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The Wave Equation with Energy Dependent Potential - the Linear Case

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Abstract

We summarize recent works devoted to energy dependent potentials. It concerns the class of potentials having a coupling constant depending linearly on the energy. Introduced in the Schrödinger equation, it produces non-linear effects. Few cases admit analytical solutions. They are of great help to get acquainted with this non-linearity. The harmonic oscillator and the Coulomb potential are presented as typical examples. Applications concern heavy. quark-anti-quark systems, as well as the many-body problem with harmonic interactions. Finally, we show that the energy dependence does not modify the number of bound states of attractive potentials. It can regularize some potentials singular at the origin. For instance, the ground state energy of the $-1/r^2$ potential in D=3 dimensional space becomes finite for finite energy dependence.

Motivations and generalities

Wave equations with potentials depending on the energy are known since the early days of relativistic quantum mechanics. The Pauli-Schrödinger equation is a typical example [1]. Recently they appear in the Hamiltonian formalism of the relativistic many-body problem, i.e. in the manifestly covariant formalism with constraints [2,3,4]. More generally, energy dependent potentials have been used in the Schrödinger

equation to simulate non-linear effect, for the soliton propagation or interacting clusters [5,6,7] (see also [8,9,10] and [11]).

This presentation summarizes works done in collaboration with my colleagues [12-15]. The aim was to build simple examples, toy models, with analytical solutions if possible, in order to get acquainted with the effect of the energy dependence of the potential and show the differences with respect to the usual case.

The starting point is the time dependent wave equation ($m = \hbar = 1$)

$$i\,\partial\Psi(\mathbf{r},t)/\partial t = [-1/2\,\Delta + V(\mathbf{r},i\partial/\partial t)\,]\,\Psi(\mathbf{r},t), \quad (1)$$

$V(\mathbf{r},i\partial/\partial t)$: a real function of 2 variables. Setting $\Psi(\mathbf{r},t) = e^{-iEt}\,\Psi(\mathbf{r})$ yields

$$H\,\Psi(\mathbf{r}) = [-1/2\,\Delta + V(\mathbf{r},E)\,]\,\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}). \quad (2)$$

The first modification of the usual rules of quantum mechanics concerns the scalar product [16, 17]. Consider two solutions of energies E and E'

$$\Phi_\varepsilon(\mathbf{r},t) = e^{-i(E-i\varepsilon)t}\,\Phi(\mathbf{r}) \text{ and } \Psi_\varepsilon(\mathbf{r},t) = e^{-i(E'-i\varepsilon)t}\,\Psi(\mathbf{r}) \quad (3)$$

with $E - E' = \varepsilon \rightarrow 0$, and the continuity equation

$$\partial P/\partial t = - \boldsymbol{\partial}\mathbf{j}. \quad (4)$$

Here,

$$P = \Psi^*_\varepsilon(\mathbf{r},t)\,\Phi_\varepsilon(\mathbf{r},t) \quad \text{and} \quad \mathbf{j} = - i/2\,[\Psi^*_\varepsilon(\mathbf{r},t)\,\boldsymbol{\partial}\Phi_\varepsilon(\mathbf{r},t) - \boldsymbol{\partial}\Psi^*_\varepsilon(\mathbf{r},t)\,\Phi_\varepsilon(\mathbf{r},t)].$$

In the case of $V(\mathbf{r},E)$, an additional term is needed, namely

$$\partial P_a/\partial t = i\,\Psi^*_\varepsilon(\mathbf{r},t)[V(\mathbf{r},E) - V(\mathbf{r},E')]\,\Phi_\varepsilon(\mathbf{r},t). \quad (5)$$

After integration, it yields

$$P_a = - \Psi^*_\varepsilon(\mathbf{r},t)[(V(\mathbf{r},E') - V(\mathbf{r},E))/(E' - E)]\,\Phi_\varepsilon(\mathbf{r},t)\,. \quad (6)$$

In the limit $E' \rightarrow E$, the scalar product (the norm) is given by

$$N = \int \Psi^*(\mathbf{r})[1 - \partial V/\partial E]\,\Psi(\mathbf{r})d\mathbf{r}\,. \quad (7)$$

Note here an essential point: in order that $\Psi^*(\mathbf{r})[1 - \partial V/\partial E]\ \Psi(\mathbf{r})$ represents a density, it has to be positive definite. This imposes constraints on the energy dependence for the theory to be coherent.

