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Quasi Centralizers and Inner Derivations in a Closed Ideal of a Complex Banach Algebra

أشباه الممركز والاشتقاقات الداخلية في مثالي مغلق في جبر بناخ العقدي

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Abstract

In this paper we show that, for an ideal J of a unital complex Banach algebra A, we have (i) under certain conditions the σ -quasi centralizer, the quasi centralizer, and the centralizer of J are all identical, and so they are subsets of the ρ -quasi centralizer of J. (ii) If J is closed and a is a quasi centralizer element of J, then D_a^J, a restriction of the inner derivation of a to J is topologically nilpotent. (iii) For each complex number λ and each x in J we have, (λ – a) x = 0 if and only if x (λ – a) = 0.

ملخص

في هذا البحث تم إثبات أنه إذا كان **J** مثالياً في جبربناخ الوحدوي العقدي فإن: (١) في حال تحقق شروط معينة تكون مجموعات شبه الممركز من نوع σ ، وشبه الممركز، والممركز جميعها متساوية وبذلك تصبح هذه المجموعات جزئية من مجموعة شبه الممركز من نوع ρ وذلك للمثالي **J**. (٢) إذا كان **J** مغلق و a عنصر ممركزي لـــــ **J** فإن الإشتقاق الداخلي لـــــ a على **J** يكون نيلبوتنت تبولوجيا. (٣) لأي عدد مركب λ ولأي عنصر *x* ينتمي للمثالي **J** يتحقق الآتي: $(\lambda - a)x = 0$ إذا وفقط إذا $x(\lambda - a) = 0$.

1. Introduction

In this paper we study quasi centralizers and inner derivations in a closed ideal of a complex Banach algebra, where we see that some results of Rennison in [6] remain true whenever the quasi centrality conditions with respect to all the elements in the algebra given by Rennison is replaced by the same quasi centrality conditions with respect to all the elements in a closed ideal.

Throughout this paper all linear spaces and algebras are assumed to be defined over $\mathbb{C}$, the field of complex numbers.

Let A be any complex normed algebra. Then we denote the center of A by Z(A) = {a∈A : ax = xa for all x ∈ A}, and the centralizer of a subset B of A by C(B) = {a∈A: ax = xa for all x∈B}. For a∈A, the spectrum in A of a will be denoted by $\sigma_A(a)$ and is defined by $\sigma_A(a)$ = {λ∈ $\mathbb{C}$: $(\lambda - a)^{-1}$ does not exist}. The resolvent set, its complement, will be denoted by $\rho_A(a)$.

In [6] Rennison defined the set of all quasi central elements in a complex Banach algebra A by Q (A) = $\bigcup_{k\geq 1} Q(k, A)$, where Q(k, A) = {a∈A: || x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈ $\mathbb{C}$ }.

Also he defined the set of all σ-quasi central elements in A by $Q_\sigma(A)$ = $\bigcup_{k\geq 1} Q_\sigma(k, A)$, where $Q_\sigma(k, A)$ = {a∈A: || x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈ $\rho_A(a)$}.

In [4] we defined the set of all ρ-quasi central elements in A by $Q_\rho(A)$ = $\bigcup_{k\geq 1} Q_\rho(k, A)$, where $Q_\rho(k,A)$ = {a∈A: || x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ ∈$\sigma_A(a)$}.

Similarly, for a subset B of a complex normed algebra A we defined in [1] the following three concepts.

1. The quasi centralizer (quasi-commutant) of B is QC(B) = $\bigcup_{k\geq 1} QC(k,B)$, where QC(k, B) = {a∈A: ||x (λ - a)|| ≤ k ||(λ – a) x|| for all x∈B and all λ∈ $\mathbb{C}$ }.
2. The σ-quasi centralizer (σ -quasi-commutant) of B is $QC_\sigma(B)$ = $\bigcup_{k\geq 1} QC_\sigma(k,B)$, where QC_σ (k, B) = {a∈A: || x (λ – a) || ≤ k || (λ – a) x || for all x∈B and all λ∈$\rho_A(a)$}.

3. The ρ-quasi centralizer (ρ-quasi commutant) of B is $QC_\rho(B) = \bigcup_{k\geq 1} QC_\rho(k,B)$, where $QC_\rho(k, B) = \{a\in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x\in B$ and all $\lambda \in \sigma_A(a)\}$.

