An - Najah Univ. J. Res. (N. Sc.) Vol. 27, 2013

On Interpolation in Hardy- Orlicz Spaces

حول الاستكمال في فضاءات هاردي -اورلكس

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Received: (27/9/2010), Accepted: (11/11/2012)

Abstract

The Hardy-Orlicz space H_ϕ is the space of all analytic functions f on the open unit disk D such that the subharmonic function $\phi(|f|)$ has a harmonic majorant on D, where ϕ is a modulus function. H_ϕ^+ is the subspace of H_ϕ consisting of all $f \in H_\phi$ such that $\phi(|f|)$ has a quasi-bounded harmonic majorant on D. If $\phi(x) = x^p$, $0 < p \le 1$, then H_ϕ is the Hardy space H^p and if $\phi(x) = \log(1+x)$, then H_ϕ is the Nevanlinna class N and H_ϕ^+ is the Smirnov class N^+. In this paper we generalize some of N. Yanagihara's and A. Hartmann's and others interpolation results from N and N^+ to H_ϕ and H_ϕ^+. For that purpose we generalize a canonical factorization theorem to functions in H_ϕ or H_ϕ^+ and introduce an F-space of complex sequences.

AMS subject Classification: Primary: 46Axx.Secondary: 46E10, 30H05.

Key words: Hardy-Orlicz space, F-space, modulus function, canonical factorization, interpolation.

ملخص

فضاء هاردي-أورلكز H_ϕ هو فضاء جميع الدوال التحليلة f على قرص الوحدة المفتوح D بحيث أن الدالة $\phi(|f|)$ المتوافقة جزئيا على D يكون لها داله توافقيه تحدها من أعلى، علما بأن ϕ هي داله مطلقه القيمه. H_ϕ^+ هو الفضاء الجزئي من H_ϕ والمحتوي على جميع الدوال $f \in H_\phi$ بحيث أن $\phi(|f|)$ يكون لها داله توافقيه شبه محدوده وتحدها من أعلى. إذا كان $\phi(x) = x^p$, $0 < p \leq 1$ فإن H_ϕ هو فضاءهاردي H^p ؛ وإذا كان $\phi(x) = \log(1+x)$ فإن H_ϕ هو فئة نفانلنا N بينما H_ϕ^+ هو فئة سميرنوف N^+. في هذا البحث نعمم بعض نتائج ياناجيهارا وهارتمان وآخرين في الاستكمال الدالي من N و N^+ إلى H_ϕ و H_ϕ^+. من أجل الوصول لهذا الهدف سنعمم نظريه معروفه في التحليل إلى العوامل إلى الدوال في H_ϕ و H_ϕ^+ وسنقدم فضاء- F مكون من متتاليات عقديه.

Introduction

If ϕ is a real-valued function on $[0,\infty)$ such that ϕ is increasing, subadditive, $\phi(x)=0$ *iff* $x=0$, and continuous at zero from the right (hence uniformly continuous on $[0,\infty)$), then ϕ is called a *modulus function*. Examples of modulus functions are $x^p, 0 < p \leq 1$, and $\log(1+x)$. We note that the composition of two modulus functions is a modulus function and if ϕ is a modulus function, then $\phi(|\alpha x|) \leq ([|\alpha|]+1)\phi(|x|)$ for all x in the real numbers **R** and for all α in the complex numbers **C**; where $[x]$ is the greatest integer in x.

Let D be the open unit disk in the complex plane **C** and H be the space of all analytic functions in D. Throughout this paper we assume that ϕ is a strictly increasing unbounded modulus function such that $\phi(|f|)$ is subharmonic on D for all $f \in H$. The *Hardy-Orlicz space* H_ϕ is the space of all $f \in H$ such that $\phi(|f|)$ has a *harmonic majorant* on D, i.e., there is a function u harmonic on D such that $\phi(|f(z)|) \leq u(z)$ for all $z \in D$. It follows that [8] for each $f \in H_\phi$,

$\phi(|f|)$ has a *least harmonic majorant* u_f, i.e., $\phi(|f(z)|) \le u_f(z)$, for all $z \in D$ and $u_f(z) \le \upsilon(z)$ for all $z \in D$, where υ is any harmonic majorant of $\phi(|f|)$. A non-negative harmonic function on D is called *quasi-bounded* if it is the pointwise increasing limit of non-negative bounded harmonic functions on D. The *Hardy-Orlicz space* H_ϕ^+ is the space of all $f \in H_\phi$ such that $\phi(|f|)$ has a quasi-bounded harmonic majorant on D.

The Hardy-Orlicz spaces H_ϕ and H_ϕ^+ were studied by W. Deeb and M. Marzuq in [2]. M. Masri in [8] and [10] considered these spaces when D is replaced by a domain Ω in **C**. Special cases of these spaces were studied by several authors. (See, for example, [3], [4], [7], [11], [15] and [17]).

If $\phi(x) = x^p$, $0 < p \le 1$, then $H_\phi = H^p$ and if $\phi(x) = \log(1+x)$, then $H_\phi = N$ and $H_\phi^+ = N^+$. Also, if $\phi(x) = (\log(1+x))^p$, $0 < p \le 1$, then $H_\phi = N^p$,and if $\phi(x) = \log(1+x^p)$, $0 < p \le 1$, then $H_\phi = N_p$.

We note that the space H^∞ of bounded analytic functions in D is contained in H_ϕ^+.

If z_0 is a fixed point in D, which is called the *point of reference*, then the *quasi-norm* $\|\ \|_\phi$ on H_ϕ is defined by

$$\|f\|_\phi = u_f(z_0)$$

for all $f \in H_\phi$. If $d(f,g) = \|f-g\|_\phi$ for all $f, g \in H_\phi$, then (H_ϕ, d) is a metric space . If ϕ is a strictly increasing unbounded modulus function, then (H_ϕ^+, d) is an *F-space*, i.e., a topological vector space with complete translation invariant metric (See [1], [2] and [8]).

Let $T = \partial D$ be the *boundary* of the open unit disk D in the complex plane **C** and H^+ be the set of all functions $f \in H$ such that

$$\lim_{r\to 1^-} f(re^{i\theta}) = f^*(e^{i\theta}) \text{ exists } a.e.\sigma,$$

where σ is the *normalized Lebesgue measure* on T .The function f^* is called the *radial limit* of f . When there is no ambiguity we denote the function f and its radial limit by f .The *Hardy-Orlicz spaces* H_ϕ and H_ϕ^+ are given by

$$H_\phi = \{ f \in H : \sup_{0\le r<1} \int_T \phi(| f_r |) d\sigma < \infty \}$$

and

$$H_\phi^+ = \{ f \in H^+ : \sup_{0\le r<1} \int_T \phi(| f_r(z) |) d\sigma(z) = \int_T \phi(| f(z) |) d\sigma(z) < \infty \},$$

where $f_r(z) = f(rz)$, $z \in T \cup D$. [8].

