilton-Jacobi method (Guler, 1992, p. 1389), (Guler, 1992, p. 1143), (Rabi, & Guler, 1992, 3513), (Muslih, 1998, p. 277) for studying the constrained systems. Consider a system with n degrees of freedom. It may have r primary constraints, in this case the canonical formulation gives the set of Hamilton-Jacobi partial differential equation (HJPDE) (Guler, 1992, p. 1389), (Guler, 1992, p. 1143), (Muslih, & Guler, 1995, p. 307).

$$H'_\alpha\left(x_\beta,\phi_\alpha,\frac{\partial S}{\partial \phi_\alpha},\frac{\partial S}{\partial X_\beta}\right)=0 \quad \alpha,\beta=0,n-r+1,...,n. \quad a = 1,... n - r, \quad (1)$$

where

$$H'_\alpha = H_\alpha(\chi_\beta,\phi_\alpha,\pi_a)+\pi_\alpha, \quad (2)$$

and H_0 is the canonical Hamiltonian. The equations of motion is obtained as a total differential equation in many variable as follows: (Guler, 1992, p. 1389), (Guler, 1992, p. 1143)

$$d\phi_a = \frac{\delta H'_\alpha}{\delta \pi_a} d\chi_\alpha, \; d\pi_a = -\frac{\delta H'_\alpha}{\delta \phi_a} d\chi_\alpha, \; d\pi_\mu = -\frac{\delta H'_\alpha}{\delta \chi_\mu} d\chi_\mu. \; \mu = 1,...,r, \quad (3)$$

$$dz = (-H_\alpha + \pi_a \frac{\delta H'_\alpha}{\delta \pi_a})d\chi_\alpha, \quad (4)$$

where $z = S(\chi_\alpha,\phi_\alpha)$ and $\frac{\delta H}{\delta x}$ represent the variations of H with respect to x.

The set of equations (3),(4) is integrable if and only if (Guler, 1992, p. 1143) , (Muslih, & Guler, 1995, p. 307)

$$dH'_0 = 0, \quad (5)$$

$$dH'_{\mu} = 0, \qquad \mu = 1,...,r. \qquad (6)$$

If conditions (5) and (6) are not satisfied identically, one considers them as a new constraint and again tests the consistency conditions. Thus, repeating this procedure, one may obtain a set of conditions. Hence, the canonical formulation leads us to obtain the set of canonical phase- space coordinates φ_a and π_a as a function of χ_α, besides the canonical action integral is obtained in term of the canonical coordinates. The Hamiltonian H'_α can be interpreted as the infinitesimal generators of the canonical transformation given by parameters χ_α. In this case, the path integral representation may be written as (Muslih, 2000, p7), (Muslih, 2000, p. 203), (Muslih, 2000, p. 2495), (Muslih, 2002, p.1), (Muslih, 2002, p. 919), (Muslih, & *et.al.* 2004, p. 119)

$$D(\phi'_a, \chi'_\alpha, \phi_a, \varphi_\alpha) = \int_{\phi_\alpha}^{\phi'_\alpha} (D\phi^a)(D\pi^a) \exp[i\{\int_{\chi_\alpha}^{\chi'_\alpha} (-H_\alpha + \pi_a \frac{\delta H'_\alpha}{\delta \pi_a}) d^3x d\chi_\alpha\}]. \quad (7)$$

One should notice that the integral (7) is an integration over the canonical phase space coordinates ϕ_a and π_a

3. A treatment of complex scalar field as constrained system

Let us consider a complex scalar field described by the Lagrangian density

$$\pounds = \partial_\mu \phi^\dagger \partial_\mu \phi - \mu_0^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2. \qquad (8)$$

Here ϕ and $\phi^\dagger$ are functions of x_μ, $\mu = 0,1,2,3$ and t is a function of independent parameter τ. The action integral for this system may be written as

$$S[\phi, \phi^\dagger] = \int dx dt [\frac{\partial \phi}{\partial t} \frac{\partial \phi^\dagger}{\partial t} - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi^\dagger}{\partial x_i} - \mu_0^2 \phi \phi^\dagger - \lambda(\phi\phi^\dagger)^2]. \qquad (9)$$

