# ON SEMILOCAL RINGS AND FINITELY GENERATED PROJECTIVE MODULES

*BY*

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## ملخـــــص

نطوّر في هذه الورقـة العلمية بعض النتائج المتعلقة بما يسمى بـ Cyclic projective modules بالاضافة الى نتائج جديدة أخرى كما يعطي المؤلف مثالاً يثبت من خلاله أن النتائج التي حصل عليها لا يمكن تعميمها الى أنواع أعم من Cyclic projective modules .

## Abstract

We establish the equivalence between the existence of a cyclic projective right C−module P such that P/J ($_B$ P) is semisimple and $_B$P is finitely generated (B = End ($P_c$)) and the existaence of an idempotent element e in c such that trace ($P_c$) is finitely generated (and equals CeC) and C/S is semisimple where S is the intersection of all maximal left ideals of C not containing the trace ideal of $P_c$ .

## 1. Introduction , Notations and Preliminaries :

In this paper we study further the situation where P is finitely generated projective right C−module (C is a ring) with trace ideal I , B = End ($P_c$) , J = J(B) and P is finitely generated as a left B−module .

The existence of a cyclic projective right C−module P such that P/J ($_B$P) is semisimple and $_B$P is finitely generated (B = End ($P_c$)) is shown to be equivalent to the existence of an idempotent element in C such that trace ($P_c$) is finitely generated and equals CeC and C/S is semisimple , where S is the intersection of all maximal left ideals of C not containing the trace ideal of $P_c$ . It is also demonstrated that the above result can not be generalized to the case $P_c$ is finitely generated projective .

The following terminology , notations and results are needed .

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Given t a left exact radical on A$-$Mod and M in A$-$Mod we say that M is t$-$torsion (t$-$torision free) provided that $t(M) = M(t(M) = 0)$.

A submodule N of M is t$-$closed if M/N is t$-$torsion free. A non$-$zero left A$-$module M is t$-$simple if M/N is t$-$torsion for all submodules N of M such that $N \subsetneq t(M)$. A submodule N of M is t$-$maximal if M/N is t$-$simple and t$-$torsion free. (for torsion theoretical concepts we refer the reader to Stenstrom, B., 1971).

Define $\mathbb{M}$ to be the collection of all maximal left ideals M of C such that the trace ideal, I, of $P_c$ is not contained in M, $S = \bigcap_{\mathbb{M}} M$, $\mathbb{K}$ is the collection of all t$-$maximal left ideals of C, $N = \bigcap_{\mathbb{K}} K$ and $C(S) = \{ c + S \in C/S : c+S \text{ is regular} \}$.

The following results may be found in Morita, K., 1970, Sandomierski, F.L., 1972 and Mohammad A., 1987.

**Lemma 1.1 :** The following hold

1- $J(P_c) = PS$

2- $J({}_BP) = PN$

**Lemma 1.2 :** Let $G = Hom(P_c, C_c)$, then :

1- G is a finitely generated projective left C$-$module.

2- $B = End({}_cG)$.

3- The trace ideal of ${}_cG$ = the trace ideal of $P_c$, and

4- $P \otimes .$ is naturally equivalent to $Hom({}_cG, .)$.

**Proposition 1.3 :** The following hold :

1- If U is a simple left C$-$module and $IU = O$, then $SU = O$.

2- If H is a t$-$maximal submodule of ${}_cG$, then H is a maximal submodule of G.

3- If U is a t$-$simple left C$-$module, then U contains a simple submodule, and

4- $SI \subset N$

If in addition to the hypothesis in the situation under consideration we have B/J is semisimple then we have the following [ see Mohammad, A., 1987].

**Proposition 1.4 :** The following hold :

i - C/S semisimple,

ii - C/N has finite left Goldie dimension as a left C/N modul and is nonsingular,and

iii - $I + S = C$.

The following lemma is needed and can be found in Sandomierski, F.L. 1972.