For each $f \in H_\phi$, the quasi-norm of f is given by

$$\| f \|_\phi = u_f(0) = \sup_{0\le r<1} \int_T \phi(| f_r |) d\sigma = \lim_{r\to 1^-} \int_T \phi(| f_r |) d\sigma$$

and for each $f \in H_\phi^+$

$$\| f \|_\phi = \int_T \phi(| f |) d\sigma \text{. (See [8]).}$$

Moreover, $f \in H_\phi^+$ iff $u_f = P[\phi(| f |)]$, where P denotes the Poisson kernel.

Using Harnack's inequality, it follows that:

$$\phi(| f(z) |) \le \frac{2 \| f \|_\phi}{1 - | z |} \quad \text{for all } f \in H_\phi \text{ and } z \in D \text{. (See [10]).}$$

Thus, if a sequence $\{f_n\}$ converges to f in H_ϕ or H_ϕ^+, then it converges uniformly to f on compact subsets of D.

Let $\Lambda = (\lambda_n)$ be a sequence in D such that $\sum_{n=1}^{\infty}(1-|\lambda_n|) < \infty$. If Λ has non- zero terms, m is a non-negative integer and

$$B(z) = z^m \prod_{n=1}^{\infty} (\frac{|\lambda_n|}{\lambda_n})(\frac{\lambda_n - z}{1-\bar{\lambda}_n z}), z \in D,$$

then the function B is called a *Blaschke product*. The term *Blaschke product* will also be used if there are only finitely many factors of B.

In section 2 of this paper, we give a canonical factorization theorem for functions in H_ϕ or H_ϕ^+ when ϕ is a strictly increasing unbounded modulus function which is a generalization of the special cases $\phi(x) = x^p, 0 < p \le 1$ and $\phi(x) = \log(1+x)$. Other similar canonical factorization theorems involving Blaschke products, singular inner functions and outer functions are still open problems even when ϕ is *strongly modulus*, which is defined in [2] as a modulus function satisfying $\int_1^\infty \frac{\phi(x)}{x^2} dx < \infty$, $\lim_{x\to\infty} \frac{\phi(x)}{\log x} > 0$ and $\phi(|f|)$ is subharmonic on D for all $f \in H$. Some consequences of the constraint $\lim_{x\to\infty} \frac{\phi(x)}{\log x} = \alpha \in [0,\infty]$ on H_ϕ and H_ϕ^+ are given in section 4 of this paper.

When $\Lambda = (\lambda_n)$ is a sequence of distinct points in D such that $\sum_{n=1}^{\infty}(1-|\lambda_n|) < \infty$ we introduce the following class of complex sequences:

$$\ell_\Lambda(\phi) = \left\{(c_n) : \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|c_n|) < \infty\right\}.$$

If $\phi(x) = x^p, 0 < p \leq 1$, then $\ell_\Lambda(\phi) = \ell_\Lambda^p$ and if $\phi(x) = \log(1 + x)$, then $\ell_\Lambda(\phi) = \ell_\Lambda^+$. For these special cases one can see [15] and [16] where $\Lambda = (\lambda_n) = (z_n) = Z$.

In section 3 of this paper we proved that $\ell_\Lambda(\phi)$ is an F-space. Also, when $\phi(ab) \leq \phi(a) + \phi(b)$, $a, b \geq 0$, we give a characterization of the bounded subsets of $\ell_\Lambda(\phi)$, a generalization to that of ℓ_Λ^+ in [16].

A space ℓ of complex sequences is called an *ideal* if $\ell^\infty l \subseteq l$, i.e., $(w_n c_n) \in \ell$ whenever $(w_n) \in \ell^\infty$ and $(c_n) \in \ell$. Let $\Lambda = (\lambda_n)$ be sequence in D and X a space of analytic functions in D. The *interpolation problem* consists of describing the *trace* $X \mid \Lambda = \{(f(\lambda_n)) : f \in X\}$ of X on Λ. One approach is to fix a *target space* ℓ and look for conditions so that $X \mid \Lambda = l$. Another approach is to require that $X \mid \Lambda$ is an ideal and call Λ a *free interpolating sequence* for X. We denote this by $\Lambda \in Int(X)$. For certain spaces such as the Hardy and Bergman spaces, the two approaches of interpolation are equivalent with ℓ as an ℓ^p with the appropriate weight (See [5]).

For any function algebra X containing the constants it is easily seen that [5] $\ell^\infty \subseteq X \mid \Lambda$ iff $\Lambda \in Int(X)$. This implies that if Y is a subalgebra of a function algebra X, then $\Lambda \in Int(X)$ whenever $\Lambda \in Int(Y)$.

Free interpolation for H_ϕ and H_ϕ^+ requires the existence of nonzero functions vanishing on all the terms of the sequence Λ except one. Thus, we assume that $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$.

The *linear operators T and* T_ϕ, given by:

$$Tf = \left(f(\lambda_n)\right) \text{ and } T_\phi f = \left(\frac{f(\lambda_n)}{\phi^{-1}\left(\frac{1}{1-|\lambda_n|^2}\right)}\right) \text{ for all } f \in H_\phi ,$$

are related to interpolation. We note that $X \mid \Lambda = \{(f(\lambda_n)) : f \in X\} = T(X)$. When $\phi(x) = x^p$, $0 < p \le 1$, T_ϕ is the operator T_p in [3, Theorem 9.1], where it is shown that $T_p(H^p) = \ell^p$, $0 < p \le \infty, T_\infty = T$ *iff* (λ_n) is *uniformly separated (Carleson sequence).* i.e., there exists $\delta > 0$ such that

$$\prod_{\substack{m=1 \\ m\ne n}}^{\infty} \left|\frac{\lambda_m - \lambda_n}{1-\bar{\lambda}_m \lambda_n}\right| \ge \delta \ , \ n = 1,2,3,....$$

Some of the interpolation techniques of N. Yanagihara in [17] for N and N^+ carry over to H_ϕ and H_ϕ^+. A. Hartmann's characterizations of free interpolation in [5] are based on the canonical factorization of functions in N and N^+ in terms of Blaschke products, singular inner functions and outer functions which is not available in H_ϕ and H_ϕ^+ in general.

In section 4 of this paper, we extend some of their results to H_ϕ and H_ϕ^+ and give some consequences in interpolation under certain restrictions on ϕ.

Finally, the following version of the dominated convergence theorem [12] is found out to be useful:

Let $\{g_n\}$ be a sequence of integrable functions which converges a.e. to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\}$ converges to f a.e. . If $\int g = \lim \int g_n$, then $\int f = \lim \int f_n$.

Canonical factorization of functions in H_ϕ or H_ϕ^+

The following canonical factorization theorem for functions in H_ϕ or H_ϕ^+ is an extension of those in [3] and [4] for N^+ and $H^p, 0 < p \leq 1$.

Canonical factorization theorem *Let $f \in H_\phi$ be not identically zero. Then $f = Bg$, where B is a Blaschke product, $g \in H_\phi$ is unique with no zeros in D and*

(1) $$\| f \|_\phi \leq \| g \|_\phi \leq 2 \| f \|_\phi$$

Moreover , if $f \in H_\phi^+$, then $g \in H_\phi^+$ and $\| g \|_\phi = \| f \|_\phi$.