The orthogonality relation between two states n and n', $n \neq n'$, is given by

$$\int \Psi_n^*(\mathbf{r})[1-\{V(\mathbf{r},E_{n'})-V(\mathbf{r},E_n)\}/\{E_{n'}- E_n\}]\Psi_{n'}(\mathbf{r})d\mathbf{r} = 0. \quad (8)$$

The modification of the scalar product is necessary but not sufficient to ensure the coherence of the theory.

Simple examples

It has been shown that a linear energy dependence affecting the coupling constant leads to a coherent theory. Moreover, it can be reformulated as an ordinary quantum theory with a non-local potential [12,15]. We consider potentials of the following form

$$V(\mathbf{r},E) = \lambda(1 + \gamma E)V_0(\mathbf{r}). \quad (9)$$

In particular we shall discuss the effect of the energy dependence on two types of power-law radial shapes:

$$V_0(r) = r^{\alpha}. \quad (10)$$

A. $\alpha = 2$: the harmonic oscillator potential in D=1 dimensional space :

$$[- 1/2\ d^2/dx^2 + 1/2(1 + \gamma E_n)x^2\ ]\Psi_n(x) = E_n\Psi_n(x). \quad (11)$$

Setting

$$\lambda_n^2 = 1 + \gamma\ E_n\ ;\ k_n^2 = 2\ E_n\ , \quad (12)$$

leads to

$$[d^2/dx^2 + k_n^2 - \lambda_n^2\ x^2\ ]\ \Psi_n(x) = 0. \quad (13)$$

The solutions are obtained by the ansatz

$$\Psi_n(z) = C_n\ h_n(z)\ \exp(-z^2/2) \quad \text{with} \quad z = \sqrt{(\lambda_n)}\ x, \quad (14)$$

which yields

$$d^2/dz^2\ h_n(z) - 2z d/dz\ h_n(z) + (k_n^2/\lambda_n^2 - 1)h_n(z) = 0. \quad (15)$$

The parameter λ_n depends on the state, and the solutions $h_n(z)$ are the Hermite polynomials. Their orthogonality is ensured by the weight function

$$\exp[-(\lambda_n + \lambda_{n'})x^2/2]\ [1 - \gamma x^2/2]. \quad (16)$$

The coherence of the theory requires $\gamma < 0$. The energies and the λ_n's are obtained from Eq. (15) together with

$$k_n^2 = (2n + 1)\ \lambda_n = 2\ E_n\ . \quad (17)$$

The eigenvalues are the roots of a second order equation. Actually, only the positive roots are retained, the negative ones lead to non-normalizable functions. Thus, the energy spectrum is given by

$$E_n = (2n+1)^2/\gamma + (n + 1/2)\sqrt{[1 + (2n + 1)^2\ \gamma^2/16]}\ ;\ \gamma < 0. \quad (18)$$

It is easy to check that E_n is a monotonically increasing function of n. Furthermore, a saturation effect is observed : $lim\_\{n\to\infty\}\ E_n = 1/|\gamma|$. As the quantum number increases, the corresponding eigenvalues reach an upper bound, and the density of states tends to infinity.

The extension to the D=3 dimensional space is straightforward but some care must be taken regarding the definition of the angular momentum. This one has its usual form if the potential is spherically symmetric.

Illustrative examples are displayed in Fig1. Spectra with positive are shown for comparison, although the model is not coherent in this case.

