An-Najah Univ. J. Res., Vol. 12, (1998) 1-6.

The Multiplier Algebra of Orlicz Spaces

جبر مضاعفات فضاء اورلكس

Abdallah Hakawati

عبد الله حكواتي

Department of Mathematics, An-Najah N. University - Nablus, Palestine.

Received: (21/1/1998). Accepted: (19/7/1998).

Abstract

In this paper we prove that if ϕ is a modulus function and if $X = [0,1]$ is given the Lebesgue measure, then $M(L_\phi) = L_\phi$ if and only if $\overline{\lim_{x \to 0}} \frac{\phi(x^2)}{\phi(x)} < \infty$; L_ϕ being the Orlicz space $L_\phi(X)$; and $M(L_\phi)$ its multiplier algebra.

في بحثنا هذا نثبت أنه إذا أعطيت الفترة $X = [0,1]$ قياس لبيج وإذا كان ϕ اقترانــا قياسيا، وكان $L_\phi(X)$ فضاء أورلكس المصاحب والـذي نختصـره علـى الصيغـة L_ϕ فـان $M(L_\phi) = L_\phi$ إذا وفقط إذا كان $\overline{\lim_{x\to 0}} \frac{\phi(x^2)}{\phi(x)} < \infty$ حيث أن $M(L_\phi)$ ترمـز إلـى جـبر مضاعفات L_ϕ.

---

1. Introduction

Let ϕ be a strictly increasing continuous subadditive function defined on $[0, \infty]$ with $\phi(0) = 0$. Such a function is called a modulus function. Let (X, μ) be a finite measure space. The Orlicz space $L_\phi(X)$ is the set of all complex-valued measurable functions f which are defined on X and satisfy

$$|f|_\phi = \int_X \phi(|f|)d\mu < \infty$$

With the metric $|\,.\,|\phi$, the space $L_\phi(X)$ becomes a complete linear topological space [1]. We will supress X, unless otherwise specified, and write L_ϕ for $L_\phi(X)$.

If $\phi(x) = x^P$, $0 < P \leq 1$, then $L\phi$ is the space L^p, and $|\,.\,|\phi$ is a norm if and only if $L_\phi = L^1$. If ϕ is bounded then L_ϕ becomes the space of all measurable functions [1].

Being increasing and subadditive, ϕ is easily seen to satisfy $\overline{\lim_{x \to \infty}} \frac{\phi(x)}{x} \leq k$ for some real constant k. It then follows that $L^1 \subset L\phi$ for all modulus functions ϕ.

If $\phi(x) = \log(1+x^p)$, $0 < P \leq 1$, then ϕ is modulus and $L\phi$ will be donoted by Np. For more on Np-spaces, see[2]. It is not hard to see that the function defined as $\phi(x) = \frac{x}{1+x}$ is modulus and that the composition of two modulus functions is again modulus [3].

A multiplier of $L\phi$ is a measurable function g on X for which $f.g \in L\phi$ for all $f \in L\phi$. $M(L\phi)$ will denote the space of all multipliers of $L\phi$.

In [1], Deep introduced two classes of modulus functions which resemble a natural interplay between $L\infty$, $L\phi$ and $M(L\phi)$. Specifically, assuming ϕ unbounded and $\phi(1) = 1$, (we may do this without loss of generality); he, on the one hand, proved that, if $\phi(xy) \geq \phi(x)\,\phi(y)$ for all $x \geq 1$ and $y \geq 0$ then $M(L\phi) = L^\infty$, and as an immediate corollary that $M(L^p) = L^\infty$ for all $0<p\leq\infty$. On the other hand, if $\phi(xy) \leq \phi(x) + \phi(y)$ for all x and y, he, then proved that $M(L\phi) = L\phi$, and hence concluded that $M(Np) = Np$ for $0<p\leq 1$.

In this paper, we characterize those modulus functions ϕ so that $M(L\phi) = L\phi$.

Since the function f(x) = 1 for all x ∈ X is in Lϕ, then M(Lϕ)⊂ Lϕ and it is clear that L∞ ⊂ M(Lϕ) for all modulus function ϕ. In what follows, we will assume that ϕ is a modulus function which is increasing without bound, ϕ(1) = 1, and X is our measure space with finite measure μ

2. The Multiplier Agebra of Lϕ

It was pointed out earlier that Lϕ is a linear space, and so it is evident that M(Lϕ) = Lϕ if and only if Lϕ is an algebra. We establish the following.