Proof: First assume that f has infinitely many zeros $\lambda_1, \lambda_2, \lambda_3, ...$ in D repeated according to their respective multiplicities and $\lambda_n \neq 0$ for all $n = 1,2,3,...$. Let

$$b_n(z) = \prod_{j=1}^{n} \left(\frac{|\lambda_j|}{\lambda_j}\right)\left(\frac{\lambda_j - z}{1 - \bar{\lambda}_j z}\right) \text{ and } g_n = \frac{f}{b_n}, \; n = 1,2,3,... \; .$$

Then $|b_n|$ is continuous on the closure $\overline{D}$ of D and $\equiv 1$ on T .Thus for a fixed n and $\varepsilon \in (0,1), |b_n(z)| > 1 - \varepsilon$ when $r = |z|$ is sufficiently close to 1. Hence,

$$(2)\, \phi(|g_n(re^{i\theta})|) = \phi(|\frac{f(re^{i\theta})}{b_n(re^{i\theta})}|) \leq \phi(|\frac{1}{1-\varepsilon} f(re^{i\theta})|) \leq ([\frac{1}{1-\varepsilon}]+1)\phi(|f(re^{i\theta})|,$$

which implies that $\| g_n \|_\phi \leq 2 \| f \|_\phi$, by integrating (2), letting $r \to 1^-$ and then $\varepsilon \to 0^+$. Noting that $| g_n | = | \frac{f}{b_n} | \geq | f |$, we obtain

$\| f \|_\phi \leq \| g_n \|_\phi \leq 2 \| f \|_\phi$.

The subharmonicity of $\phi(| g_n |)$ and $f(0) \neq 0$ give

$$0 < \phi(| \frac{f(0)}{\prod_{j=1}^{n} | \lambda_j |} |) = \phi(| g_n(0) |) \leq \int_T \phi(| (g_n)_r |) d\sigma \leq 2 \| f \|_\phi .$$

Therefore, for all $n = 1,2,3,\ldots$, $\prod_{j=1}^{n} | \lambda_j | \geq \frac{| f(0) |}{\phi^{-1}(2 \| f \|_\phi)} > 0.$

Letting $n \to \infty$ it follows that

(3) $$\prod_{j=1}^{\infty} | \lambda_j | \geq \frac{| f(0) |}{\phi^{-1}(2 \| f \|_\phi)} > 0.$$

Thus, by [14, Theorem 15.5], (3) is equivalent to $\sum_{j=1}^{\infty} (1 - | \lambda_j |) < \infty$. Hence,

$B(z) = \prod_{n=1}^{\infty} (\frac{| \lambda_n |}{\lambda_n})(\frac{\lambda_n - z}{1 - \bar{\lambda}_n z}), z \in D$ is a Blaschke product and $\{ b_n \}$ converges uniformly on compact subsets of D to B. Therefore, [4, p. 56], $\{ g_n \}$ converges uniformly on compact subsets of D to $g = \frac{f}{B}$. Thus,

(4) $$\int_T \phi(| g_r |) d\sigma = \lim_{n \to \infty} \int_T \phi(| (g_n)_r |) d\sigma \leq \lim_{n \to \infty} \| g_n \|_\phi \leq 2 \| f \|_\phi .$$

Therefore, we obtain (1) from (4) and noting that $| g | \geq | f |$.

In case, $f \in H_\phi^+$ the dominated convergence theorem and (2) imply that

$$\| g_n \|_\phi = \lim_{r \to 1^-} \int_T \phi(|(g_n)_r|) d\sigma = \int_T \phi(|g_n|) d\sigma = \int_T \phi(|\frac{f}{b_n}|) d\sigma$$

$$= \int_T \phi(|(f|) d\sigma = \| f \|_\phi .$$

Also, by Fatou's lemma we get

$$\| f \|_\phi = \int_T \phi(|g|) d\sigma = \int_T \lim_{r \to 1^-} \phi(|g_r|) d\sigma \le \lim_{r \to 1^-} \int_T \phi(|g_r|) d\sigma = \| g \|_\phi$$

$$= \lim_{r \to 1^-} \lim_{n \to \infty} \int_T \phi(|(g_n)_r|) d\sigma \le \lim_{r \to 1^-} \lim_{n \to \infty} \| g_n \|_\phi = \| f \|_\phi .$$

Therefore, $g \in H_\phi^+$ and $\| f \|_\phi = \| g \|_\phi$.

The above argument easily shows that the same results hold when f has finitely many zeros in D or $f(0) = 0$. The uniqueness of g follows from properties of zeros of analytic functions.

The space $\ell_\Lambda(\phi)$ and its bounded subsets

As in H_ϕ, Λ and ϕ induce on $\ell_\Lambda(\phi)$ a quasi-norm $\| \; \|_{\phi,\Lambda}$ given by:

$$\|u\|_{\phi,\Lambda} = \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right) \phi\left(|c_n(u)|\right), \; \forall u = (c_n(u)) \in \ell_\Lambda(\phi).$$

For notation convenience we write $\| \; \|_\phi$ instead of $\| \; \|_{\phi,\Lambda}$.

Theorem 3.1 *The space $\ell_\Lambda(\phi)$ is an F-space with the distance function σ defined by*

$$\sigma(u,v) = \|u - v\|_\phi \; , \; \forall u, \; v \in \ell_\Lambda(\phi).$$

That is,

(i) $\sigma(u,v)=\sigma(u-v,0)$

(ii) If $\sigma(u_k,u)\to 0$ *as* $k\to\infty$, *then* $\sigma(\alpha u_k,\alpha u)\to 0$ *as* $k\to\infty$ *for all* $\alpha\in\boldsymbol{C}$.

(iii) If $\alpha_k\to\alpha$ *as* $k\to\infty$, *then* $\sigma(\alpha_k u,\alpha u)\to 0$ *as* $k\to\infty$ *for all* $u\in\ell_\Lambda(\phi)$.

(iv) $\ell_\Lambda(\phi)$ *is complete.*

Proof: The linearity of $\ell_\Lambda(\phi)$ and (ii) follow from $\phi(|\alpha x|)\le([|\alpha|]+1)\phi(x)$, $x\ge 0$, while (i) is obvious from the definition of σ. To prove (iii), let $u\in\ell_\Lambda(\phi)$ be fixed. Then, for all $\varepsilon>0$, there exists $n_0\in\mathbf{N}$ such that $\sum_{n=n_0+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|c_n(u)|)<\frac{\varepsilon}{2}$. Let $K>\max_{1\le n\le n_0}|c_n(u)|$ and $0<\delta_1=\min\left\{1,\frac{1}{K}\phi^{-1}(\frac{\varepsilon}{2n_0})\right\}$. Then there exists $k_o\in\mathbf{N}$ such that $|\alpha_k-\alpha|<\delta_1$ for all $k\ge k_o$. Thus,

$$\sigma(\alpha_k u,\alpha u)=\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|(\alpha_k-\alpha)c_n(u)|)$$

$$\le\sum_{n=1}^{n_o}\phi(K|\alpha_k-\alpha|)+\sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|c_n(u)|)$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

for all $k\ge k_o$. Thus, (ii) holds.