**Lemma 1.5 :** If $P_c$ is finitely generated projective with trace ideal I, then ${}_cI$ is finitely generated iff ${}_BP$ is finitely generated, where $B = End(P_c)$.

Although we really need the remark after the following theorem , but we must write the statement of the theorem which was proved by Ghazi, M. Eid in his Ph.D. thesis in 1983 .

**Theorem 1.6 :** Let P in A−Mod be flat of hype FP, $B = \text{End}(P_A)$ , $C = \text{End}({}_BP)$ , M in A−Mod and ${}_BX = T(M) = P \otimes M$ , then

1 – If N is a submodule of M and Y a left B−submodule of X, let $\overline{T}(N) = \text{Img}(T(N)) \to T(M)$ and $\overline{S}(Y) = \{ m \in M : P\hat{m} \subset Y \}$ , then $\overline{S}$ and $\overline{T}$ are lattice isomorphisms of the lattice of t−closed submodules of M and the lattice of submodules of X .

2 – ${}_A$ M is a t−simple submodule of ${}_AM$ if and only if $\overline{T}(N)$ is a maximal submodule of T(M) .

3 – ${}_AM$ has a unique t−maximal submodule if and only if T(M) has a unique maximal submodule .

4 – ${}_AN$ is a t−simple submodule of ${}_AM$ if and only if T(N) is a simple submodule of T(M) .

**Remark :** For $P_A$ a flat right A−Module of type FP and $B = \text{End}(P_A)$ theorem 1.6 above yields a one−to−one correspondence between the submodules of P(as a left B−module) and the t−closed left ideals of A. In particular , if X is a submodule of P and $D = \{ a \text{ in } A: Pa \subset X \}$ , then clearly D is a t−closed ideal of A and X = PD .

## 2. Semilocal Rings

Recall that a ring R is semilocal provided that R/J , J the Jacobson radical of R, is a semisimple ring .

**Proposition 2.1 :** Let C be a ring and P a finitely generated projective right C−module with trace ideal I satisfying :

i – $\phi = \{ L \subset C : PL = P$, where L is a left ideal of C $\}$ is an idempotent topologizing filter on C .

ii – For $B = \text{End}(P_c)$ , B/J is semisimple , and

iii – ${}_BP$ is finitely enerated .

Then the following hold :

i – C/S is semisimple , I+S = C and ${}_cI$ is finitely generated .

ii – $\phi = \{ L \subset C : I \subset L \}$ .

iii – If $\{ D_k \}_K$ is chain of proper left ideals of C such that $D_k \notin \phi$ for all $k \in K$ , then $U_K D_k \notin \phi$

**Proof** : In view of proposition 1.4 and lemma 1.5 i) holds and we listed it only for completion.
To prove ii), let L be a left ideal of C such that $I \subset L$, then $PI \subset PL$. Since $PI = P$, then $P \subset PL$. Since the other inclusion is trivial then $PL = P$. On the otherhand if $PL = P$ then $P/PL = 0$. But $P/PL$ is isomorphic to $P \otimes (C/L)$, thus $P \otimes (C/L) = \circ$ which in turn implies that $I(C/L) = \circ$, hence $I \subset L$.

Finally for iii), assume on the contrary that $D_k \notin \Phi$ for all $k \in K$, but $U_K D_k \in \Phi$. Then $P(UD_k) = U_K (PD_k) = P$. Since $\{ D_k \}_K$ is a chain of proper left ideals of C, then $\{ PD_k \}_K$ is a totally ordered collection of proper submodules of $_BP$. Since $_BP$ is finitely generated then $U(PD_k) \subsetneq P$. Acontradication and iii) follows.

The following lemmas are needed for later consideration.

**Lemma 2.2**:For a left C−module M the following are equivalent.

i − $\dim ({}_cM) < \infty$.

ii − whenever $M_1 \supset M_2 \supset M_3 \supset \ldots\ldots$, where $M_i$ is a left C−submodule of M, then $\exists N$, a positive integer, such that $M_{N+1}$ is large in M.