Since $\mathcal{L}$ is a regular Lagrangian density, parameterize the time $t \longrightarrow \tau$ with $\dot{\tau} = \frac{\partial \tau}{\partial t} > 0$, then the action integral (9) may be written as

$$S[\phi, \phi^{\dagger}] = \int dx d\tau \mathcal{L}_{\tau}, \tag{10}$$

where the singular Lagrangian density $\mathcal{L}_{\tau}$ is given by

$$\mathcal{L}_{\tau} = \dot{\tau} \frac{\partial \phi}{\partial \tau} \frac{\partial \phi^{\dagger}}{\partial \tau} - \frac{\nabla \phi \nabla \phi^{\dagger}}{\dot{\tau}} - \frac{\mu_0^2 \phi \phi^{\dagger}}{\dot{\tau}} - \frac{\lambda (\phi \phi^{\dagger})^2}{\dot{\tau}}. \tag{11}$$

The generalized momenta conjugated to the generalized coordinate are defined as

$$\pi = \frac{\partial \mathcal{L}_{\tau}}{\partial (\frac{\partial \phi}{\partial \tau})} = \dot{\tau} \frac{\partial \phi^{\dagger}}{\partial \tau}, \tag{12}$$

$$\pi^{\dagger} = \frac{\partial \mathcal{L}_{\tau}}{\partial (\frac{\partial \phi^{\dagger}}{\partial \tau})} = \dot{\tau} \frac{\partial \phi}{\partial \tau}, \tag{13}$$

$$\pi_t = \frac{\partial \mathcal{L}_{\tau}}{\partial \dot{t}} = (\dot{\tau})^2 \frac{\partial \phi}{\partial \tau} \frac{\partial \phi^{\dagger}}{\partial \tau} - \nabla \phi \nabla \phi^{\dagger} - \mu_0^2 \phi \phi^{\dagger} - \lambda (\phi \phi^{\dagger})^2. \tag{14}$$

Using (12) and (13), the equation (14) can be rewritten as

$$\pi_t = -[\pi \pi^{\dagger} + \nabla \phi \nabla \phi^{\dagger} + \mu_0^2 \phi \phi^{\dagger} + \lambda (\phi \phi^{\dagger})^2] = -\mathcal{H}_t. \tag{15}$$

Hence, the primary constraint is

$$\mathcal{H}_t^{'} = \pi_t + \mathcal{H}_t = 0, \tag{16}$$

where

$$\mathcal{H}_t = \pi \pi^{\dagger} + \nabla \phi \nabla \phi^{\dagger} + \mu_0^2 \phi \phi^{\dagger} + \lambda (\phi \phi^{\dagger})^2. \tag{17}$$

Besides the canonical Hamiltonian H_τ is defined as

$$H_\tau = \int d^3x[-\mathcal{L}_\tau + \pi\frac{\partial\phi}{\partial\tau} + \pi^\dagger\frac{\partial\phi^\dagger}{\partial\tau} + \pi_t\frac{\partial t}{\partial\tau}] = 0, \tag{18}$$

calculations show that H_τ is identically zero.

Now the set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$\mathcal{H}_\tau^{'} = \pi^0 + \mathcal{H}_\tau = 0, \qquad \pi^0 = \frac{\partial S}{\partial\tau} = 0, \tag{19}$$

$$\mathcal{H}_t^{'} = \pi_t + \mathcal{H}_t = 0, \qquad \pi_t = \frac{\partial S}{\partial t} = 0. \tag{20}$$

The equations of motion are obtained as a total differential equation in many variables as follow

$$d\phi = \frac{\delta\mathcal{H}_\tau^{'}}{\delta\pi}d\tau + \frac{\delta\mathcal{H}_t^{'}}{\delta\pi}dt = \pi^\dagger dt, \tag{21}$$

$$d\phi^\dagger = \frac{\delta\mathcal{H}_\tau^{'}}{\delta\pi^\dagger}d\tau + \frac{\delta\mathcal{H}_t^{'}}{\delta\pi^\dagger}dt = \pi dt, \tag{22}$$

$$d\pi = -\frac{\delta\mathcal{H}_\tau^{'}}{\delta\phi}d\tau - \frac{\delta\mathcal{H}_t^{'}}{\delta\phi}dt = -[\mu_0^2\phi^\dagger + 2\lambda\phi\phi^\dagger + \nabla\phi^\dagger\nabla^2\phi]dt, \tag{23}$$

$$d\pi^\dagger = -\frac{\delta\mathcal{H}_\tau^{'}}{\delta\phi^\dagger}d\tau - \frac{\delta\mathcal{H}_t^{'}}{\delta\phi^\dagger}dt = -[\mu_0^2\phi + 2\lambda\phi\phi^\dagger + \nabla\phi\nabla^2\phi^\dagger]dt, \tag{24}$$

$$d\pi_t = -\frac{\delta\mathcal{H}_\tau^{'}}{\delta t}d\tau - \frac{\delta\mathcal{H}_t^{'}}{\delta t}dt = 0. \tag{25}$$