To prove (iv) let (u_k) be a Cauchy sequence in $\ell_\Lambda(\phi)$. First we show that for each fixed $j\in\mathbf{N}$, the complex sequence $(c_j(u_k))$ is Cauchy. Let $\varepsilon>0$ be given. Then there exists $k_o\in\mathbf{N}$ such that for all $k,m\ge k_o$ we have

$$\sigma(u_k, u_m) = \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right) \phi\left(|c_n(u_k) - c_n(u_m)|\right) < \left(1 - |\lambda_j|^2\right) \phi(\varepsilon).$$

Hence, for all $k, m \geq k_o$, we have $|c_j(u_k) - c_j(u_m)| < \varepsilon$, *i.e.*, $(c_j(u_k))$ is Cauchy.

Let $c_n = \lim_{k \to \infty} c_n(u_k)$. For all $\varepsilon > 0$, there exists $k_o \in \mathbf{N}$ such that, for all $k, m \geq k_o$, we have $\sigma(u_k, u_m) = \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right) \varphi\left(|c_n(u_k) - c_n(u_m)|\right) < \frac{\varepsilon}{2}$. Therefore, for each

$j \in \mathbf{N}$ and for all $k, m \geq k_o$ we have

$$\sum_{n=1}^{j} \left(1 - |\lambda_n|^2\right) \phi\left(|c_n(u_k) - c_n(u_m)|\right) < \frac{\varepsilon}{2}.$$

Letting $m \to \infty$ and then $j \to \infty$, it follows that $u = (c_n) \in \ell_\Lambda(\phi)$ and $\sigma(u_k, u) \to 0$ *as* $k \to \infty$. Thus, $\ell_\Lambda(\phi)$ is complete.

In an F-space X with topology induced by a complete translation invariant metric ρ, there are two none equivalent notions of bounded sets. The first is in the metric sense, i.e., a subset E of X is ρ - *bounded* if there exists a constant M such that $\rho(x, y) \leq M < \infty$ for all $x, y \in E$ .The second is in the topological vector space sense, i.e., a subset E of X is *topologically bounded* if for each neighborhood V of zero there exists a number $t_0 > 0$ such that $E \subseteq t\,V$ for all $t \geq t_0$. We refer the interested reader to [13].

The bounded subsets of N^+ and ℓ_Z^+ are studied by N. Yanagihara in [16]. When $\phi(ab) \leq \phi(a) + \phi(b)$ for all $a, b \geq 0$, his results about N^+ were generalized to H_ϕ^+ in [9] where it is shown that a subset E of H_ϕ^+ is topologically bounded iff ($\forall \varepsilon > 0 \, \exists \delta = \delta(\varepsilon) > 0$ such that for

each subset A of T with $\sigma(A) < \delta$ we have $\int_A \phi(|f|)\, d\sigma < \varepsilon, \forall f \in E$).
Moreover, as a corollary of this, a necessary but not sufficient condition for topological bounded sets is given, namely if a subset E of H_ϕ^+ is topologically bounded, then there exists a positive continuous function $\omega(r)$, independent of $f \in E$, $\omega(r) \downarrow 0$ *as* $r \to 1^-$, and such that

$$M(r, f) \leq \phi^{-1}\left(\frac{\omega(r)}{1-r}\right), \text{ for all } f \in E \text{ and } 0 < r < 1,$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Here, we prove the corresponding results for $\ell_\Lambda(\phi)$.

Theorem 3.2 *Let* $\phi(ab) \leq \phi(a) + \phi(b)$ *for all* $a, b \geq 0$ *and E be a subset of* $\ell_\Lambda(\phi)$. *Then E is topologically bounded iff*

(i) $\|u\|_\phi < M = M(E) < \infty, \forall u \in E$

and

(ii) $\forall \varepsilon > 0 \; \exists \; n_o = n_o(\varepsilon, E) \in \mathbf{N}$ *such that*

$$\sum_{n=n_o+1}^{\infty} \left(1 - |\lambda_n|^2\right) \phi(|c_n(u)|) < \varepsilon, \; \forall \; u = (c_n(u)) \in E .$$

Proof: Assume that E is a topologically bounded subset of $\ell_\Lambda(\phi)$. Therefore, $\forall \; \eta > 0$, $\exists \; \alpha = \alpha(\eta) > 0$, $0 < \alpha \leq 1$, such that $\beta E \subseteq V(\eta) = \{u \in \ell_\Lambda(\phi) ; \; \|u\|_\phi < \eta\}$ whenever $0 < \beta \leq \alpha$. Let $\eta = 1$. Then $\exists \; \alpha = \alpha(1)$, $0 < \alpha \leq 1$, such that $\beta E \subseteq V(1)$ whenever $0 < \beta \leq \alpha$. Let $M = 1 + \left[\frac{1}{\alpha}\right]$. Then for all $u = (c_n(u)) \in E$, we have

$$\|u\|_\phi = \left\|\frac{1}{\alpha}.\alpha u\right\|_\phi \leq \left(\left[\frac{1}{\alpha}\right]+1\right)\|\alpha u\|_\phi < \left(\left[\frac{1}{\alpha}\right]+1\right) = M .$$

Thus, (i) holds.

Next, let $\varepsilon > 0$ be given. Choose η such that $0 < \eta < \frac{\varepsilon}{2}$ and $\alpha = \alpha(\eta)$ as above. Since $\sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right) < \infty$, there exists $n_o \in \mathbf{N}$ such that $\sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right) < (\varepsilon / 2\phi(\alpha^{-1}))$.

Then, (ii) holds, since for all $u = (c_n(u)) \in E$ we have

$$\sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|c_n(u)|) \leq \sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi\left(\frac{1}{\alpha}\right) + \sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi(|\alpha\, c_n(u)|)$$

$$< \frac{\varepsilon}{2} + \|\alpha u\|_\phi < \frac{\varepsilon}{2} + \eta < \varepsilon .$$

Conversely, assume that (i) and (ii) hold. Let $V(\eta)$ be any neighborhood of zero. Continuity of ϕ at zero from the right implies that there exists $\varepsilon > 0$ such $\varepsilon < \eta/2$ and $\phi(x) < \eta/2K$ whenever $0 \leq x < \varepsilon$ where $K = \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right) < \infty$.