Remark

In this remark we state Theorem 2.1 in [1] which will be used frequently in this paper. The theorem states that:

If A is a complex normed algebra and $D \subseteq B \subseteq A$, then for $k \geq 1$,

1. $C(B) \subseteq QC(k, B) = QC_\sigma(k, B) \cap QC_\rho(k, B)$.
2. $Q(k, A) = QC(k, A) \subseteq QC(k, B) \subseteq QC(k, D)$.
3. $Q_\sigma(k, A) = QC_\sigma(k, A) \subseteq QC_\sigma(k, B) \subseteq QC_\sigma(k, D)$.
4. $Q_\rho(k, A) = QC_\rho(k, A) \subseteq QC_\rho(k, B) \subseteq QC_\rho(k, D)$.

2. The Results

Let F be a compact subset of complex numbers and let $\mathfrak{I}$ be the class of all bounded and analytic complex valued functions on the unbounded component of ¢ -F. If each element in $\mathfrak{I}$ is constant, then F is called a Painleve null set. Painleve null sets coincide with compact sets of zero analytic capacity (7, p. 198).

If A is a complex normed algebra and $a\in A$, then the inner derivation corresponding to a is denoted by D_a, which is a bounded linear operator on A defined by $D_a x = ax - xa$. We define the bounded linear operators L_a and R_a on A by $L_a x = ax$ and $R_a x = xa$. For J an ideal of A, we will use the symbols D_a^J, L_a^J, and R_a^J to denote the restriction of these operators to J.

In [6] Rennison proved that if A is a complex Banach algebra with unity and $\sigma_A(a)$ has a zero analytic capacity for every a in $Q_\sigma(A)$, then $Q_\sigma(A) = Q(A) = Z(A)$. In this section we prove a similar result for quasi centralizers (Corollary1 of Theorem 2.1).

In the proof of the following theorem (which is similar to the proof of Theorem 3.7 in [(6)]) we need a lemma that is referred to Rennison [6, Lemma 3.6], which states that "Suppose that D is a domain in $\mathbb{C}$ and that K is a compact subset of D having zero planar measure. If f is a complex valued function analytic on D\ K and satisfies $|f(\mu)- f(\lambda)| \leq C |\mu - \lambda|$, for all μ, λ belonging to D \ K, then f extends to a function analytic on D".

2.1 Theorem

Let A be a complex Banach algebra with unity, J be an ideal of A, and $a \in QC_\sigma(J)$.

1. If $\sigma_A(a)$ has zero analytic capacity, then $D_a^J = 0$, and so $a \in C(J)$.
2. If $\sigma_A(a)$ has zero planar measure, then $(D_a^J)^2 = 0$.

Proof

First: Fix any $x \in J$ and a bounded linear functional Φ on J. Let $a \in QC_\sigma(J)$, then there exists $k \geq 1$ such that $\| y(\lambda - a) \| \leq k \| (\lambda - a) y \|$ for all $y \in J$ and $\lambda \in \rho_A(a)$. But J is an ideal, then $y = (\lambda - a)^{-1} x \in J$, hence $\| (\lambda - a)^{-1} x (\lambda - a) \| \leq k \| x \|$.

Define $f: \rho_A(a) \to \mathbb{C}$ by $f(\lambda) = \Phi((\lambda - a)^{-1} D_a x)$. Then for all $\lambda \in \rho_A(a)$, we have, $| f(\lambda) | = | \Phi((\lambda - a)^{-1} D_a x) | \leq \| \Phi \| \| (\lambda - a)^{-1} D_a x \| = \| \Phi \| \| (\lambda - a)^{-1} (ax - xa) \| = \| \Phi \| \| (\lambda - a)^{-1} (ax - \lambda x + x\lambda - xa) \| = \|\Phi\| \| (\lambda - a)^{-1} x (\lambda - a) - x \| \leq (k + 1) \|\Phi\| \| x \|$ ……………………..…. (1)