Lemma 1: Lϕ is an algebra if and only if f^2 ∈ Lϕ whenever f ∈ Lϕ

Proof: If Lϕ is an algebra, then, obviously f^2 ∈ Lϕ when ever f ∈ Lϕ. Conversely; suppose f^2 ∈ Lϕ for all f∈ Lϕ, and let f, g∈ Lϕ be arbitrary.

$$(f+g)^2 = f^2 + 2fg + g^2$$

$$\text{so, } fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$$

hence fg ∈ Lϕ

Since f, g were arbitrary, Lϕ is an algebra.

In the following theorem, which is our main result we assume X = [0,1] and equipped with the lebesgue measure.

Theorem 1: $\overline{\lim_{x\to\infty}} \dfrac{\phi(x^2)}{\phi(x)} < \infty$ if and only if f^2 ∈ Lϕ for all f ∈ Lϕ.

Proof: Suppose first that $\overline{\lim_{x\to\infty}} \dfrac{\phi(x^2)}{\phi(x)} < \infty$, and let f ∈ Lϕ.

There are positive numbers M and K so that

$\phi(x^2) < M\phi(x)$ for all $x \geq k$.

Let A = {x ∈X: f(x) ≤ K} and

$B = \{x \in X: f(x) > K\}$

$$\int_X \phi(|f^2|)d\mu = \int_A (|f^2|)d\mu + \int_B \phi(|f^2|)d\mu$$

$$\leq \int_X \phi(K^2)d\mu + \int_X M\phi(|f|)d\mu$$

$$= \phi(K^2)\mu(x) + M\int_X \phi(|f|)d\mu$$

$<$ since $f \in L\phi$.

Therefore, $f^2 \in L\phi$.

Conversely, suppose that $\overline{\lim_{x\to\infty}} \frac{\phi(x^2)}{\phi(x)} < \infty$

There is a sequence $\{x_n\}$ of real numbers such that $\lim_{X\to\infty} \frac{\phi(x_n^2)}{\phi(x_n)} = \infty$

So, for each M, there is a positive integer N so that $\frac{\phi(x_n^2)}{\phi(x_n)} > M$ for all $n \geq N$.

Since ϕ is continuous, for each m, there is an interval $I_m = (a_m, b_m)$ such that $\frac{\phi(x_n^2)}{\phi(x_n)} > m$ for all $x \in I_m$ and such that $I_m \cap I_n = \phi$ if $m \neq n$.

Let, for each m, $J_m = (\alpha_m, \beta_m)$ such that the length $\ell(J_m)$ of J_m satisfies.

$$\ell(J_m)\,\phi(b_m) = \frac{1}{m^2}$$

and that $J_m \cap J_n = \phi$ if $m \neq n$.

Define f on X as,

$$f(X) = \begin{cases} \dfrac{x-\alpha_n}{\beta_n-\alpha_n}(b_n-a_n)+a & \text{if } x \in J_n \\ 0 & \text{elsewhere} \end{cases}$$

Now, $\int_X \phi(|f|)d\mu = \sum_n \int_{\alpha n}^{\beta n} \phi(|f|)d\mu$

$$= \sum_n \int_{an}^{bn} \phi(|y|)\frac{\beta n-\alpha n}{bn-an}dy$$

$$= \sum_n \ell(Jn)\frac{1}{bn-an}\int_{an}^{bn}\phi(|y|)dy$$

$= \sum_n \ell(Jn).\phi(yn)$ (where, for each n, yn is appropriate for the Mean Value Theorem for integrals).

$$= \sum_n \frac{1}{n^2} < \infty$$

It therefore follows that $f \in L\phi$.

But, $\int_X \phi(|f^2|)d\mu = \sum \int_{J_n} \phi(|f^2(x)|)dx \geq \sum_n n\int_{J_n} \phi(|f(x)|)dx$

$= \sum_n n\dfrac{\beta n-\alpha n}{bn-an}\int_{an}^{bn}\phi(y)dy = \sum n\phi(y_n)\ell(J_n)$; ($y_n$ is as above) $= \sum \dfrac{1}{n} = \infty$

Hence, $f^2 \notin L\phi$.

This completes the proof of the theorem.

With X as in theorem 1, we get the following:

Corollary 1.: $M(L\phi) = L\phi$ if and only if $\overline{\lim_{x\to\infty}} \frac{\phi(x^2)}{\phi(x)} < \infty$

References

[1] W. Deeb, Multipliers and isometries of Orlisz Spaces, in "Proceedings of the Conference on Mathematical Analysis and its Applications, Kuwait Univ., Kuwait, Feb. 18-21", 1985.

[2] L. Waelbrock, "Topological Vector Spaces, Lecuture Notes in Mathematics", Springer-Verlag, No. 331(1972), 1-40.

[3] W. Deeb and R. Khalil, Best Approximation in L^p (I,X), 0<p<1, *Journal of Approximation Theory*, **58, (1)**, (1989), 68-77.