**Proof :**

i − implies ii :

Given i) suppose that ii) does not hold. Rearrange and renumber the descending sequence so that $M_{i+1}$ is not large in $M_i \forall i$. Thus in particular we get $M_2$ (say) is not large in $M_1$, hence $\exists M_1' \subset M_1$ such that $M_1' \cap M_2 = \circ$. Notice that $M_1' + M_2 \subset M_1$. Since $M_3$ is not large in $M_2$, then $\exists M_2 \subset M_2'$ such that $M_3 \cap M_2' = \circ$. Again notice that $M_2' + M_3 \subset M_2$.
Now we have the direct sum $M_1 \oplus M_2 \oplus M_3$. Continue the above process. In view of i) this process must terminate, while if the contrary of ii) holds, the process above must be infinite, an obsurdity. Thus ii) holds.

ii) implies i) :

Suppose on the contrary that ii) holds but $\dim({}_cM)$ is not finite. Write $M = \oplus L_i$, Where $L_i$ is a submodule of M $\forall i$. Then $\oplus_{i=1} L_i \supset \oplus_{i=2} L_i \supset \ldots\ldots$ with each term being not large in the preceeding one, a contradication to ii). Thus ii) implies i).

**Lemma 2.3** : Let M be a left C−module such that $t({}_c M) = \circ$, $\dim ({}_cM) < \infty$ then $_cM$ has the descending chain condition on t−closed submodules.

**Proof** : Let $M_1 \supset M_2 \supset \ldots\ldots \supset M_n \ldots\ldots$ be a descending chain of t−closed left C−submodules of M. be Lemma 2.2 $\exists$ N, a positive integer such that $M_{N+1}$ is large in $M_N$. Thus $M_{N+1}$ is t−dense in $M_N$. By hypothesis $M_{N+1}$ is also t-closed in $M_N$.The only way out is that $M_{N+1} = M$ and the proof is complete.

**Remark 2.4 :** Replacing M above by C/N we get $_c(C/N)$ to have the descending chain condition on t−closed left C−submodules . Thus C has the descending chain condition on t−closed left ideals containing N .

**Lemma 2.5 :** $N = \bigcap_{i=1}^{n} K_i \ (K_i \in \mathbb{K})$ .

**Proof :** By Remark 2.4 , C has the descending chain condition on left ideals containing N . By definition $K_i \supset N \ \forall i$ . Now , pick $K_1 \in \mathbb{K}$ if $K_1 = N$ we are done , if not , then $K_1 \supsetneqq N$ and in the latter case $\exists K_2 \in \mathbb{K}$ such that $K_1 \cap K_2 \supseteqq N$ . If equality holds we are done , otherwize , $K_1 \cap K_2 \supsetneqq N$ and hence $\exists K_3 \in \mathbb{K}$ such that $K_1 \supset K_1 \cap K_2 \supset K_1 \cap K_2 \cap K_3 \supset N$ . Continuing in the above manner we must have a positive integer n such that $N = \bigcap_{i=1}^{n} K_i$ .

**Proposition 2.6 :**

Let $P_c$ be a finitely generated projective right C−module with trace ideal I that is finitely generated and suppose that dim $(_{C/N}(C/N))$ is finite .

Then $_B P$ is finitely generated and B/J (B) is semisimple , where $B = \mathrm{End}\,(P_c)$ .

**Proof :** Since $J(_BP) = PN \supset J(B)\,P$ , then

P/PN is a left B/J (B) module . Next ,
$P/PN = P/P(\bigcap_{i=1}^{n} K_i)) = P/(\bigcap_{i=1}^{n} (PK_i))$ .

Thus the B−homomorphism $P/PN \rightarrow P/PK_1 \times \ldots\ldots \times P/PK_n$ is one-to-one . Since $\prod_{i=1}^{n} (P/PK_i)$ is semisimple , then so is $_B(P/PN)$ .