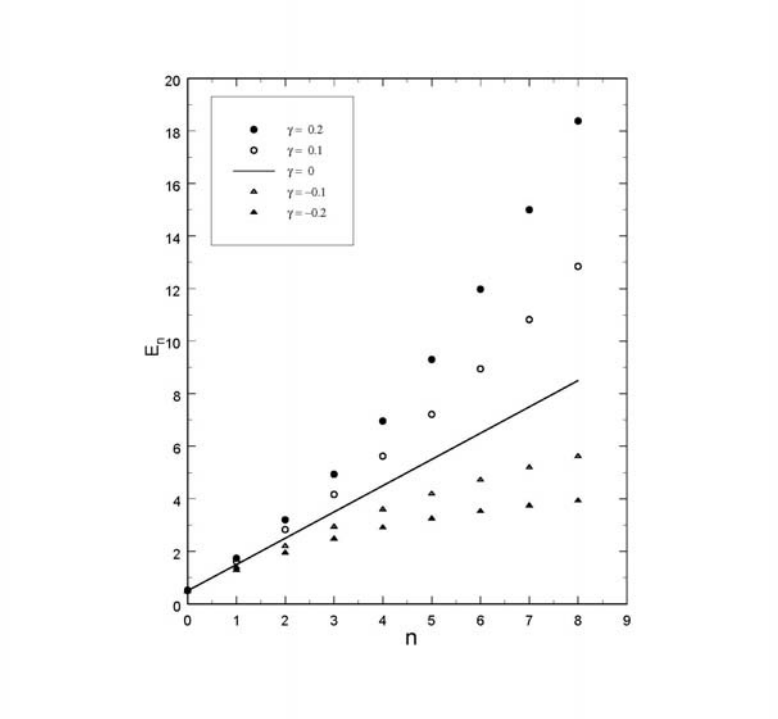

Figure (1): E_n as function of n for the linear E-dependent harmonic oscillator.

Only negative values of γ lead to a coherent theory [12].

B : α = -1: the Coulomb potential (D=3).

$$V(r, E_{nl}) = \lambda(1 + \gamma E_{nl})/r \quad \text{with } \lambda < 0. \tag{19}$$

The reduced radial wave functions take the form

$$\Phi_{nl}(r) = C_{nl}\, r^{l+1} \exp(- a_{nl} r/2) P_{nl}(r), \tag{20}$$

with $P_{nl}(r)$: the polynomial form of the confluent hyper-geometric functions. The quantization condition implies

$$a_{nl} = - 2\lambda(1 + \gamma E_{nl})/(n + l),\ n = 1,2,3,\ldots\ E_{nl} = -a_{nl}^2/8. \tag{21}$$

The eigenvalues are given by a second order equation with roots

$$E^{\pm}_{nl} = [-(n+l)^2 - \gamma\lambda^2 \pm (n+l)\sqrt{\{(n+l)^2 + 2\gamma\lambda^2\}}]/\lambda^2\gamma^2. \quad (22)$$

Only the E^{+}_{nl} roots are retained, the E^{-}_{nl} ones corresponding to non-normalizable solutions. The energy dependence of the potential affects essentially the lowest levels, making them less bound than in the case of $\gamma = 0$. This is easily seen by taking the large quantum number limit, which is independent on γ :

$$lim\_\{n,l \to \infty\} \quad E_{nl} \approx -\lambda^2/(2(n+l)^2) \quad (23)$$

For illustrative purposes, we list here eigenvalues for three values of γ corresponding to the equations (18) and (22).

Table (1): The harmonic oscillator in the one-dimensional space : Eq. (18).

n	**γ = 0.**	**γ = -0.25**	**γ = -0.50**
0	0.500	0.470	0.441
1	1.500	1.245	1.040
2	2.500	1.837	1.386
3	3.500	2.289	1.588
4	4.500	2.632	1.711
5	5.500	2.893	1.788
6	6,500	3.094	1.840
7	7.500	3.249	1.875
8	8.500	3.371	1.900
9	9.500	3.467	1.918
10	10.500	3.544	1.932

Table (2): The Coulomb potential: Eq. (22) for $\lambda = 1.0$.

n	**$\gamma = 0.$**	**$\gamma = -0.25$**	**$\gamma = -0.50$**
0	0.50000	0.40408	0.34315
1	0.12500	0.11775	0.11146
2	0.05556	0.05406	0.05267
3	0.03125	0.03077	0.03031
4	0.02000	0.01980	0.01961
5	0.01389	0.01379	0.01370
6	0.01020	0.01015	0.01010
7	0.00781	0.00778	0.00775
8	0.00617	0.00615	0.00614
9	0.00500	0.00499	0.00498
10	0.00413	0.00412	0.00412

Application to heavy quark-anti-quark systems

The spectra of the heavy mesons, based on a quark-anti-quark model $q\bar{q}$, constitute a natural application. As a toy model, used can be made of

$$V(\mathbf{r},E_{nl}) = [\ \mu\omega^2\mathbf{r}^2/2 + D_0\,\boldsymbol{l}^2](1 + E_{nl}) \qquad (24)$$

for the $q\bar{q}$ potential. Here, μ is the reduced mass, $\boldsymbol{l}$ is the angular momentum operator and the $\boldsymbol{l}^2$ term is known to simulate a potential well intermediate between the harmonic shape and the square well potential at low angular momentum. This model has been applied to the $c\bar{c}$ (charmonium) and $b\bar{b}$ (bottomium) spectra.

Table (3): The $c\bar{c}$ and $b\bar{b}$ spectra.