To check whether the set of equations (21-25) is integrable or not, we have to consider the total variation of the constraints. Since

$$d\mathcal{H}_t^{'} = d\pi_t + d\mathcal{H}_t = 0, \tag{26}$$

vanishes identically, the set of equations is integrable and the canonical phase-space coordinates $\phi, \phi^{\dagger}, \pi, \pi^{\dagger}$ are obtained in terms of t.

4. Hamilton-Jacobi Analysis for the time "t" as an evolution parameter

In this section, we shall investigate the model (8), by treating the time "t" as an evolution parameter. The Lagrangian for this model is given by

$$\mathcal{L} = \frac{\partial \phi}{\partial t}\frac{\partial \phi^{\dagger}}{\partial t} - \frac{\partial \phi}{\partial x_i}\frac{\partial \phi^{\dagger}}{\partial x_i} - \mu_0^2 \phi\phi^{\dagger} - \lambda(\phi\phi^{\dagger})^2. \tag{27}$$

The generalized canonical momenta are calculated as follows:

$$\pi = \frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial t})} = \frac{\partial \phi^{\dagger}}{\partial t}, \tag{28}$$

$$\pi^{\dagger} = \frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi^{\dagger}}{\partial t})} = \frac{\partial \phi}{\partial t}. \tag{29}$$

The canonical Hamiltonian is calculated as

$$\mathcal{H}_t = \pi\pi^{\dagger} + \nabla\phi\nabla\phi^{\dagger} + \mu_0^2\phi\phi^{\dagger} + \lambda(\phi\phi^{\dagger})^2. \tag{30}$$

The Hamilton-Jacobi partial differential equation is

$$\mathcal{H}_t^{'} = \pi_t + \mathcal{H}_t = 0, \qquad \pi_t = \frac{\partial S}{\partial t} = 0. \tag{31}$$

Besides, the equation of motion is the same as the once obtained in equations (21-25). The equivalence between (31) and (20), shows that by using the Hamilton-Jacobi method we obtain the canonical Hamiltonian $\mathcal{H}_t$ for the complex scalar field in a gauge independent manner.

5 Path integral quantization method

To obtain the path integral quantization for this system, we can use the canonical path integral formulation. Making use of (4), the canonical action integral is calculated as

$$S[\phi,\phi^{\dagger}] = \int[-\mathcal{H}_t + \pi\frac{\delta\mathcal{H}_t^{'}}{\delta\pi} + \pi^{\dagger}\frac{\delta\mathcal{H}_t^{'}}{\delta\pi^{\dagger}}]d^3xdt. \quad (32)$$

Then the path integral for the complex scalar field system is given as

$$\langle Out|S|In\rangle = \int\prod d\phi d\phi^{\dagger} d\pi d\pi^{\dagger} \exp[i\{\int[-\mathcal{H}_t + \pi\frac{\delta\mathcal{H}_t^{'}}{\delta\pi} + \pi^{\dagger}\frac{\delta\mathcal{H}_t^{'}}{\delta\pi^{\dagger}}]d^3xdt\}]. \quad (33)$$

One should notice that the integral (33) is an integration over the canonical phase space coordinates $\phi,\phi^{\dagger},\pi,\pi^{\dagger}$.

6 Conclusion

Reparametrization invariant theories have a vanishing canonical Hamiltonian (Gitman & Tyutin, 1990), (Gavrilo, & Gitman, 1993, p. 57) and enforce there dynamics through constraints. To obtain the correct physical Hamiltonian and the correct equations of motion by using Dirac's method (Dirac, 1950, p. 129) one has to impose gauge fixing of the form $t - f(t) = 0$. Such gauge fixing is not always an easy task. While, in the Hamilton-Jacobi method treatment of this system, there is no need to introduce Lagrange multipliers to the Hamiltonian H_{τ} as well as no need to use any gauge fixing condition. On the other hand, the path integral for the system is obtained as an integration over the canonical phase space coordinates $\phi,\phi^{\dagger},\pi,\pi^{\dagger}$ without using any gauge fixing conditions.

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