Therefore, there exists $n_o \in \mathbf{N}$ as in (ii). For each $u = (c_n(u)) \in E$, let $\delta = \min_{1 \leq n \leq n_o}\left(1-|\lambda_n|^2\right) > 0$ and $A_u = \{n \in \mathbf{N} : \phi(|\, c_n(u)\,|) < \frac{M}{\delta}\}$. Thus,

$$M > \|u\|_\phi \geq \sum_{n \notin A_u}\left(1-|\lambda_n|^2\right)\phi(|c_n(u)|) \geq \frac{M}{\delta}\sum_{n \notin A_u}\left(1-|\lambda_n|^2\right).$$

Hence, $\sum_{n\notin A_u}\left(1-|\lambda_n|^2\right)<\delta$. This implies that $\{1,2,...,n_o\}\subseteq A_u$. Letting $0<\alpha<\min\left\{1,\ \frac{\varepsilon}{2\phi^{-1}(M/\delta)}\right\}$, we have

$$\|\alpha u\|_\phi \le \sum_{n=1}^{n_o}\left(1-|\lambda_n|^2\right)\phi\left(|\alpha\, c_n(u)|\right)+\sum_{n=n_o+1}^{\infty}\left(1-|\lambda_n|^2\right)\phi\left(|c_n(u)|\right)$$

$$<\phi\left(\frac{\varepsilon}{2}\right)\sum_{n=1}^{n_o}\left(1-|\lambda_n|^2\right)+\varepsilon<\phi\left(\frac{\varepsilon}{2}\right)K+\frac{\eta}{2}<\eta\,.$$

Therefore, $\alpha E\subseteq V(\eta)$, which shows that E is topologically bounded.

Corollary 3.3 *Let* $\phi(ab)\le\phi(a)+\phi(b)$, $a,b\ge 0$ *and E be a topologically bounded subset of* $\ell_\Lambda(\phi)$.*Then there exists a positive sequence* (ω_n) , $\omega_n\downarrow 0$ *as* $n\to\infty$ *and*

$$|c_n(u)|\le\phi^{-1}\left(\frac{\omega_n}{1-|\lambda_n|^2}\right),\ \forall\, u=(c_n(u))\in E \text{ and } \forall\, n\in \mathbf{N}.$$

Proof: From the proof of theorem 3.2 it follows that, for all $\eta>0$, there exists $\delta=\delta(\eta)>0$ such that

$$\left(1-|\lambda_n|\right)\phi\left(|c_n(u)|\right)\le\left(1-|\lambda_n|\right)\frac{M}{\delta}+\frac{\eta}{2}$$

for all $u=(c_n(u))\in E$ and for all $n\in\mathbf{N}$.

Let (η_k) be a positive sequence with $\eta_k\downarrow 0$ as $k\to\infty$. Hence, for each $k\in\mathbf{N}$ there exists $\delta_k=\delta(\eta_k)>0$ such that

$$\left(1-|\lambda_n|\right)\phi\left(|c_n(u)|\right)\le\left(1-|\lambda_n|\right)\frac{M}{\delta_k}+\frac{\eta_k}{2}$$

for all $u = (c_n(u)) \in E$ and for all $n \in \mathbf{N}$.

Choose a strictly increasing sequence (n_k) in **N** such that $n_k \uparrow \infty$ as $k \to \infty$ and

$$(1-|\lambda_n|)\phi(|c_n(u)|) \leq (1-|\lambda_n|)\frac{M}{\delta_k} + \frac{\eta_k}{2} < \frac{\eta_k}{2} + \frac{\eta_k}{2} = \eta_k$$

for all $n \geq n_k$ and for all $u = (c_n(u)) \in E$.

Define the positive sequence (ω_n) by:

$$\omega_n = \begin{cases} \dfrac{M}{\delta_1} + \eta_1, 1 \leq n < n_1 \\ \eta_k, n_k \leq n < n_{k+1}, k = 1,2,3,... \end{cases}.$$

Then, (ω_n) satisfies the required properties.

We mention that although the spaces H_ϕ^+ and $\ell_\Lambda(\phi)$ look similar, a topologically bounded subset of $\ell_\Lambda(\phi)$ could be relatively compact while a topologically bounded subset of H_ϕ^+ need not be relatively compact. This is the case when $\phi(x) = \log(1+x)$ as in [16].

Interpolation in H_ϕ and H_ϕ^+

The first result of this section is a generalization from N^+ [17, Theorem 1(second part)] to H_ϕ^+ while the second is a generalization from N [17, Theorem 4] to H_ϕ.

Theorem 4.1 *If* $\ell_\Lambda(\phi) \subseteq T(H_\phi^+)$, *then* $\lim_{n\to\infty}(1-|\lambda_n|^2)\phi\left(\frac{1}{|B_n(\lambda_n)|}\right) = 0$

where

$$B_n(z) = \prod_{\substack{m=1 \\ m \neq n}} \frac{|\lambda_m|}{\lambda_m} \frac{\lambda_m - z}{1 - \overline{\lambda}_m z}, \quad z \in D.$$

Proof: Let $K = \ker T = \{ f \in H_\phi^+ : f(\lambda_n) = 0, \forall n \in \mathbf{N} \}$. Then K is a closed subspace of H_ϕ^+ since the convergence of a sequence in H_ϕ^+ implies its convergence on compact subsets of D. Thus, by [13, Theorem 1.41] the quotient space $H_\phi^+ / K = \{ f + K : f \in H_\phi^+ \}$ is an F-space. Let ρ be the metric on H_ϕ^+ / K and $\pi : H_\phi^+ \to H_\phi^+ / K$ be the quotient map where $\pi(f) = f + K$ for all $f \in H_\phi^+$.For each $u = (c_n(u)) \in \ell_\Lambda(\phi)$ there exists $f \in H_\phi^+$ such that $T\,f = (f(\lambda_n)) = u$.

Let $\widetilde{T}u = \pi(f)$. Then it is easy to see that $\widetilde{T} : \ell_\Lambda(\phi) \to H_\phi^+ / K$ is a well defined linear operator. Using the closed graph theorem we prove that it is continuous and hence bounded (See [13]).

Let $u_k \to 0$ in $\ell_\Lambda(\phi)$ as $k \to \infty$ and $\widetilde{T}u_k = \pi(f_k) \to \pi(f^*)$ in H_ϕ^+ / K as $k \to \infty$.

We show that $f^* \in K$ i.e. $\pi(f^*) = K$.

Let $Tf^* = (f^*(\lambda_n)) = (c_n)$, $Tf_k = (f_k(\lambda_n)) = (c_n(u_k)) = u_k$ and $n_0 \in \mathbf{N}$ be fixed .Then $\forall \varepsilon > 0, \exists k_o$, $k_1 \in \mathbf{N}$ such that if $k \geq k_o$, then

$$\|u_k\|_\phi = \sum_{n=1}^{\infty} \left(1 - |\lambda_n|^2\right) \phi(|c_n(u_k)|) < \left(1 - |\lambda_{n_o}|^2\right) \phi(\varepsilon).$$

Thus, if $k \geq k_o$, then $|f_k(\lambda_{n_o})| = |c_{n_o}(u_k)| < \varepsilon$. Therefore, $f_k(\lambda_n) \to 0$ as $k \to \infty$ for all $n \in \mathbf{N}$. Also, if $k \geq k_1$, then $\rho(\pi(f_k), \pi(f^*)) = \rho(\pi(f_k - f^*), \pi(0)) < \frac{\varepsilon}{4}$.

For each $k \ge k_1$, choose $g_k \in H_\phi^+$ such that $\pi(g_k) = \pi(f_k - f^*)$ and $\|g_k\|_\phi < \frac{\varepsilon}{4}$.