	Charmonium		**Bottonium**	
Spectrum		**exp.**		**exp.**
1p	397	397	428	428
2s	589	589	563	563
1d	713	710	759	701
2p	819	776	805	792
3s	941	943	897	895
4s	1153	1063	1098	1120
parameters				
ω (fm^{-1})	2.75		2.65	
D_0 (MeV)	85.3		153.0	
$\gamma\ 10^{-4}$ (MeV)	-4.336		$-4.558\ 10^{-4}$	
m (GeV)	1.207		4.401	

In each case, the quark mass is linked to the total ground state energy, i.e. to the physical mass of the $q\bar{q}$ system: $M(q,\bar{q}) = 2\ m_q + E_{1s}$. The results are summarized in table 3 and compared to experimental values. The agreement is satisfactory, although the potential has a radial shape quite different from those currently used in this topic [13]. For this reason, the low levels, used to fix the parameters, are very well reproduced, whereas the higher levels show some deviation. Note that without the energy dependence, the harmonic oscillator potential does not even qualitatively fit these two spectra.

The many-body problem with an energy dependent confining potential.

Harmonic interactions have the property of producing soluble many-body problems. In this respect, it is interesting to check the effect of the energy dependence. We consider N bosons of equal mass m in D=3 interacting through

$$V(\mathbf{r}_i - \mathbf{r}_J) = \omega^2(1 + \gamma E)\,(\mathbf{r}_i - \mathbf{r}_J)^2. \quad (25)$$

Here, E is the total energy of the system.

The Hamiltonian is given by ($\hbar = 1$)

$$H(\mathbf{p}_i, \mathbf{r}_i) = 1/2m \sum_i^N \mathbf{p}_i^2 + \{m\,\omega^2/2\}(1 + \gamma E) \sum_{i<j}^N (\mathbf{r}_i - \mathbf{r}_J)^2. \quad (26)$$

The use of Jacobi coordinates allows the separation of centre of mass motion, and leaves us with an Hamiltonian of N - 1 separable harmonic oscillators. It reads

$$H(\boldsymbol{\pi}_i,\boldsymbol{\xi}_i) = \{1/4m\} \sum_i^{N-1}(i + 1)\boldsymbol{\pi}_i^2/i + \{m\,\omega^2/2\}(1 + \gamma E)N\sum_i^{N-1} i\boldsymbol{\xi}_i^2/(I + 1)\,. \quad (27)$$

The total wave function is a product of harmonic oscillator wave functions. The energy of each oscillator is given by

$$E_{ni,li} = \alpha_i\,(i + 1)(2\,l_i + 4n_i + 3)/(4\,m\,i) \quad (28)$$

with

$$\alpha_i = 2i\,\alpha_1/(I + 1). \quad (29)$$

The total energy becomes

$$E = \{\alpha_1/2\,m\} \sum_i^{N-1} (2\,l_i + 4\,n_i + 3) = \{\alpha_1/2\,m\}[3(N-1) + 2 \sum_i^{N-1}(l_i + 2n_i) = \alpha_i\,S/2\,m. \quad (30)$$

Taking into account that

$$\alpha_1^2 = m^2\omega^2\,N(1 + \gamma E)/2, \quad (31)$$

the energy is given by

$$8\,E^2 - \omega^2 NS^2\,\gamma\,E - \omega^2 NS^2 = 0. \quad (32)$$

The only root yielding a square integrable solution is

$$E = \omega^2 NS^2 \gamma \left[1 - (1 + 32/\omega^2 NS^2 \gamma^2)^{1/2}\right]/16 . \quad (33)$$

This result has 2 implications. First, for large quantum numbers, i.e. as $S \rightarrow \infty$, the energy reaches an upper limit independent on N : $lim\__{S \rightarrow \infty} E = 1/|\gamma|$.

Secondly, as N increases, the ground state energy increases. Thus, the density of states increases up to the point where the concept of individual states looses its meaning :

transitions among states cost no energy. This situation is illustrated in the figure below.