Therefore, f is bounded on $\rho_A(a)$. Let $\lambda, \mu \in \rho_A(a)$. Then $(\mu - a)^{-1} (\lambda - a) = (\lambda - a)(\mu - a)^{-1}$, hence $(\mu - \lambda)(\lambda - a)^{-1} (\mu - a)^{-1} = [(\mu - a) - (\lambda - a)](\lambda - a)^{-1}(\mu - a)^{-1} = (\lambda - a)^{-1} - (\mu - a)^{-1}$,

and

$$\frac{df}{d\lambda}(\mu) = \lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \lim_{\lambda \to \mu} \frac{\Phi([(\lambda - a)^{-1} - (\mu - a)^{-1}] D_a x)}{\lambda - \mu}$$

$$= \lim_{\lambda \to \mu} \frac{(\mu - \lambda)\Phi((\lambda - a)^{-1} (\mu - a)^{-1} D_a x)}{\lambda - \mu} = -\Phi((\mu - a)^{-2} D_a x)$$

Hence f is analytic and bounded on ρ_A (a). However σ_A (a) is a compact subset of complex numbers which has zero analytic capacity. Then by [7, p. 198], σ_A (a) is a Painleve null set, hence f is constant on ρ_A (a).

Let $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$, then by [5, p. 398], $(\lambda - a)^{-1} = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$, and so $f(\lambda) = \Phi\left(\sum_{n=0}^{\infty} a^n \lambda^{-n-1} D_a x\right) = \sum_{n=0}^{\infty} \lambda^{-n-1} \Phi(a^n D_a x)$. But f is constant, therefore, $\Phi(D_a x) = 0$, which implies $D_a x = 0$, since Φ is an arbitrary bounded linear functional on J, and x is also arbitrary in J, hence $D_a^J = 0$ and so $a \in C(J)$ □

Second: Let x and Φ as in (i) and let $a \in QC_\sigma(J)$. Define f: $\rho_A(a) \to \mathbb{C}$ by $f(\lambda) = \Phi((\lambda - a)^{-1} D_a^2 x)$. For all $\lambda, \mu \in \rho_A(a)$, $f(\lambda) - f(\mu) = \Phi([(\lambda - a)^{-1} - (\mu - a)^{-1}] D_a^2 x) = (\mu - \lambda)\Phi((\lambda - a)^{-1}(\mu - a)^{-1} D_a^2 x)$. As in (i) we proceed as follow: $|f(\lambda) - f(\mu)| = |(\mu - \lambda)\Phi((\lambda - a)^{-1}(\mu - a)^{-1} D_a^2 x)| \le |\mu - \lambda| \|\Phi\| \|(\lambda - a)^{-1}(\mu - a)^{-1} D_a D_a x\| = |\mu - \lambda| \|\Phi\| \|(\lambda - a)^{-1}(\mu - a)^{-1}(aD_a x - D_a xa)\| = |\mu - \lambda| \|\Phi\| \|(\lambda - a)^{-1}(\mu - a)^{-1}(aD_a x - \mu D_a x + \mu D_a x - D_a xa)\| = |\mu - \lambda| \|\Phi\| \|(\lambda - a)^{-1}(\mu - a)^{-1}[(a - \mu)D_a x + D_a x(\mu - a)]\| = |\mu - \lambda| \|\Phi\| \|(\lambda - a)^{-1}[- D_a x + (\mu - a)^{-1} D_a x(\mu - a)]\|$

$\le |\mu - \lambda| \|\Phi\| (\|(\lambda - a)^{-1} D_a x\| + \|(\lambda - a)^{-1}(\mu - a)^{-1} D_a x(\mu - a)\|)$

$\le |\mu - \lambda| \|\Phi\| (\|(\lambda - a)^{-1} D_a x\| + k\|(\mu - a)(\lambda - a)^{-1}(\mu - a)^{-1} D_a x\|)$

$= (k + 1)\|\Phi\| |\mu - \lambda| \|(\lambda - a)^{-1} D_a x\| \le (k+1)^2 \|\Phi\| \|x\| |\mu - \lambda|$ (see (1)).

Similarly, $|f(\lambda)| \le (k + 1)\|\Phi\| \|D_a x\|$. Thus f is bounded and uniformly Lipschitz on ρ_A (a). Also as in (i) it can be shown that f is analytic on ρ_A (a). Since σ_A (a) has zero planar measure, then, by [6, Lemma 3.6], f extends to a bounded entire function and hence is constant. As in (i) it follows that $(D_a^J)^2 = 0$ □

In [1, Example 2.5] we show that each of $QC_\sigma(B)$ and QC(B) need not be equal to C(B), where B is a closed subalgebra of a Banach algebra A. But in the following two corollaries we show that the three sets are the same under certain conditions.