(See [13, p. 30]).Thus, for each $k \ge k_1$ and $\forall n \in \mathbf{N}$ we have

$$\left(1-|\lambda_n|^2\right)\phi\left(|c_n(u_k)-c_n|\right)=\left(1-|\lambda_n|^2\right)\phi\left(\left|f_k(\lambda_n)-f^*(\lambda_n)\right|\right)$$
$$=\left(1-|\lambda_n|^2\right)\phi\left(|g_k(\lambda_n)| \le 4\|g_k\|_\phi < \varepsilon.\right.$$

Let $k \to \infty$ and then $\varepsilon \to 0$ we get $f^*(\lambda_n) = c_n = 0\ \forall n \in \mathbf{N}$, i.e. , $f^* \in K$.

Next, let $e_k = (c_n(e_k))$, where $c_n(e_k) = 1$ if $n = k$ and $c_n(e_k) = 0$ if $n \ne k$. Then $\|e_k\|_\phi = \left(1-|\lambda_k|^2\right)\phi(1) \to 0$ as $k \to \infty$. Therefore, the continuity of $\widetilde{T}$ implies that $\rho(\pi(f_k),\ \pi(0)) \to 0$ as $k \to \infty$, where $\widetilde{T} e_k = \pi(f_k)$ and $T f_k = e_k$.

Thus, $\forall \varepsilon > 0$, $\exists\ k_2 \in \mathbf{N}$ such that if $k \ge k_2$, then $\rho(\pi(f_k),\ \pi(0)) < \varepsilon$. For each $k \ge k_2$, choose $h_k \in H_\phi^+$ such that $\pi(h_k) = \pi(f_k)$ and $\|h_k\|_\phi < \varepsilon$. Therefore, there exists a sequence (h_k) in H_ϕ^+ which converges to zero and $(h_k(\lambda_n)) = (f_k(\lambda_n))$ for all $k, n \in \mathbf{N}$. For $k > n$, let

$$B_{n,k}(z) = \prod_{\substack{m=1 \\ m \ne n}}^{k} \frac{|\lambda_m|}{\lambda_m} \frac{\lambda_m - z}{1-\bar{\lambda}_m z} \text{ and } H_{n,k} = h_n / B_{n,k}.$$

Then, $H_{n,k} \in H_\phi^+$ and $\|H_{n,k}\|_\phi = \|h_n\|_\phi$. Hence,

$$\left(1-|\lambda_n|^2\right)\phi\left(\frac{1}{|B_{n,k}(\lambda_n)|}\right) = \left(1-|\lambda_n|^2\right)\phi\left(\left|\frac{f_n(\lambda_n)}{B_{n,k}(\lambda_n)}\right|\right)$$
$$= \left(1-|\lambda_n|^2\right)\phi\left(|H_{n,k}(\lambda_n)|\right) \le 4\|h_n\|_\phi.$$

Letting $k \to \infty$ and then $n \to \infty$ we get

$$\lim_{n\to\infty}\left(1-|\lambda_n|^2\right)\phi\left(\frac{1}{|B_n(\lambda_n)|}\right)=0 . \qquad \blacksquare$$

Theorem 4.2 *Let (λ_n) be uniformly separated. Then $T(H_\phi)\subseteq \ell_\Lambda(\psi)$, where $\psi(x)=(\phi(x))^p$, $x \geq 0$ and $0<p<1$. Moreover, the above inclusion could be proper.*

Proof: Let $f \in H_\phi$ .Then $f = Bg$ where B is the Blaschke product of the zeros of f in D and $g \in H_\phi$ with no zeros in D. Let $h = u_g + i v_g$ where v_g is a harmonic conjugate of u_g the least harmonic majorant of $\phi(|g|)$. Since the analytic function h has a positive real part, then by [3, Theorem 3.2] $h \in H^p$, $0<p<1$ .Then, $T h \in \ell^p_\Lambda$, $0<p<1$ (See [17, p. 429]). Therefore,

$$\|T f\|_\psi = \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\left(\phi(|f(\lambda_n)|)\right)^p \leq \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\left(\phi(|g(\lambda_n)|)\right)^p$$

$$\leq \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)\left(u_g(\lambda_n)\right)^p \leq \sum_{n=1}^{\infty}\left(1-|\lambda_n|^2\right)|h(\lambda_n)|^p < \infty$$

Next, let $\lambda_n = 1-b^n$ where $0<b<1$ and $(c_n)=\left(\phi^{-1}(n/b^n)\right)$. Then, by [3, Theorem 9.2, p. 155] (λ_n) is uniformly separated and $\|(c_n)\|_\psi \leq 2\sum_{n=1}^{\infty} b^n\left(\frac{n}{b^n}\right)^p = 2\sum_{n=1}^{\infty} n^p b^{n(1-p)} < \infty$, i.e., $(c_n)\in \ell_\Lambda(\psi)$. Also, $\lim_{n\to\infty}(1-|\lambda_n|)\phi(|c_n|)=\infty$ .Thus, there is no $f \in H_\phi$ such that $Tf = (c_n)$ since

$$(1-|z|)\phi(|f(z)|) \leq 2\|f\|_\phi , \ \forall f \in H_\phi \text{ and } \forall z \in D .$$

We note that $l^p \subseteq l^\infty \subseteq l_\Lambda(\phi), 0 < p \le \infty$. Moreover, for each $f \in H_\phi^+$ we have [9, Theorem 2.1, p. 14] $\lim_{r\to 1^-}(1-r)\phi(M(r,f)) = 0$. Hence, it follows that $T_\phi(H_\phi^+) \subseteq l^\infty$. Also, when $\phi(x) = x^p$, $0 < p \le 1$, it is easy to see that

$T_\phi(H_\phi^+) = l^p$ iff $T(H_\phi^+) = l_\Lambda(\phi)$.

Thus, in this case, theorem 9.1 in [3] can be restated as $T(H_\phi^+) = l_\Lambda(\phi)$ iff (λ_n) is uniformly separated. So, it is natural to ask for what other kinds of ϕ this is true. When $\phi(x) = \log(1+x)$, it is shown [17, Theorem 3] that there exists a uniformly separated sequence (λ_n) and $f \in H_\phi^+$ such that $T(f) \notin l_\Lambda(\phi)$. Furthermore, if (λ_n) is uniformly separated, then $l_\Lambda(\phi) \subseteq T(H_\phi^+)$

(See [17, Theorem 1]). This motivated the following theorem whose converse is still open.