Now, $_BP$ is a generator for B−Mod, thus $\exists\ n \in Z^+$ and an epimorphism $f : P^n \rightarrow B$. Since $J(P^n) = (J(P))^n$ and f maps the radical to the radical then f induces the epimorphism :
$\bar{f} : P^n / J(P^n) \rightarrow B/J(B)$ .

Being the homomorphic image of a semisimple module , B/J(B) is semisimple . In view of lemma 1.5 the other conclusion is trivial .

In case $P_c$ is cyclic projective we get better results as the theorem below shows . But for that we recall that $_cG = \mathrm{Hom}\,(P_c, C_c)$ is finitely generated projective which has the same trace ideal as that of $P_c$ . We also need the following lemma .

**Lemma 2.7 :** $J(_cG) = SG$

**Proof :** $J(_CG) = J(C)G = (\bigcap M)G$ , where the intersection runs over the maximal left ideals of C .

Since $(\bigcap M)\, C \subseteq (\bigcap_m M)G = SG$ , then $J({}_cG) \subset SG$ . On the othedr hand , if ${}_cM$ is maximal in ${}_cG$ , then $G/M$ is simple . Now , either $I \not\subset M$ and hence $G/M$ is not torsion , in which case we get $S(G/M) = 0$ (by proposition 1.3.i) and hence $SG \subset M$ which implies that $SG \subset J({}_cG)$, or $I \subset M$, in which case, since $IG = G$, then $MG = G$ , so that $J({}_cG) = J(C)G = (\bigcap M)G$, where the intersection runs over all maximal left ideals of C, thus $J({}_cG) = (\bigcap MG)$ , where the intersection runs over the maximal left ideals of C not containing I. Since $(\bigcap M)G \subset MG$ for all maximal left ideals of C, then $SG \subseteq \bigcap MG = J({}_cG)$ and the lemma is proved .

For $b \in B = \text{End}(P_c)$ ( $= \text{End}\ ({}_cG)$ by lemma 1.2) we have the induced homomorphism $\bar{b} : G/(SG) \rightarrow G/(SG)$ where $(g+SG)\bar{b} = gb + SG$ . Next, given $h \in \text{End}\ (G/SG)$ we have the diagram .

$$\begin{tikzcd} & {}_cG \arrow[d, "g"] & \\ & {}_c\bar{G} \arrow[d, "h"] & \\ {}_cG \arrow[r, "g"] & {}_c\bar{G} \arrow[r] & 0 \end{tikzcd}$$

Where g is the natural epimorphism $G \rightarrow G/SG$ and $\bar{G} = G/SG$ . By the projectivity of ${}_cG$ there exist a unique $b : {}_cG \rightarrow {}_cG$ making the diagram commutative (Note that upto this point we used only the projectivity of ${}_cG$) . Thus we have $B = \text{End}\ ({}_cG) \rightarrow \text{End}(G/SG)$ is a surjective ring hamomorphism . Simple calculations show that its kernel $= \{ b \in B : Gb \text{ is small in } {}_cG \} = J(B)$ ( see also Sandomierski , F.L., 1964) . Thus we have $B/J(B)$ is isomorphic to End $(G/SG)$ .

**Theorem 2.8 :** Given a ring C, the following are equivalent :

1 – There exist a right C−module P which is cyclic projective such that for $B = \text{End}(P_c)$ we have
   a – $P/J({}_BP)$ is semisimple as a left B−module , and
   b – ${}_BP$ is finitely generated .
2 – There exist an idempotent element e in C such that :
   i – The trace ideal , I, of $P_c$ equals CeC and is finitely generated , and
   ii – C/S is semisimple .

