On the Size of Blocking Sets in $\Omega^+(12,q)$

حول حجم المجموعات المغلقة $\Omega^+(12,q)$

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Abstract

Considering half-spin geometry of type $D_{6,6}(F)$, we investigate the size of substructures of the geometry called blocking sets. We give an upper bound on size of blocking sets.

Keyword: half-spin geometry-blocking set-Covers-classical polar spaces.

ملخص

ليكن لدينا هندسة النصف مغزلية من نوع $D_{6,6}(F)$، سنتحقق من حجم تركيبات موجودة داخل الهندسة (سنثبت وجودها ونعطي وصفها) والتي تسمى بالمجموعات المغلقة، وكذلك سنقدم حدا اقصى لحجم تلك المجموعة.

Introduction

In this paper, special objects inside the half-spin geometry of type $D_{6,6}(F)$ are described, such as blocking sets and covers. We also obtain combinatorial information since the number of points, lines, etc. is finite. In (Blokhuis, & et.al. 1998), studied covers of the projective space of type PG(3,q) (and of finite generalized quadrangle) which is small. In essence, they gave a structure theorem for minimal covers S with $q^2 + 1 < |S| < q^2 + q + 1$. In (Aiden, & Drudge, 1998), studied a large minimal covers of PG (3,*q*). In (De Beule, 2004), gave an interesting study of blocking sets for some finite classical polar spaces. In (Cimrakova, & Fack, 2005), presented results on smallest blocking sets in the generalized quadrangle Q(4, *q*) for *q*=5, 7, 9, 11 and they found minimal blocking sets of size $q^2 + q - 2$.

2. Basic Definitions and Notations

Let V be a vector space over an arbitrary field F. **A bilinear form** B on V is a mapping B: V x V → F, such that for α, β∈F, *x*, *y*, *z* ∈V we have:

i. $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$.

ii. $B(z, \alpha x + \beta y) = \alpha B(z, x) + \beta B(z, y)$.

Thus a bilinear form is a linear functional in each of its coordinates.

For (a subspace) W⊂V, we set

$W^{\perp}{}_L = \{u \in V: B(u, v)=0, \text{ for all } v \in W\}$,

$W^{\perp}{}_R = \{u \in V: B(v, u)=0, \text{ for all } v \in W\}$.

$W^{\perp}{}_L$, $W^{\perp}{}_R$ are called the left and right radicals of W with respect to B.

A bilinear form B is called symmetric if B(*u*, *v*)= B(*v*, *u*) for all vectors *u*, *v*∈V. A bilinear form B is called alternate if B(*u*, *u*)= 0 for all vectors *u*∈V. If B is a symmetric form, then $V^{\perp}{}_R = V^{\perp}{}_L$ is called the radical of V with respect to B and is denoted by $V^{\perp}$. A bilinear form B is called non-degenerate if $V^{\perp} = \{0\}$. Otherwise B is called degenerate.

A vector *u*∈V is called an isotropic vector if B(*u*, *u*)=0, and a subspace W of V is called totally isotropic (abbreviated TI) if B(*u*, *v*)=0 for all *u*, *v* ∈W. A subspace W of V is called maximal totally isotropic if W is not contained properly in any TI subspace of V.

Given a set I, **a geometry** Γ over I is an ordered triple Γ=(X,∗, D), where X is a set, D is a partition $\{X_i\}$ of X indexed by I, X_i are called components, and $*$ is a symmetric and reflexive relation on X called incidence relation such that:

$x * y$ implies that either x and y belong to distinct components of the partition of X or $x = y$. Elements of X are called **object**s of the geometry, and the objects within one component X_i of the partition are called the objects of type i. The subscripts that index the components are called **types**. The obvious mappingτ: $X \rightarrow I$, which takes each object to the index of the component of the partition containing it is called the type map τ.

A point-line geometry (P, L) is simply a geometry for which $|I|=2$, one of the two types is called *points*; in this notation the points are the members of P, and the other type is called *lines.* Lines are the members of L. If $p \in P$ and $l \in L$, then $p * e$ stands for $p \in l$. In point-line geometry (P, L), we say that two points of P are *collinear* if they are incident with a common line. (We use the symbol ~ for collinear)

$x^{\perp}$ means the set of all points in P collinear with x, including x itself.

A clique of P is a set of points in which every pair of points are collinear.

A partial linear space is a point-line geometry, in which every pair of points are incident with at most one line, and all lines have cardinality at least 2.

A point-line geometry is called **singular** or (**linear)** if every pair of points are incident with a unique line.

A subspace of a point-line geometry Γ=(P, L) is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. $\langle X \rangle$ means the intersection of all subspaces containing X, where $X \subseteq P$.

Lines incident with more than two points are called **thick** lines, those incident with exactly two points are called **thin lines**.