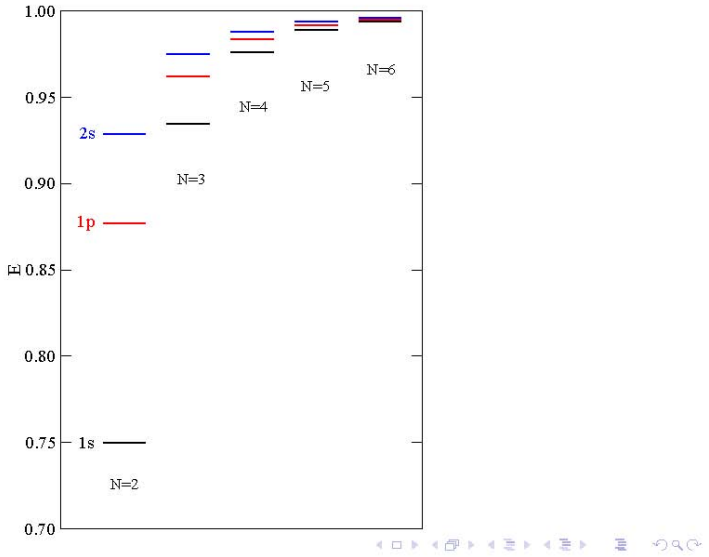

Figure (2): Few low lying eigenvalues E_{nl} as function of the number of particles N.

Critical¨ situations

In the D=3 dimensional space, short and finite range potentials are defined as having at the most a finite number of bound states. They decrease asymptotically faster then $1/r^2$, and they are not too singular at the origin. The spherical symmetry is assumed. It is well known that for these potentials the existence of a bound state is linked to the value of the coupling constant. In other words, for a given state $\{n,l\}$, a critical value of the coupling constant $\lambda_c(nl)$ exists below which this state is unbound. The critical $\lambda_c(nl)$ is obtained by solving the Schrödinger equation at E_{nl} =0. Consequently, the energy dependence of the coupling constant does not modify the $\lambda_c(nl)$.

Furthermore, for a fixed angular momentum l, the number of bound states is given by the number of nodes of the first unbound wave function. It increases by one unit when passing from $\lambda < \lambda_c(nl)$ to $\lambda > \lambda_c(nl)$. As a consequence, the number of bound states is not affected by the energy dependence of the coupling constant. The major modifications concern the eigenvalues and the level densities. The two examples given above are particularly clear.

Another interesting case is the power-law potential with $\alpha = -2$. In D=3 it leads to a collapse of the ground state energy which is infinite. Here, the energy dependence has a regularisation effect. It can be shown that the ground state wave function is proportional to $K_0(\beta r)$, the modified Bessel function [18], and the ground state energy takes the form

$$E_{1s} = - (\lambda - 1/4)/\lambda\gamma \; ; \; \gamma \geq 0. \qquad (34)$$

Thus, for any finite value of γ the ground state energy is finite.

To illustrate this situation, the E_{1s} energies have been calculated numerically for attractive power-law potentials with $-2 \leq \alpha \leq -0.5$. The results are displayed in Fig 3.

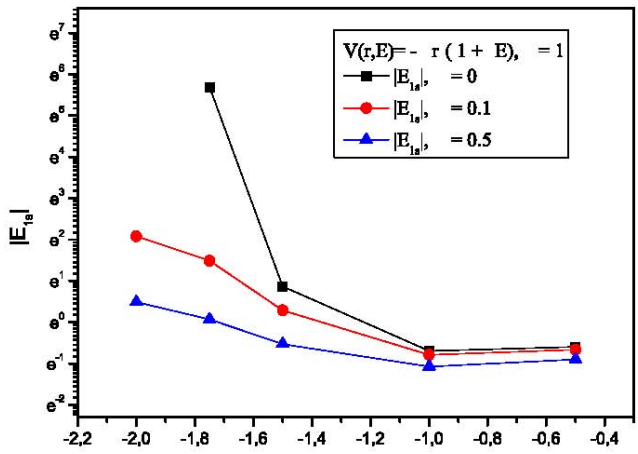

Figure (3): E_{1s} energies as function of α for different values of γ.

Conclusions

The present work is devoted to energy dependent potentials. We consider potentials having their coupling constant depending linearly on the energy. This case leads to a coherent theory. The energy dependence affects essentially the eigenvalues. The spectrum is compressed and the density of states is enlarged.

For confining potentials, we note the appearance of a saturation energy : the spectrum is bounded from above. In the many-body problem with harmonic interactions, the saturation energy is independent of the number of particles, while the ground state energy increases with N. Thus, for large N, the concept of individual levels becomes dubious.

For attractive power-law potentials, the energy dependence affects essentially the lowest states, which are less bound. It has a regularisation effect illustrated for $\alpha = -2$. In this case, the ground state energy is finite as soon as $\gamma \neq 0$.

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