Corollary 1

Let A be a complex Banach algebra with unity. If J is an ideal of A and $\sigma_A(a)$ has zero analytic capacity for all $a \in QC_\sigma(J)$, then $QC_\sigma(J) = QC(J) = C(J)$, and hence $QC_\sigma(J) \subseteq QC_\rho(J)$.

Proof

Use [1, Theorem 2.1 (i)] and Theorem 2.1 (i) to get the result □

Corollary 2

Let $A = M_n(\mathbb{C})$ be the complex Banach algebra of all $n \times n$ matrices $a = (a_{ij})$ over $\mathbb{C}$ with the norm $\| a \| = \max\{\sum_{i=1}^{n} |a_{1i}|, \sum_{i=1}^{n} |a_{2i}|, \ldots\ldots\ldots, \sum_{i=1}^{n} |a_{ni}|\}$ and J be an ideal of A. Then $QC_\sigma(J) = QC(J) = C(J)$, and hence $QC_\sigma(J) \subseteq QC_\rho(J)$.

Proof

Since $\sigma_A(a)$ is countable for all $a \in A$, then $\sigma_A(a)$ has a zero analytic capacity. Hence the result follows from Corollary 1 □

Corollary 3

[6, Theorem 3.7]. *Let A be a complex Banach algebra with unity.*

1. If $\sigma_A(a)$ has zero analytic capacity for every $a \in Q_\sigma(A)$, then $Q_\sigma(A) = Q(A) = Z(A)$.
2. If $a \in Q_\sigma(A)$ and $\sigma_A(a)$ has zero planar measure then $D_a^2 = 0$.

Proof

1. Let J = A in Corollary 1. Then use [1, Theorem 2.1] □

2. Let J = A in Theorem 2.1 (ii). Then use [1, Theorem 2.1] □

Corollary 2 can also be seen from the well-known fact that $A = M_n(\mathbb{C})$ is a simple algebra. So, either J = 0 and the result is obvious, or J $= M_n(\mathbb{C})$ and the result is a corollary of Rennison's result [6, Theorem 3.7].

If X is a complex Banach space, then we write BL(X) to represent the complex Banach algebra of bounded linear operators on X with pointwise addition and scalar multiplication but the product as a composition. The following lemma is similar to Lemma 4.1 in [6].

2.2 Lemma

Let A be a complex Banach algebra with unity, J be a closed ideal of A, $k \geq 1$, and $a \in QC(k, J)$. If M is a closed commutative subalgebra of BL(J) containing L_a^J, R_a^J, and the identity operator Id_J, then $\|D_a^J T\| \leq (k + 1) \|(\lambda - L_a^J) T\|$ for all $T \in M$ and $\lambda \in \mathbb{C}$.

Proof

Since $a \in QC(k, J)$, for all $x \in J$ and $\lambda \in \mathbb{C}$ we have $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$. However, $\| x (\lambda - a) \| = \| (\lambda - R_a) x \|$ and $\| (\lambda - a) x \| = \| (\lambda - L_a) x \|$. Then $\| (\lambda - R_a) x \| \leq k \| (\lambda - L_a) x \|$. So that $\| D_a x \| = \| (\lambda - R_a) x - (\lambda - L_a) x \| \leq (k + 1) \| (\lambda - L_a) x \|$.

Finally, since $Tx \in J$ for all $x \in J$, then the result follows by replacing x by Tx in the above inequality and taking the supremum over all x in J with $\| x \| = 1$□

Similarly one can easily prove the following remark.

Remark

Let A be a complex Banach algebra with unity, J be a closed ideal of A, $k \geq 1$, and $a \in A$. Assume that M is a closed commutative subalgebra of BL(J) containing L_a^J, R_a^J, and Id_J. Then:

1. If $a \in QC_\sigma (k, J)$, then $\|D_a^J T\| \leq (k + 1) \| (\lambda - L_a^J) T \|$ for all $T \in M$ and $\lambda \in \rho_A (a)$.

2. If $a \in QC_\rho(k, J)$, then $\|D_a^J T\| \leq (k+1)\,\|(\lambda - L_a^J)T\|$ for all $T \in M$ and $\lambda \in \sigma_A(a)$.