The singular rank of a space Γ is the maximal number *n* (possibly ∞) for which there exists a chain of distinct subspaces $\varnothing \neq X_0 \subset X_1 \subset \ldots \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$. For example rank (∅) = -1, rank({*p*}) = 0 where *p* is a point and rank(L) = 1 where L a line.

In a point-line geometry Γ=(P, L), **a path of length** *n* is a sequence of *n*+ 1 points $(x_0,x_1,..,x_n)$ where, (x_i,x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point.

A geodesic from a point *x* to a point *y* is a path of minimal possible length with initial point *x* and end point *y*. We denote this length by $d_\Gamma(x, y)$.

A geometry Γ is called **connected** if for any two of its points there is a path connecting them.

A subset X of P is said to be **convex** if X contains all points of all geodesics connecting two points of X.

A gamma space is a point-line geometry such that for every point-line pair (*p*, *l*), *p* is collinear with either no point, exactly one point, or all points of *l*, i.e., $p^{\perp} \cap l$ is empty, consists of a single point, or equal *l*.

A polar space is a point-line geometry Γ= (P, L) satisfying the Buekenhout-Shult axiom:

For each point-line pair (*p, l*) with *p* not incident with *l*, *p* is collinear with one or all points of *l*, that is $|p^{\perp} \cap l| = 1$ or else $p^{\perp} \supset l$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of Γ for every point $p \in$ P.

We write Rad (Γ) for the set $\{p: p^{\perp}=P\}$, and we called it the radical of Γ.

A polar space Γ=(P, L) is said to be **non-degenerate** if Rad Γ=∅.

A projective plane is a point-line geometry Γ= (P, L) which satisfies the following conditions:

(i) Γ is a linear space i.e, every two distinct points x, y in P lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of which lie on a line.

A projective space is a point-line geometry in which the following conditions are satisfied:

(i) every two distinct points lie exactly on one line ,

(ii) if l_1, l_2 are two lines with $l_1 \cap l_2 \neq \varnothing$, then $<l_1, l_2>$ is a projective plane. ($<l_1, l_2>$ means the smallest subspace of Γ containing l_1 and l_2.)

A parapolar space is a point-line geometry Γ=(P, L) of rank $r + 1$, $r \geq 2$; and satisfies the following conditions:

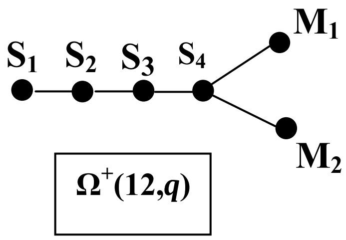
(pp1) Γ is a connected gamma space.

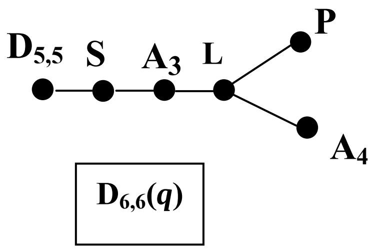
(pp2) for every line l; $l^{\perp}$ is not a singular space.

(pp3) for every pair of distinct points x, y; $x^{\perp} \cap y^{\perp}$ is either empty, a point, or a nondegenerate polar space of rank r.

A strong parapolar space is a parapolar space in which $x^{\perp} \cap y^{\perp}$ is a polar space for every pair of distinct points x, y of distance 2 apart.

3. Definition of the half-spin geometry $D_{n,n}(F)$





Now we give a construction of $D_{6,6}(q)$. Let B be a symmetric bilinear form on a vector space of dimension 12 over a finite field F=GF(q). Define the polar space $\Omega^{+}(12,q)$. Let S_i be the of all TI idimensional subspaces of V, $1 \le i \le 4$. Let S_6 be the class that consists of all maximal TI subspaces of dimension 6. S_6 is partitioned into two classes denoted by M_1, M_2 subjected to the following rule:

Two TI 6-subspaces m_1 and m_2 fall in the same class if their intersection is of even dimension. So the dimension of the intersection $m_1 \cap m_2$ is 0, 2, or 4 for distinct m_1, m_2. Thus the points of $D_{6,6}(q)$ consist of one class (M_1, say) of the two classes of MTI 6-spaces, and whose set of lines corresponds to the set of all TI 4-spaces, where a line l that corresponds to a 4-subspace X is incident with the set of all points that corresponds to all TI 6-spaces that contains X.

Symplecta (that are convex non-degenerate polar spaces of rank at least 2) correspond to the set of all TI 2-subspaces, where a symplecton S that corresponds to a TI 2-subspace Y is the set of all TI 6-subspaces that contains Y. The half-spin geometries $D_{5,5}(q)$ correspond to TI 1-subspaces. TI 3-subspaces correspond to projective subspaces of singular rank 3; A_3's. TI 6-subspaces of the second class M_2 corresponds to projective subspaces of singular rank 5; A_5's.