Theorem 4.3 *If* $T_\phi(H_\phi^+) = \ell^p$, $0 < p \le \infty$, $\phi(ab) \le \phi(a) + \phi(b)$, $a, b \ge 0$, *then* (λ_n) *is uniformly separated.*

Proof: The closed graph theorem implies that $T_\phi : H_\phi^+ \to \ell^p$ for $0 < p \le \infty$ is continuous since $T_\phi(H_\phi^+) \subseteq \ell^p$.Then $K_\phi =$ kernel of T_ϕ is a closed subspace of H_ϕ^+ and the quotient space H_ϕ^+ / K_ϕ is an F-space. Since $T_\phi(H_\phi^+) = \ell^p$, T_ϕ induces a bijective bounded linear operator $\widetilde{T}_\phi : H_\phi^+ / K_\phi \to \ell^p$ such that $T_\phi = \widetilde{T}_\phi \, o \, \pi$ where $\pi : H_\phi^+ \to H_\phi^+ / K_\phi$ is the quotient map (see [13, p. 37]).The open mapping theorem [13] implies that $\widetilde{T}_\phi^{-1} : \ell^p \to H_\phi^+ / K_\phi$, the inverse of $\widetilde{T}_\phi$, is bounded, i.e., continuous. Let $e_k = (c_n(e_k))$ be as before and $E = \{e_k : k = 1, 2, ...\}$. For each $k \in \mathbf{N}$ there exists $f_k \in H_\phi^+$ such that

$T_\phi f_k = e_k$.Let $E_1 = \{f_k \in H_\phi^+ : T_\phi f_k = e_k\}$.We prove that E_1 is a bounded subset of H_ϕ^+. Let $V = V(\eta) = \{ f \in H_\phi^+ : \|f\|_\phi < \eta\}, \eta > 0$, be a neighborhood of zero in H_ϕ^+. Since π and $\widetilde{T}_\phi$ are open there exists $\alpha_1 > 0$ such that $W = \{u \in \ell^p : \|u\|_p < \alpha_1\} \subseteq (\widetilde{T}_\phi o \pi)(V)$

(See [13]). Let $0 < \alpha < \min\{1, \alpha_1\}$.Then $0 < \alpha \le 1$ and $\beta E \subseteq W$ whenever $0 < \beta \le \alpha$. Thus, $\beta E \subseteq (\widetilde{T}_\phi o \pi)(V) = T_\phi(V)$. Hence, $E \subseteq T_\phi\left(\frac{1}{\beta} V\right)$.Therefore,

$$\beta E_1 \subseteq \beta T_\phi^{-1}(E) \subseteq \beta T_\phi^{-1}\left(T_\phi\left(\frac{1}{\beta} V\right)\right) \subseteq V$$

whenever $0 < \beta \le \alpha$.Thus E_1 is a topologically bounded subset of H_ϕ^+.

Clearly, $E_2 = \{f_n / B_{n,k} : n, k = 1, 2, ..., k > n\}$ is bounded since $f_n / B_{n,k} \in H_\phi^+$ and $\|f_n / B_{n,k}\|_\phi = \|f_n\|_\phi$. Therefore, by [9, Corollary 3.2, p. 18], there exists a positive continuous function $\omega(r) \downarrow 0 \ as \ r \to 1^-$ and

$$M(r, f_n / B_{n,k}) \le \phi^{-1}\left(\frac{2\omega(r)}{1 - r^2}\right)$$

$\forall\, r \in (0,1)$ and $\forall\, k, n \in \mathbf{N}$ where $k > n$. Since $r_n = |\lambda_n| \to 1$ as $n \to \infty$, there exists $n_o \in \mathbf{N}$ such that

$$\left|\frac{f_n(\lambda_n)}{B_{n,k}(\lambda_n)}\right| \le M(r_n, f_n / B_{n,k}) \le \phi^{-1}\left(\frac{1}{1 - r_n^2}\right)$$

for all $n \ge n_o$.Thus,

$$\frac{1}{\left|B_{n,k}(\lambda_n)\right|} = \left| \frac{f_n(\lambda_n)}{\phi^{-1}\left(\frac{1}{1-r_n^2}\right) B_{n,k}(\lambda_n)} \right| \leq 1$$

for all $n \geq n_o$. If $|z| \leq r < 1$, then

$$\sum_{m=1}^{\infty} \left| 1 - \left| \frac{z-\lambda_m}{1-\bar{\lambda}_m z} \right| \right| \leq \sum_{m=1}^{\infty} \left| 1 - \frac{|\lambda_m|}{\lambda_m} \left(\frac{\lambda_m - z}{1-\bar{\lambda}_m z} \right) \right| \leq \frac{2}{1-r} \sum_{m=1}^{\infty} (1-|\lambda_m|) < \infty .$$

Therefore, by [14, Theorem 15.5], $\left|B_n(\lambda_n)\right| > 0$ for $n = 1, 2, ..., n_o - 1$. Let $\delta = \min_{1 \leq n \leq n_o - 1} \left\{1, \left|B_n(\lambda_n)\right|\right\}$.Then (λ_n) is uniformly separated since $\left|B_n(\lambda_n)\right| \geq \delta > 0$ for all $n \in \mathbf{N}$.

Next we consider the relation between free interpolation and harmonic functions. Let $Har(D)$ denote the space of harmonic functions in D and $Har_+(D)$ the subspace of its positive functions.

When $\Lambda = (\lambda_n)$ is a sequence in D such that $\sum_{n=1}^{\infty}(1-|\lambda_n|)<\infty$, we define

$$\ell_{\phi} = \{(c_n) : \exists h \in Har_+(D) \text{ such that } \phi(|c_n|) \leq h(\lambda_n), n = 1,2,3,...\}$$

and

$$\ell_{\phi}^{+} = \{(c_n) : \exists \text{a quaqsi--bounded} h \in Har_+(D) \text{such that} \phi(|c_n|) \leq h(\lambda_n), n = 1,2,3,...\}$$

The main results of A. Hartmann [5] is giving equivalent conditions for free interpolation in N and N^+ depending on the canonical factorization of functions in them in terms of Blaschke products, singular inner functions and outer functions which is not available in H_ϕ and H_ϕ^+ in general. Also, in [6] he defined big Hardy-Orlicz spaces and characterized free interpolation in them. Here we prove the following results noting that, according to his results, when $\phi(x)=\log(1+x)$, equivalence holds in theorem 4.4(i) and (iii) below.

Theorem 4.4 *Let* $\phi(ab)\le\phi(a)+\phi(b)$, $a, b\ge 0$.

(i) If $\ell_\phi=(H_\phi\mid\Lambda)$, *then* $\Lambda\in Int(H_\phi)$

(ii) If $\Lambda\in Int(H_\phi)$, *then* $(H_\phi\mid\Lambda)\subseteq l_\phi$

(iii) If $\ell_\phi^+=(H_\phi^+\mid\Lambda)$ *, then* $\Lambda\in Int(H_\phi^+)$

(iv) If $\Lambda\in Int(H_\phi^+)$ *), then* $(H_\phi^+\mid\Lambda)\subseteq l_\phi^+$

(v) Let $\phi(x)=\psi(\log(1+x)), x\ge 0$, *where* ψ *is a modulus function.*

If $\Lambda\in Int(N)$, *then* $\Lambda\in Int(H_\phi)$. *Moreover, If* $\Lambda\in Int(N^+)$, *then* $\Lambda\in Int(H_\phi^+)$.