**Proof : 1 – implies 2**

$P_c$ cyclic implies that there exists an idempotent e in C such that $P = eC$ . Now , easily [ see Anderson and Fuller PP 266 ] the trace ideal of $P_c = CeC$ . The other conclusions of 2. are clear [see proposition 1.4 and lemma 1.5]

2 – implies 1 :

By 2 there exists an idempotent e in C . Define $P_c = eC$ . Then by choice $P_c$ is cyclic and projective [ see Anderson and Fuller PP 199 ] with trace ideal = CeC . By lemma 1.5 , $_BP$ is finitely generated . Observe that $B = End(P_c) = End\ (eC)$ is isomorphic to eCe . Now $G = Hom(P_c,C_c) = Hom\ (eC,C) = Ce$ . Also B/J(B) is isomorphic to End(G/J(G) = End(Ce/Se) which is the endomorphism ring of a semisimple ring and hence is semisimple . Thus B/J(B) is semisimple and the proof is complete .

**Remark 2.9** : 1. (and hence 2.) in Theorem 2.8 implies that

i – $I + S = C$ .
ii – $SI \subset N$, and
iii – $Se = Ne$ .

**Proof :**

i – Suppose on the contrary that $I + S \subsetneqq C$ , then there exist a maximal left ideal of C such that $(I + S) \subset M \subset C$ . Thus C/M is simple with $I(C/M) = \circ = S(C/M)$ which is impossible .
ii – See proposition 1.3 .
iii – $SI \subset N$ implies that $SIG \subset NG$ with $IG = G$ and $NG \subset SG$. Thus $SG = NG$. But $SG = SCe = Se$ and $NG = NCe = Ne$ and iii) follows .

We demonstrate now that Theorem 2.8 above can not be generalized to the case $P_c$ finitely generated projective .

If $P_c$ is a finitely generated projective module , then there is a positive integer n such that $C_c^{(n)} . \rightarrow P_c \rightarrow \circ$ is exact . Since End $({}_cC^{(n)})$ is isomorphic to $C_{nxn}$ with ${}_cC^{(n)}$ finitely generated projective generator , then $C^{(n)} \otimes_C C^{(n)} \rightarrow P \otimes_C C^{(n)} \rightarrow \circ$ is exact with $C^{(n)} \otimes C^{(n)}$ is isomorphic to $(C_{nxn})_{Cnxn}$ and $(P \otimes_C C^{(n)})_{Cnxn}$ is isomorphic to $P^{(n)}_{Cnxn}$ which we claim to be cyclic .

To prove this claim , write $P = Cx_1 + \ldots + Cx_n$ and let $y = [y_1, \ldots, y_n] \in P^{(n)}$ . Since $y_j \in P$ for every $j = 1,2,\ldots,n$, then each $y_j$ is a linear combination of $x_1,\ldots,x_n$ with coefficents from C . Form an nxn matrix , $M_{nxn}$ , by putting the coefficients of the linear combination for $y_j$ in the $j_{th}$ column , then , as simple calculations show , we have $[x_1,\ldots,x_n]\,M_{nxn} = [y_1,\ldots,y_n]$ . Thus we have shown that $P^{(n)}_{Cnxn} \,.= [x_1,\ldots,x_n]$ $C_{nxn}$ and the claim is proved . Observe now that :

1 – The trace ideal of $P_c$ , I, corresponds to the trace ideal $I_{nxn}$ of $P^{(n)}_{Cnxn}$ and $_c I$ is finitely generated (iff $_B P$ is finitely generated) iff $I_{nxn}$ is finitely generated .
2 – As ideals , S also corresponds to $S_{nxn}$ and C/S is semisimple iff $C_{nxn} / S_{nxn}$ is semisimple . But $C_{nxn} /S_{nxn}$ is isomorphic to $(C/S)_{nxn}$ , thus C/S is semisimple iff $(C/S)_{nxn}$ is semisimple .

Nevertheless , for $I_{nxn} .= C_{nxn}\, eC_{nxn}$ for some idempotent element e of $C_{nxn}$ , it is NOT necessary the case that e corresponds to an idempotent in C (for example, take $C = F$ , a field , and $C_{nxn} .= F_{nxn}$ where we don't know how to find an idempotent element e in C such that $I = CeC$) .

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