Let the map Ψ: $P \to V$ that forms a correspondence between the half–spin geometry $D_{6,6}(q)$ and the classical polar space of type $\Omega^+(12,q)$ which is defined above, i.e., $\Psi(p)$ is the TI 6-space corresponding to the point p. We will use Ψ for the rest of the varieties of the geometry; for example $\Psi(l)$ is the TI 4-space corresponding to the line l, and $\Psi(S)$ is the TI 2-space corresponding to the symplecton S. The inverse map Ψ^{-1} will be used for the inverse; for example $\Psi^{-1}(\pi)$ is the symplecton corresponding to the TI 2-spaceπ.

We summarize the most important properties of the half-spin geometry $D_{6,6}(q)$ in the following theorems

1. $D_{6,6}(q)$ is a strong parapolar space of Diameter 3.
2. If S_1 and S_2 are two distinct symplecta, then either $S_1 \cap S_2$ is empty, a line or a maximal singular subspace of both (a member of A_3).
3. If (p, S) is a non-incident pair of point and symplecton S, then $p^{\perp} \cap S$ is either a single point or a maximal singular subspace of S.

4. The main result

Most papers are interested in the cardinality of blocking sets but in projective spaces, and in this paper we present a general definition of the blocking sets. To apply this idea on some kinds of finite geometries such as half-spin geometry $D_{6,6}(q)$, description of blocking sets and upper bound of its cardinality will be investigated.

A (t, s)-**blocking set** of PG (n, q), where $n\geq 2$, $n\geq s\geq 1$ and $n-1\geq t\geq 0$, is a set B of points of PG(n, q) satisfying the following properties :

i. any subspace of dimension n-t of PG(n, q) intersects B in at least one point;

ii. any s-dimensional subspace of PG(n, q) contains at least one point not in B

A blocking set of PG (n, q), $n\geq 2$, is a set B of points of PG(n, q) satisfying:

i. any hyperplane (a subspace of dimension n-1) of PG(n, q) intersects B in at least one point;

ii. any line of PG(n, q) contains at least one point not in B.

So a blocking set is the same as a (1, 1)-blocking set.

Now we generalize the definition of the blocking set by applying it at half-spin geometry $D_{6,6}(q)$.

Firstly, we give a first part of the result by describing a blocking set of $D_{6,6}(q)$:

4.1 Theorem A blocking set of $D_{6,6}(q)$ is the set of all points that are of distance at most 2 from a fixed point; namely

$\Delta^*_2(p) = \{x\in P\text{: } d(x, p)\leq 2\}$.

Proof. Let l be a line in $D_{6,6}(q)$. Let U be the correspondent TI 4-space, i.e., U = $\Psi(l)$. We take a fixed point of $D_{6,6}(q)$, say p, then $\Psi(p)$ is a MTI 6-space. Now there are 2 cases for the intersection $\Psi(l)\cap\Psi(p)$:

1. $\Psi(l) \cap \Psi(p)$ = 2-space , say W; In this case the 2-space $\Psi(l)\backslash W$ has the property that $(\Psi(l)\backslash W)^{\perp} \cap \Psi(p)$ = 4 –space which is equal exactly to $W \cup D$, where D is a TI 2-space contained in $\Psi(p)\backslash W$. Then we have a point $s = \Psi^{-1} < \Psi(l) \cup D >$ such that $\Psi^{-1} < \Psi(l) \cup D > \cap \Psi(p) = < W \cup D >$= 4-space and $\Psi(l) \subseteq \Psi(s)$, i.,e., the point s lies on the line l and $s \in \Delta^*_2(p)$.
2. $\Psi(l) \cap \Psi(p)$ = 0-space , then $\Psi(l)^{\perp} \cap \Psi(p)$ is at most a TI 2-space, then we get the TI 6- space $<\Psi(l), \Psi(l)^{\perp} \cap \Psi(p)>$ which is a point, say r, where r lies on the line l and of a distance at most 2 of the point p.), i.,e., the point r lies on the line l and $r \in \Delta^*_2(p)$.