Proof: (i) Assume that $\ell_\phi=(H_\phi\mid\Lambda)$ and $(c_n)\in l^\infty$. Then there exists a positive constant c such that $\phi(|c_n|)\le\phi(c)<\infty, n=1,2,3,\ldots$. Therefore, $(c_n)\in\ell_\phi=(H_\phi\mid\Lambda)$.

This implies that $\Lambda\in Int(H_\phi)$ since $\ell^\infty\subseteq(H_\phi\mid\Lambda)$.

(ii) Assume that $\Lambda\in Int(H_\phi)$ and $(c_n)\in(H_\phi\mid\Lambda)$. Then there exists $f\in H_\phi$ such that $(c_n)=(f(\lambda_n))$. Let $h=u_f$. Then $\phi(|c_n|)=\phi(|f(\lambda_n)|)\le h(\lambda_n), n=1,2,3,\ldots$.

Therefore, $(H_\phi\mid\Lambda)\subseteq l_\phi$.

The proof of (iii) and (iv) is similar to (i) and (ii).

For (v) the inequalities $x\le 1+[x]\le 1+x, x\ge 0$ imply that $\psi(x)\le\psi(1)(1+x), x\ge 0$.

Hence, $\phi(x) = \psi(\log(1+x)) \le \psi(1)(1+\log(1+x)), x \ge 0$ .Thus,
$N \subseteq H_\phi$ and $N^+ \subseteq H_\phi^+$

which implies (v).

Finally under certain constraints on ϕ we get the following results.

Theorem 4.5 *Let* $\phi(ab) \le \phi(a) + \phi(b)$, $a, b \ge 0$ *and* $\lim_{x\to\infty} \frac{\phi(x)}{\log x} = \alpha$.

(i) If $\alpha \in (0,\infty)$*, then* $\Lambda \in Int(H_\phi)$ *iff* $\Lambda \in Int(N)$ *and* $\Lambda \in Int(H_\phi^+)$ *iff* $\Lambda \in Int(N^+)$

(ii) If $\alpha = \infty$*, then* $\Lambda \in Int(H_\phi) \Rightarrow \Lambda \in Int(N)$ *and* $\Lambda \in Int(H_\phi^+) \Rightarrow \Lambda \in Int(N^+)$.

(iii) If $\alpha = 0$*, then* $\Lambda \in Int(N) \Rightarrow \Lambda \in Int(H_\phi)$ *and* $\Lambda \in Int(N^+) \Rightarrow \Lambda \in Int(H_\phi^+)$.

Proof: (i) Let $\alpha \in (0,\infty)$ **.**Then there exists $x_0 > 1$ such that

$$\frac{\alpha}{2}\log x < \phi(x) < \frac{3\alpha}{2}\log x, \text{ for all } x \ge x_0.$$

Hence,

$$\log(1+x) \le 1+\log^+ x = 1+\log x \le 1+\frac{2}{\alpha}\phi(x), \text{ for all } x \ge x_0.$$

Thus,

$$\log(1+x) \le 1+\log(1+x_0)+\frac{2}{\alpha}\phi(x), \text{ for all } x \ge 0,$$

implies that $H_\phi \subseteq N$ and $H_\phi^+ \subseteq N^+$. Also, we have

$$\phi(x) \le \frac{3\alpha}{2}\log(1+x)+\phi(x_0), \text{ for all } x \ge 0$$

implies that $N \subseteq H_\phi$ and $N^+ \subseteq H_\phi^+$.Therefore, (i) holds since $N = H_\phi$ and $N^+ = H_\phi^+$.

(ii) Let $\alpha = \infty$. Then there exists $x_0 > 1$ such that

$\log x < \phi(x)$, for all $x \geq x_0$.

Hence,

$\log(1+x) \leq 1 + \log(1+x_0) + \phi(x)$, for all $x \geq 0$.

Thus, $H_\phi \subseteq N$ and $H_\phi^+ \subseteq N^+$ which implies (ii).

(iii) Let $\alpha = 0$. Then there exists $x_0 > 1$ such that

$\phi(x) < \log x$, for all $x \geq x_0$.

Hence,

$\phi(x) < \log x < \log(1+x) < \log(1+x_0) + \log(1+x)$, for all $x \geq 0$.

Thus, $N \subseteq H_\phi$ and $N^+ \subseteq H_\phi^+$ which implies (iii).

References

[1] Deeb, W. Khalil, R. & Marzuq, M. (1986). "Isometric Multiplication of Hardy- Orlicz spaces". <u>Bull. Austral. Math. Soc. (34)</u>. 177-189.

[2] Deeb, W. & Marzuq, M. (1986). "$H(\phi)$ spaces". <u>Canad. Math. Bull. 29(3)</u>. 295-301.

[3] Duren, P. L. (1970). "Theory of H^p-Spaces". Academic Press. New York.

[4] Fisher, S. D. (1983). <u>Function Theory on Planar Domains</u>. John Wiley and Sons. Inc.

[5] Hartmann, A. Massenda, X. Nicolau, A. & Thomas, P. (2004). "Interpolation in the Nevanlinna and Smirnov classes and harmonic majorants". <u>J. Funct. Anal. (217)</u>. 1-37.

[6] Hartmann, A. (2006). "Interpolation and harmonic majorants in big Hardy- Orlicz space". arXiv: math. CV/0603624 v2. 1-19.

[7] Masri, M. (1985). "Compact Composition Operators on the Nevanlinna and Smirnov Classes". Dissertation. Univer. of North Carolina at Chapel Hill.

[8] M. Masri. $H_\phi(\Omega)$ *and* $H_\phi^+(\Omega)$ *Spaces*. An- Najah Univ. J. Res. (N. Sc.). Vol. **16** (1). (2002). 1-21.

[9] Masri, M. (2002). "The Space H_ϕ^+ and its Containing Frechet Space F_ϕ^+". Mu'tah Lil- Buhuth Wad Dirasat. Natural and Applied Sciences Series. 17(3). 11-26.

[10] Masri, M. (2003). "Compact Composition Operators on $H_\phi(\Omega)$ and $H_\phi^+(\Omega)$ Spaces". Abhath Al-Yarmouk. Basic Sc. and Engin. 12(1). 155-167.

[11] Roberts, J. W. & Stoll, M. (1976). "Prime and Principal Ideals in the Algebra N^+". Arich der Mathematik (27). 387-393.

[12] Royden, H. L. (1968). Real Analysis. 2^{nd} ed. Macmillan Company. New York.

[13] Rudin, W. (1973). Functional Analysis. McGraw-Hill.

[14] Rudin, W. (1974). Real and Complex Analysis. 2^{nd} ed. McGraw-Hill.

[15] Shapiro, H. S. & Shields, A. L. (1961). "On some interpolation problems for analytic functions". Amer. J. Math. (83). 513-532.

[16] Yanagihara, N. (1973). "Bounded Subsets of Some Spaces of Holomorphic Functions". Sci. Papers College Gen. Ed. Univ. Tokyo. (23). 19-28.

[17] Yanagihara, N. (1974). "Interpolation Theorems for the Class N^+". Illinois Journal of Math. (18). 427-435.