Corollary

Let A be a complex Banach algebra with unity and $k \geq 1$. Assume that M is a closed commutative subalgebra of BL(A) containing L_a, R_a, and the identity operator I. Then:

1. If $a \in Q(k, A)$, then $\|D_a T\| \leq (k+1)\,\|(\lambda - L_a)T\|$ for all $T \in M$ and $\lambda \in ¢$ [6, Lemma 4.1]
2. If $a \in Q_\sigma(k, A)$, then $\|D_a T\| \leq (k+1)\,\|(\lambda - L_a)T\|$ for all $T \in M$ and $\lambda \in \rho_A(a)$.
3. If $a \in Q_\rho(k, A)$, then $\|D_a T\| \leq (k+1)\,\|(\lambda - L_a)T\|$ for all $T \in M$ and $\lambda \in \sigma_A(a)$ [4, Proposition 2]

Proof

Use [1, Theorem 2.1], Lemma 2.2 and the above Remark □

If M is a commutative complex Banach algebra, then the radical of M is given by $Rad(M) = \{a \in M : \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0\}$ [6, p. 83]. We call an element a in a complex Banach algebra topologically nilpotent if $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0$.

In the proof of the following theorem we need a theorem that is referred to Rennison [6, Theorem 4.2], which says "Suppose that M is a commutative complex Banach algebra with unity, that u and v are elements of M, and that for some $c \geq 0$, $\|ux\| \leq c\,\|(\lambda - v)x\|$ for all $x \in M$ and all $\lambda \in ¢$. Then $u \in Rad(M)$".

2.3 Theorem

Let A be a complex Banach algebra with unity, J be a closed ideal of A, and $a \in QC(J)$. Then D_a^J is topologically nilpotent.

Proof

Since J is closed in the Banach algebra A, then J is complete, and so BL(J) is a complex Banach algebra. Since $a \in QC(J)$, then there exists $k \geq 1$ such that $a \in QC(k, J)$. Now, let M be a closed commutative subalgebra of BL(J) containing the identity operator Id_J, L_a^J, and R_a^J, then by Lemma 2.2, $\| D_a^J T \| \leq (k+1) \| (\lambda - L_a^J) T \|$ for all $T \in M$ and $\lambda \in \mathbb{C}$. But M is closed in BL(J) and BL(J) is complete, then M is complete. Now, by [6, Theorem 4.2] we see that $D_a^J \in Rad(M)$, and so $\lim \| (D_a^J)^n \|^{1/n} = 0$. That means, D_a^J is topologically nilpotent □

Corollary

[6, Theorem 4.3]. Let A be a complex Banach algebra with unity and $a \in Q(A)$. Then D_a is topologically nilpotent.

Proof

Use [1, Theorem 2.1 (ii)] and Theorem 2.3 □

In [6, Proposition 4.5], Rennison proved that if A is a complex Banach algebra with unity and if $a \in Q(A)$, $\lambda \in \mathbb{C}$, and $x \in A$, then $(\lambda - a)\, x = 0$ if and only if $x\, (\lambda - a) = 0$. Now we give a similar result with a similar proof for quasi centralizers.

2.4 Proposition

Let A be a complex Banach algebra with unity, J be a closed ideal of A, and $a \in QC(J)$. Then for each $\lambda \in \mathbb{C}$, and $x \in J$, $(\lambda - a)\, x = 0$ if and only if $x\, (\lambda - a) = 0$.

Proof

Let $a \in QC(J)$ and assume that $k \geq 1$ is such that $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in J$ and $\lambda \in \mathbb{C}$. Then, we see that $\| x (\mu - (\lambda - a)) \| = \| -x ((\lambda - \mu) - a) \| \leq k \| ((\lambda - \mu) - a)(-x) \| = \| (\mu - (\lambda - a)) x \|$ for all $\lambda, \mu \in \mathbb{C}$ and $x \in J$. Hence, $(\lambda e - a) \in QC(J)$ for all $\lambda \in \mathbb{C}$. ….……..……..... (1)