The remaining part is to prove that the line l has at least a point not in $\Delta^*_2(p)$. Let $\Psi(l) = <x_1, x_2, x_3, x_4>$, let p be a point such that $\Psi(p) = <y_1, y_2, y_3, y_4, y_5, y_6>$ and take the case at which $K = \Psi(l) \cap \Psi(p)$ = 2-space. Since the TI 4-space $\Psi(l)$ contained in maximal TI 6-spaces, say, $\Psi(s) =< x_1, x_2, x_3, x_4, u_1, u_2 >$ and $\Psi(r) =< x_1, x_2, x_3, x_4, w_1, w_2 >$, then they are considered to be the corresponding two points r and s that are incident to the line l and at the same time r is collinear to s. Now we find out another point q that is collinear to the points r and p, since $< w_1, w_2 >^{\perp} \cap \Psi(p)$= TI 4-space that is: $< K \cup <y_3, y_4> >$, then we get a TI 6-space $\Psi(q) = < K \cup <y_3, y_4> \cup < w_1, w_2 > >$ that corresponds to the point q. Thus we found out a point s incident to the line l such that $d(s, p)=3$ So, the point does not belong to $\Delta^*_2(p)$.

Then $\Delta^*_2(p)$ is a blocking set of $D_{6,6}(q)$. ■

Propositions 4.2 and 4.3 will be used to prove the second part of the main results. The propositions and their proofs can be found in (Cameron, 1992) and (Cimrakova, & Fack, 2005).

4.2 Proposition (Cameron, 1992). The number of subspaces of dimension k in a vector space of dimension n over GF(q) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \quad \frac{(q^n-1)(q^n-q)\ \dots\ (q^n-q^{k-1})}{(q^k-1)(q^k-q)\ \dots\ (q^k-q^{k-1})}$$

Proof. This is Proposition 1.4.1 in (Cameron, 1992).

Remark. This number in Proposition 4.2 is called **a Gaussian coefficient**, and is denoted by

4.3 proposition (Cimrakova, & Fack, 2005). Let V be equipped with a bilinear form. Then the number of totally isotropic k-subspaces is the following:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i}+1)$$ in the symplectic case W(2n,q).

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i}+1)$$ in the orthogonal case Ω(2n+1,q).

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i-1}+1)$$ in the hyperbolic case $\Omega^{+}(2n,q)$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1}(q^{n-i-1}+1)$$ in the elliptic case $\Omega^{-}(2n+2,q)$.

Proof. See (Cimrakova, & Fack, 2005).

Now we present an upper bound of a blocking set in $D_{6,6}(q)$.

Theorem 4.4 Let **B** be a blocking set in $D_{6,6}(q)$. Then

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q \prod_{i=0}^{1}(q^{6-i-1}+1)\,|B| \leq \frac{(q^6-1)\,(q^{10}-1)\,(q^4+1)}{(q^2-1)\,(q-1)}$$

Proof. In blocking set $\Delta^{*}_{2}(p)$, we showed that any TI 4-space which intersects Ψ(p) in a TI 2-space, is contained in a maximal TI 6-space. Then by the correspondence between the half-spin geometry $D_{6,6}(q)$ and the hyperbolic case $\Omega^{+}(12,q)$, any line has a point in $\Delta^{*}_{2}(p)$. Then any 2-space that can be found in Ψ(p) gives a TI 6-space which intersects Ψ(p) in a 4-space. So the maximal number of 2-spaces in Ψ(p) determines the maximal number of point in $\Delta^{*}_{2}(p)$. Using Propositions 4.2 and 4.3, the number of 2-spaces that are contained in a 6-space can be determined by the formula

Then the upper bound of the size of B is given by: $|B| \le \begin{bmatrix} 6 \\ 2 \end{bmatrix}_q \prod_{i=0}^{1} (q^{6-i-1}+1)$

and since $\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q \prod_{i=0}^{1} (q^{6-i-1}+1) = \dfrac{(q^6-1)\,(q^{10}-1)\,(q^4+1)}{(q^2-1)\,(q-1)}$

it follows that $|B| \le \dfrac{(q^6-1)\,(q^{10}-1)\,(q^4+1)}{(q^2-1)\,(q-1)}$

References

- Blokhuis, A. O'keefe, C.M.S. Payne, Storme, L. & Wilbrink, H. (1998). "Minimal covers of PG(3,*q*) and finite generalized quadrangles". In Proceedings of the third international conference on finite Geometry and Combinatorics at Deinze, F. Declerck et al. (eds). *Bell. Belg. Math Soc*.- Simon Stevin **5.** 141-157.
- Aiden, A. Bruen. & Drudge, Keldon. (1998). "Large minimal covers of PG (3,*q*)". Bull. Belg. Math. Soc. (5). 207-212.
- De Beule, Jan. (2004). "Blocking sets and partial spreads in finite spaces" PHD thesis Faculteit Wetenschappen Vakgroep Zuivere Wiskunde en Computeralgebra
- Cameron, P.J. (1992). "Projective and polar spaces". QMW Maths Notes. (13). Queen Mary and Westfield College, University of London.
- Cimrakova, Miroslava & Fack,Veerle. (2005)."On minimal blocking sets of the generalized quadrangle Q(4,*q*)" *Discrete Mathematics and Theoretical Computer Science* (DMTCS), AE, 299-302.