Now for any $x \in J$, if $ax = 0$, then $xa = 0$; Conversely, if $xa = 0$, then $D_a\, x = ax$. From which by induction we have, $D_a^n\, x = a^n\, x$ for all natural numbers n. Since Theorem 2.3 shows that $\lim \| (D_a^J)^n \|^{1/n} = 0$, and since $\| a^n x \|^{1/n} = \| (D_a^J)^n (x) \|^{1/n} \leq \| (D_a^J)^n \|^{1/n} \| x \|^{1/n}$, taking into a count that J is a closed ideal of A, we have $\Phi(\lambda) = \sum_{n=0}^{\infty} a^n x \lambda^{-n-1}$ as a convergent series in J. So that $(\lambda - a)\, \mathbf{\Phi}(\lambda) = x$, and $\mathbf{\Phi}(\lambda)(\lambda - a) = \sum_{n=0}^{\infty} a^n x \lambda^{-n}$. However , $a \in QC(k, J)$ and $\mathbf{\Phi}(\lambda) \in J$. Then $\| \mathbf{\Phi}(\lambda)(\lambda - a) \| \leq k \| (\lambda - a)\, \mathbf{\Phi}(\lambda) \| \leq k \| x \|$. Therefore, $f(\lambda) = \mathbf{\Phi}(\lambda)(\lambda - a)$ is a bounded J-valued function on $\mathbb{C} \setminus \{0\}$ which can easily be seen analytic there. But $\{0\}$ is a countable compact subset of $\mathbb{C}$, then by (7) it has zero analytic capacity and so, by [3, Theorem 1.10VIII], f extends to be analytic on $\mathbb{C}$. Hence, by Liouville's Theorem f is constant, that means $f(\lambda) = \mathbf{\Phi}(\lambda)(\lambda - a)$ has one value for all $\lambda \in \mathbb{C} \setminus \{0\}$. Equating to zero the coefficient of λ^{-1} in its Laurent expansion gives $ax = 0$. Therefore, $ax = 0$ if and only if $xa = 0$. Hence by (1) we get the result □

Corollary

[6, Proposition 4.5]. Let A be a complex Banach algebra with unity. If $a \in Q(A)$, $\lambda \in \mathbb{C}$, and $x \in A$, then $(\lambda - a)\, x = 0$ if and only if $x\, (\lambda - a) = 0$.

Proof

Use [1, Theorem 2.1(ii)] and Proposition 2.4 □

2.5 Proposition

Let A be a complex Banach algebra, J be an ideal of A, and $k \geq 1$. Then for each integer n:

1. If $a \in QC(k, J)$, then $\| (D_a^J)^n \| \leq (k+1)^n \| a \|^n$.
2. If $a \in QC\sigma(k, J)$ and $0 \in \rho_A(a)$, then $\| (D_a^J)^n \| \leq (k+1)^n \| a \|^n$.

3. If $a \in QC\rho$ (k, J) and $0 \in \sigma_A$ (a), then $\| (D_a^J)^n \| \leq (k+1)^n \| a \|^n$.

Proof

We prove (ii), and omit the similar proofs of (i) and (iii).

We prove by induction on n that $\| D_a^n x \| \leq (k+1)^n \| a \|^n \| x \|$ for all x in J.

Since $a \in QC_\sigma$ (k, J) and $0 \in \rho_A$ (a), we have, $\| xa \| \leq k \| ax \|$, and so $\| D_a x \| \leq \| ax \| + \| xa \| \leq (k+1) \| ax \| \leq (k+1) \| a \| \| x \|$ for all x in J. Next, we assume that for some integer n, $\| D_a^n x \| \leq (k+1)^n \| a \|^n \| x \|$. Then $\| D_a^{n+1} x \| = \| D_a^n D_a x \| \leq (k+1)^n \| a \|^n \| D_a x \| \leq (k+1)^{n+1} \| a \|^{n+1} \| x \|$. Therefore, for each integer n, $\| D_a^n x \| \leq (k+1)^n \| a \|^n \| x \|$ for all x in J. Hence, for each integer n, $\| (D_a^J)^n \| \leq (k+1)^n \| a \|^n$ □

Corollary

Let A be a complex Banach algebra and $k \geq 1$. Then for each integer n:

1. If $a \in Q(k, A)$, then $\| D_a^n \| \leq (k+1)^n \| a \|^n$.
2. If $a \in QC_\sigma$ (k, A) and $0 \in \rho_A$ (a), then $\| D_a^n \| \leq (k+1)^n \| a \|^n$.
3. If $a \in QC_\rho$ (k, A) and $0 \in \sigma_A$ (a), then $\| D_a^n \| \leq (k+1)^n \| a \|^n$ [4, Proposition 3].

Proof

Use [1, Theorem 2.1] and Proposition 2.5 □

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